## 5.5 Substitution Rule

Given just about any differentiable function, with enough know-how and persistence, you can compute its derivative. But the same cannot be said of antiderivatives. Many functions, even relatively simple ones, do not have antiderivatives that can be expressed in terms of familiar functions. Examples are $\sin x^2$, and $(\sin x)/x$. The immediate goal of this section is to enlarge the family of functions for which we can find antiderivatives. This campaign resumes in Chapter 8, where additional integration methods are developed.

### Indefinite Integrals »

One way to find new antiderivative rules is to start with familiar derivative rules and work backward. When applied to the Chain Rule, this strategy leads to the Substitution Rule. For example, consider the indefinite integral $\int \cos 2x\, dx$. The closest familiar integral related to this problem is

$$\int \cos x\, dx = \sin x + C,$$

**Note »**

We assume $C$ is an arbitrary constant without stating so each time it appears.

which is true because

$$\frac{d}{dx}(\sin x + C) = \cos x.$$

Therefore, we might *incorrectly* conclude that the indefinite integral of $\cos 2x$ is $\sin 2x + C$. However, by the Chain Rule,

$$\frac{d}{dx}(\sin 2x + C) = 2\cos 2x \neq \cos 2x.$$

Note that $\sin 2x$ fails to be an antiderivative of $\cos 2x$ by a multiplicative factor of 2. A small adjustment corrects this problem. Let's try $\frac{1}{2}\sin 2x$:

$$\frac{d}{dx}\left(\frac{1}{2}\sin 2x\right) = \frac{1}{2}\cdot 2\cos 2x = \cos 2x.$$

It works! So we have

$$\int \cos 2x\, dx = \frac{1}{2}\sin 2x + C.$$

The trial-and-error approach of the previous example does not work for complicated integrals. To develop a systematic method, consider a composite function $F(g(x))$, where $F$ is an antiderivative of $f$; that is, $F' = f$. Using the Chain Rule to differentiate the composite function $F(g(x))$, we find that

$$\frac{d}{dx}(F(g(x))) = \underbrace{F'(g(x))}_{f(g(x))}\, g'(x) = f(g(x))\, g'(x).$$

This equation says that $F(g(x))$ is an antiderivative of $f(g(x))\,g'(x)$, which is written

$$\int f(g(x))\,g'(x)\,dx = F(g(x)) + C, \tag{1}$$

where $F$ is any antiderivative of $f$.

Why is this approach called the *Substitution Rule* (or *Change of Variables Rule*)? In the composite function $f(g(x))$ in equation (1), we identify the inner function as $u = g(x)$, which implies that $du = g'(x)\,dx$. Making this identification, the integral in equation (1) is written

$$\int \underbrace{f(g(x))}_{f(u)}\,\underbrace{g'(x)\,dx}_{du} = \int f(u)\,du = F(u) + C.$$

**Note »**

> You can call the new variable anything you want because it is just another variable of integration. Typically, $u$ is a standard choice for the new variable.

We see that the integral $\int f(g(x))\,g'(x)\,dx$ with respect to $x$ is replaced by a new integral $\int f(u)\,du$ with respect to the new variable $u$. In other words, we have substituted the new variable $u$ for the old variable $x$. Of course, if the new integral with respect to $u$ is no easier to find than the original integral, then the change of variables has not helped. The Substitution Rule requires some practice until certain patterns become familiar.

**THEOREM 5.6 Substitution Rule for Indefinite Integrals**

Let $u = g(x)$, where $g$ is differentiable on an interval, and let $f$ be continuous on the corresponding range of $g$. On that interval,

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du.$$

In practice, Theorem 5.6 is applied using the following procedure.

**PROCEDURE Substitution Rule (Change of Variables)**

**1.** Given an indefinite integral involving a composite function $f(g(x))$, identify an inner function $u = g(x)$ such that a constant multiple of $u'(x)$ (equivalently, $g'(x)$) appears in the integrand.

**2.** Substitute $u = g(x)$ and $du = u'(x)\,dx$ in the integral.

**3.** Evaluate the new indefinite integral with respect to $u$.

**4.** Write the result in terms of $x$ using $u = g(x)$.

*Disclaimer: Not all integrals yield to the Substitution Rule.*

### EXAMPLE 1 Perfect substitutions

Use the Substitution Rule to find the following indefinite integrals. Check your work by differentiating.

**a.** $\int 2\,(2x + 1)^3\,dx$

**b.** $\int 2x \cos x^2 \, dx$

**SOLUTION »**

**a.** We identify $u = 2x + 1$ as the inner function of the composite function $(2x+1)^3$. Therefore, we choose the new variable $u = 2x + 1$, which implies that $\frac{du}{dx} = 2$, or $du = 2\,dx$. Notice that $du = 2\,dx$ appears as a factor in the integrand. The change of variables looks like this:

$$\int \underbrace{(2x+1)^3}_{u^3} \cdot \underbrace{2\,dx}_{du} = \int u^3 \, du \qquad \text{Substitute } u = 2x+1,\; du = 2\,dx.$$

$$= \frac{u^4}{4} + C \qquad \text{Antiderivative}$$

$$= \frac{(2x+1)^4}{4} + C. \qquad \text{Replace } u \text{ by } 2x+1.$$

Notice that the final step uses $u = 2x + 1$ to return to the original variable.

**Note »**

Use the Chain Rule to check that $\frac{d}{dx}\left(\frac{(2x+1)^4}{4} + C\right) = 2(2x+1)^3$.

**b.** The composite function $\cos x^2$ has the inner function $u = x^2$, which implies that $du = 2x\,dx$. The change of variables appears as

$$\int \underbrace{\left(\cos x^2\right)}_{\cos u} \underbrace{2x\,dx}_{du} = \int \cos u \, du \qquad \text{Substitute } u = x^2,\; du = 2x\,dx.$$

$$= \sin u + C \qquad \text{Antiderivative}$$

$$= \sin x^2 + C. \qquad \text{Replace } u \text{ by } x^2.$$

To check the result, we compute $\frac{d}{dx}\left(\sin x^2 + C\right) = \left(\cos x^2\right) 2x = 2x \cos x^2$.

*Related Exercises 17, 20–21* ◆

**Quick Check 1** Find a new variable $u$ so that $\int 4x^3\left(x^4+5\right)^{10} dx = \int u^{10} \, du$. ◆

**Answer »**

$u = x^4 + 5$

Most substitutions are not perfect. The remaining examples show more typical situations that require introducing a constant factor.

## EXAMPLE 2 Introducing a constant

Find the following indefinite integrals.

**a.** $\int x^4 (x^5+6)^9 \, dx$

**b.** $\int \cos^3 x \sin x \, dx$

**SOLUTION »**

**a.** The inner function of the composite function $(x^5+6)^9$ is $x^5+6$ and its derivative $5x^4$ also appears in the integrand (up to a multiplicative factor). Therefore, we use the substitution $u = x^5 + 6$, which implies that $du = 5x^4 \, dx$ or $x^4 \, dx = \frac{1}{5} \, du$. By the Substitution Rule,

$$\int \underbrace{(x^5+6)^9}_{u^9} \underbrace{x^4 \, dx}_{\frac{1}{5} du} = \int u^9 \cdot \frac{1}{5} \, du \qquad \text{Substitute } u = x^5+6, \; du = 5x^4 \, dx \Rightarrow x^4 \, dx = \frac{1}{5} \, du.$$

$$= \frac{1}{5} \int u^9 \, du \qquad \int c f(x) \, dx = c \int f(x) \, dx$$

$$= \frac{1}{5} \cdot \frac{u^{10}}{10} + C \qquad \text{Antiderivative}$$

$$= \frac{1}{50} (x^5+6)^{10} + C. \qquad \text{Replace } u \text{ by } x^5+6.$$

**b.** The integrand can be written as $(\cos x)^3 \sin x$. The inner function in the composition $(\cos x)^3$ is $\cos x$, which suggests the substitution $u = \cos x$. Note that $du = -\sin x \, dx$ or $\sin x \, dx = -du$. The change of variables appears as

$$\int \underbrace{\cos^3 x}_{u^3} \underbrace{\sin x \, dx}_{-du} = -\int u^3 \, du \qquad \text{Substitute } u = \cos x, \; du = -\sin x \, dx.$$

$$= -\frac{u^4}{4} + C \qquad \text{Antiderivative}$$

$$= -\frac{\cos^4 x}{4} + C. \qquad \text{Replace } u \text{ by } \cos x.$$

*Related Exercises 23–24* ◆

**Quick Check 2** In Example 2a, explain why the same substitution would not work as well for the integral $\int x^3 (x^5+6)^9 \, dx$. ◆

**Answer »**

> With $u = x^5 + 6$, we have $du = 5x^4 \, dx$, and $x^4$ does not appear in the integrand.

Sometimes the choice for a $u$-substitution is not so obvious *or* more than one $u$-substitution works. The following example illustrates both of these points.

**EXAMPLE 3** **Variations on the substitution method**

Find $\int \frac{x}{\sqrt{x+1}}\,dx$.

**SOLUTION »**

**Substitution 1** The composite function $\sqrt{x+1}$ suggests the new variable $u = x+1$. You might doubt whether this choice will work because $du = dx$ and the $x$ in the numerator of the integrand is unaccounted for. But let's proceed. Letting $u = x+1$, we have $x = u-1$, $du = dx$, and

$$\begin{aligned}\int \frac{x}{\sqrt{x+1}}\,dx &= \int \frac{u-1}{\sqrt{u}}\,du && \text{Substitute } u = x+1,\ du = dx.\\ &= \int \left(\sqrt{u} - \frac{1}{\sqrt{u}}\right) du && \text{Rewrite integrand.}\\ &= \int \left(u^{1/2} - u^{-1/2}\right) du. && \text{Fractional powers}\end{aligned}$$

We integrate each term individually and then return to the original variable $x$:

$$\begin{aligned}\int \left(u^{1/2} - u^{-1/2}\right) du &= \frac{2}{3}\,u^{3/2} - 2\,u^{1/2} + C && \text{Antiderivatives}\\ &= \frac{2}{3}\,(x+1)^{3/2} - 2\,(x+1)^{1/2} + C && \text{Replace } u \text{ by } x+1.\\ &= \frac{2}{3}\,(x+1)^{1/2}\,(x-2) + C. && \text{Factor out } (x+1)^{1/2} \text{ and simplify.}\end{aligned}$$

**Substitution 2** Another possible substitution is $u = \sqrt{x+1}$. Now $u^2 = x+1$, $x = u^2 - 1$, and $dx = 2\,u\,du$. Making these substitutions leads to

$$\begin{aligned}\int \frac{x}{\sqrt{x+1}}\,dx &= \int \frac{u^2-1}{u}\,2\,u\,du && \text{Substitute } u = \sqrt{x+1},\ x = u^2-1.\\ &= 2\int \left(u^2 - 1\right) du && \text{Simplify the integrand.}\\ &= 2\left(\frac{u^3}{3} - u\right) + C && \text{Antiderivatives}\\ &= \frac{2}{3}\,(x+1)^{3/2} - 2\,(x+1)^{1/2} + C && \text{Replace } u \text{ by } \sqrt{x+1}.\\ &= \frac{2}{3}\,(x+1)^{1/2}\,(x-2) + C. && \text{Factor out } (x+1)^{1/2} \text{ and simplify.}\end{aligned}$$

Observe that the same indefinite integral is found using either substitution.

**Note »**

In Substitution 2, you could also use the fact that

$$u'(x) = \frac{1}{2\sqrt{x+1}},$$

which implies

$$du = \frac{1}{2\sqrt{x+1}}\,dx.$$

*Related Exercises 74–75* ◆

## General Formulas for Indefinite Integrals

Integrals of the form $\int f(ax)\,dx$ occur frequently in the remainder of this text, so our aim here is to generalize the integral formulas first introduced in Section 4.9. We encountered an integral of the form $\int f(ax)\,dx$ in the opening of this section, where we used trial and error to discover that $\int \cos 2x\,dx = \frac{1}{2}\sin 2x + C$. Notice that 2 could be replaced by any nonzero constant $a$ to produce the more general result

$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C.$$

Let's verify this result without resorting to trial and error by using substitution. Letting $u = ax$, we have $du = a\,dx$, or $dx = \frac{1}{a}\,du$. By the Substitution Rule,

$$\begin{aligned}
\int \cos ax\,dx &= \int \cos u \cdot \frac{1}{a}\,du && \text{Substitute } u = ax,\ dx = \frac{1}{a}\,du.\\
&= \frac{1}{a}\int \cos u\,du && \int c\,f(x)\,dx = c\int f(x)\,dx\\
&= \frac{1}{a}\sin u + C && \text{Antiderivative}\\
&= \frac{1}{a}\sin ax + C. && \text{Replace } u \text{ with } ax.
\end{aligned}$$

**Quick Check 3** Evaluate $\int \cos 6x\,dx$ without using the substitution method. ◆

**Answer »**

$\frac{1}{6}\sin 6x + C$

Now that we have established a general result for $\int \cos ax\,dx$, we can use it to evaluate integrals such as $\int \cos 6x\,dx$ without resorting to the substitution method. Table 5.6 lists additional general formulas for standard trigonometric integrals, where we assume $a \neq 0$ is a real number. The derivations of results (2)–(6) are similar to the derivation of result (1) just given. Note that all of these integration formulas can be verified by differentiation.

**Table 5.6 General Integration Formulas**

| | |
|---|---|
| **1.** $\int \cos ax\, dx = \frac{1}{a}\sin ax + C$ | **2.** $\int \sin ax\, dx = -\frac{1}{a}\cos ax + C$ |
| **3.** $\int \sec^2 ax\, dx = \frac{1}{a}\tan ax + C$ | **4.** $\int \csc^2 ax\, dx = -\frac{1}{a}\cot ax + C$ |
| **5.** $\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax + C$ | **6.** $\int \csc ax \cot ax\, dx = -\frac{1}{a}\csc ax + C$ |

## Definite Integrals »

The Substitution Rule is also used for definite integrals; in fact, there are two ways to proceed.

- You may use the Substitution Rule to find an antiderivative $F$, as described above, and then use the Fundamental Theorem to evaluate $F(b) - F(a)$.
- Alternatively, once you have changed variables from $x$ to $u$, you may also change the limits of integration and complete the integration with respect to $u$. Specifically, if $u = g(x)$, the lower limit $x = a$ is replaced by $u = g(a)$ and the upper limit $x = b$ is replaced by $u = g(b)$.

The second option tends to be more efficient, and we use it whenever possible. This approach is summarized in the following theorem, which is then applied to several definite integrals.

**THEOREM 5.7 Substitution Rule for Definite Integrals**

Let $u = g(x)$, where $g'$ is continuous on $[a, b]$, and let $f$ be continuous on the range of $g$. Then

$$\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du. \tag{2}$$

**Proof:** We apply Part 2 of the Fundamental Theorem of Calculus to both sides of equation (2). Let $F$ be an antiderivative of $f$. Then, by equation (1), we know that $F(g(x))$ is an antiderivative of $f(g(x))\, g'(x)$, which implies that

$$\int_a^b f(g(x))\, g'(x)\, dx = F(g(x))\Big|_a^b = F(g(b)) - F(g(a)). \quad \text{Fundamental Theorem, Part 2}$$

Applying the Fundamental Theorem to the right side of equation (2) leads to the same result:

$$\int_{g(a)}^{g(b)} f(u)\, du = F(u)\Big|_{g(a)}^{g(b)} = F(g(b)) - F(g(a)).$$ ◆

**EXAMPLE 4 Definite integrals**

Evaluate the following integrals.

**a.** $\displaystyle\int_0^2 \frac{dx}{(x+3)^3}$

**b.** $\displaystyle\int_{-1}^2 \frac{x^2}{\left(x^3+2\right)^3}\, dx$

**c.** $\int_0^{\pi/2} \sin^4 x \cos x \, dx$

**SOLUTION »**

**a.** Let the new variable be $u = x + 3$; then $du = dx$. Because we have changed the variable of integration from $x$ to $u$, the limits of integration must also be expressed in terms of $u$. In this case,

$$x = 0 \text{ implies } u = 0 + 3 = 3, \quad \text{Lower limit}$$
$$x = 2 \text{ implies } u = 2 + 3 = 5. \quad \text{Upper limit}$$

The entire integration is carried out as follows:

$$\int_0^2 \frac{dx}{(x+3)^3} = \int_3^5 u^{-3} \, du \qquad \text{Substitute } u = x + 3, \ du = dx.$$
$$= -\frac{u^{-2}}{2}\bigg|_3^5 \qquad \text{Fundamental Theorem}$$
$$= -\frac{1}{2}\left(5^{-2} - 3^{-2}\right) = \frac{8}{225}. \qquad \text{Simplify.}$$

**Note »**

**b.** Notice that a multiple of the derivative of $x^3 + 2$ appears in the numerator; therefore, we let $u = x^3 + 2$. Then $du = 3x^2 \, dx$, or $x^2 \, dx = \frac{1}{3} \, du$. Changing limits of integration,

$$x = -1 \text{ implies } u = -1 + 2 = 1, \quad \text{Lower limit}$$
$$x = 2 \text{ implies } u = 2^3 + 2 = 10. \quad \text{Upper limit}$$

Changing variables, we have

$$\int_{-1}^2 \frac{x^2}{\left(x^3+2\right)^3} \, dx = \frac{1}{3}\int_1^{10} u^{-3} \, du \qquad \text{Substitute } u = x^3 + 2, \ \frac{1}{3} \, du = x^2 \, dx.$$
$$= \frac{1}{3}\left(-\frac{u^{-2}}{2}\right)\bigg|_1^{10} \qquad \text{Fundamental Theorem}$$
$$= \frac{1}{3}\left(-\frac{1}{200} - \left(-\frac{1}{2}\right)\right) \qquad \text{Simplify.}$$
$$= \frac{33}{200}.$$

**c.** Let $u = \sin x$, which implies that $du = \cos x \, dx$. The lower limit of integration becomes $u = 0$ and the upper limit becomes $u = 1$. Changing variables, we have

$$\int_0^{\pi/2} \sin^4 x \cos x \, dx = \int_0^1 u^4 \, du \qquad u = \sin x, \ du = \cos x \, dx$$
$$= \frac{u^5}{5}\bigg|_0^1 = \frac{1}{5}. \qquad \text{Fundamental Theorem}$$

*Related Exercises 44, 47, 60* ◆

The Substitution Rule enables us to find two standard integrals that appear frequently in practice, $\int \sin^2 x\, dx$ and $\int \cos^2 x\, dx$. These integrals are handled using the identities

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \text{and} \quad \cos^2 x = \frac{1 + \cos 2x}{2}.$$

### EXAMPLE 5 Integral of $\cos^2 \theta$

Evaluate $\int_0^{\pi/2} \cos^2 \theta\, d\theta$.

**SOLUTION »**

Working with the indefinite integral first, we use the identity for $\cos^2 \theta$:

$$\int \cos^2 \theta\, d\theta = \int \frac{1 + \cos 2\theta}{2}\, d\theta = \frac{1}{2}\int d\theta + \frac{1}{2}\int \cos 2\theta\, d\theta.$$

Result (1) of Table 5.6 is used for the second integral, and we have

$$\int \cos^2 \theta\, d\theta = \frac{1}{2}\int d\theta + \frac{1}{2}\underbrace{\int \cos 2\theta\, d\theta}_{\frac{1}{2}\sin 2\theta + C}$$

$$= \frac{\theta}{2} + \frac{1}{4}\sin 2\theta + C. \qquad \text{Evaluate integrals; Table 5.6.}$$

Using the Fundamental Theorem of Calculus, the value of the definite integral is

$$\int_0^{\pi/2} \cos^2 \theta\, d\theta = \left(\frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right)\Bigg|_0^{\pi/2}$$

$$= \left(\frac{\pi}{4} + \frac{1}{4}\sin \pi\right) - \left(0 + \frac{1}{4}\sin 0\right) = \frac{\pi}{4}.$$

**Note »**

*Related Exercises 83, 87* ◆

## Geometry of Substitution »

The Substitution Rule may be interpreted graphically. To keep matters simple, consider the integral $\int_0^2 2(2x + 1)\, dx$. The graph of the integrand $y = 2(2x + 1)$ on the interval $[0, 2]$ is shown in **Figure 5.58a**, along with the region $R$ whose area is given by the integral. The change of variables $u = 2x + 1$, $du = 2\, dx$, $u(0) = 1$, and $u(2) = 5$ leads to the new integral

$$\int_0^2 2(2x + 1)\, dx = \int_1^5 u\, du.$$

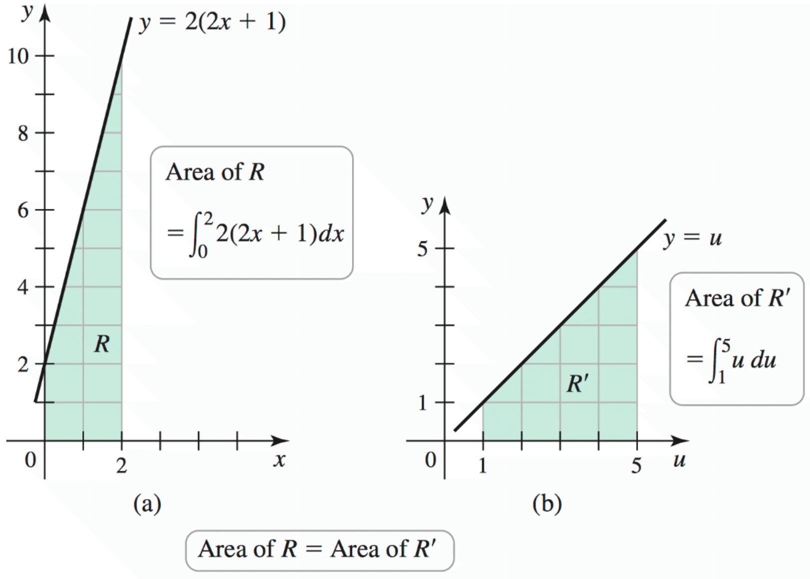



**Figure 5.58**

**Figure 5.58b** shows the graph of the new integrand $y = u$ on the interval $[1, 5]$ and the region $R'$ whose area is given by the new integral. You can check that the areas of $R$ and $R'$ are equal. An analogous interpretation may be given to more complicated integrands and substitutions.

**Quick Check 4** Changes of variables occur frequently in mathematics. For example, suppose you want to solve the equation $x^4 - 13x^2 + 36 = 0$. If you use the substitution $u = x^2$, what is the new equation that must be solved for $u$? What are the roots of the original equation? ◆

**Answer »**

## Exercises »

### Getting Started »

### Practice Exercises »

**17–40. Indefinite integrals** *Use a change of variables or Table 5.6 to find the following indefinite integrals. Check your work by differentiating.*

**17.** $\int 2x\left(x^2 - 1\right)^{99} dx$

**18.** $\int x \sin x^2\, dx$

**19.** $\int \frac{2x^2}{\sqrt{1 - 4x^3}}\, dx$

**20.** $\int \frac{\left(\sqrt{x} + 1\right)^4}{2\sqrt{x}}\, dx$

**21.** $\int \left(x^2 + x\right)^{10} (2x + 1)\, dx$

**22.** $\int \frac{1}{(10\,x-3)^2}\,dx$

**23.** $\int x^3\left(x^4+16\right)^6\,dx$

**24.** $\int \sin^{10}\theta\cos\theta\,d\theta$

**25.** $\int \frac{x}{\sqrt{4-9\,x^2}}\,dx$

**26.** $\int x^9\sin x^{10}\,dx$

**27.** $\int \left(x^6-3\,x^2\right)^4\left(x^5-x\right)dx$

**28.** $\int \frac{x}{\left(1+4\,x^2\right)^3}\,dx$

**29.** $\int x\csc x^2\cot x^2\,dx$

**30.** $\int \sec 4\,w\ \tan 4\,w\,dw$

**31.** $\int \sec^2(10\,x+7)\,dx$

**32.** $\int \csc^4 x\cot x\,dx$

**33.** $\int t^3\sin t^4\cos t^4\,dt$

**34.** $\int \left(\sin^5 x+3\sin^3 x-\sin x\right)\cos x\,dx$

**35.** $\int \frac{\csc^2 x}{\cot^3 x}\,dx$

**36.** $\int \left(x^{3/2}+8\right)^5\sqrt{x}\,dx$

**37.** $\int \sin x\sec^8 x\,dx$

**38.** $\int \frac{\cos\sqrt{t}}{\sqrt{t}}\,dt$

**39.** $\int (\sec w + 3)^9 \sec w \tan w \, dw$

**40.** $\int \left(\sec^5 x + \sec^3 x\right) \tan x \, dx$

**41–70. Definite integrals** *Use a change of variables or Table 5.6 to evaluate the following definite integrals.*

**41.** $\int_0^{\pi/8} \cos 2x \, dx$

**42.** $\int_{\pi/16}^{\pi/8} 8 \csc^2 4x \, dx$

**43.** $\int_0^1 2x\left(4 - x^2\right) dx$

**44.** $\int_0^2 \frac{2x}{\left(x^2+1\right)^2} \, dx$

**45.** $\int_0^{\pi/4} \frac{\sec^2 x}{\sqrt{\tan x + 3}} \, dx$

**46.** $\int_{-2\pi}^{2\pi} \cos \frac{\theta}{8} \, d\theta$

**47.** $\int_0^{\pi/2} \sin^2 \theta \cos \theta \, d\theta$

**48.** $\int_0^{\pi/4} \frac{\sin x}{\cos^2 x} \, dx$

**49.** $\int_0^{\pi/4} \left(\sec^3 x + \sec^2 x\right) \tan x \, dx$

**50.** $\int_{\pi/12}^{\pi/6} \csc^4 3x \cot 3x \, dx$

**51.** $\int_{-\pi/12}^{\pi/8} \sec^2 2y \, dy$

**52.** $\int_0^4 \frac{p}{\sqrt{9 + p^2}} \, dp$

**53.** $\int_{\pi/8}^{\pi/4} \left(\sin^3 2x + \sin 2x\right) \cos 2x \, dx$

**54.** $\int_0^{\pi/4} \frac{\sin \theta}{\cos^3 \theta} \, d\theta$

**55.** $\int_2^6 \frac{x}{\sqrt{2x-3}}\,dx$

**56.** $\int_0^1 \frac{v^3+1}{\sqrt{v^4+4v+4}}\,dv$

**57.** $\int_0^{\sqrt{3}} \frac{w}{\sqrt{w^2+1}}\,dw$

**58.** $\int_0^4 3\sqrt{2t+1}\,dt$

**59.** $\int_0^1 (2p+1)^3\,dp$

**60.** $\int_0^2 \frac{81q}{\left(2q^2+1\right)^3}\,dq$

**61.** $\int_0^1 x\sqrt{1-x^2}\,dx$

**62.** $\int_1^3 \frac{(1+4/x)^2}{x^2}\,dx$

**63.** $\int_2^3 \frac{x}{\sqrt[3]{x^2-1}}\,dx$

**64.** $\int_0^{6/5} \frac{x}{\left(25x^2+36\right)^2}\,dx$

**65.** $\int_0^2 x^3\sqrt{16-x^4}\,dx$

**66.** $\int_{-1}^1 (x-1)\left(x^2-2x\right)^7\,dx$

**67.** $\int_0^1 x(x+1)^{10}(x-1)^{10}\,dx$

**68.** $\int_{\pi/8}^{\pi/4} \csc^2 2x\sqrt{1+\cot 2x}\,dx$

**69.** $\int_1^2 \frac{4}{9x^2+6x+1}\,dx$

**70.** $\int_0^{\pi/2} (2+\sin x)\sin x\cos x\,dx$

**71.** **Average velocity** An object moves in one dimension with a velocity in m/s given by $v(t) = 8\sin \pi t + 2t$. Find its average velocity over the time interval from $t = 0$ to $t = 10$, where $t$ is measured in seconds.

T **72.** **Periodic motion** An object moves along a line with a velocity in m/s given by $v(t) = 8\cos \frac{\pi t}{6}$. Its initial position is $s(0) = 0$.

**a.** Graph the velocity function.

**b.** As discussed in Chapter 6, the position of the object is given by $s(t) = \int_0^t v(y)\,dy$, for $t \geq 0$. Find the position function, for $t \geq 0$.

**c.** What is the period of the motion—that is, starting at any point, how long does it take the object to return to that position?

**73.** **Population models** The population of a culture of bacteria has a growth rate given by $p'(t) = \frac{200}{(t+1)^r}$ bacteria per hour, for $t \geq 0$, where $r > 1$ is a real number. In Chapter 6 it is shown that the increase in the population over the time interval $[0, t]$ is given by $\int_0^t p'(s)\,ds$. (Note that the growth rate decreases in time, reflecting competition for space and food.)

**a.** Using the population model with $r = 2$, what is the increase in the population over the time interval $0 \leq t \leq 4$?

**b.** Using the population model with $r = 3$, what is the increase in the population over the time interval $0 \leq t \leq 6$?

**c.** Let $\Delta P$ be the increase in the population over a fixed time interval $[0, T]$. For fixed $T$, does $\Delta P$ increase or decrease with the parameter $r$? Explain.

**d.** A lab technician measures an increase in the population of 350 bacteria over the 10-hr period $[0, 10]$. Estimate the value of $r$ that best fits this data point.

**e.** Looking ahead: Use the population model in part (b) to find the increase in population over the time interval $[0, T]$, for any $T > 0$. If the culture is allowed to grow indefinitely $(T \to \infty)$, does the bacteria population increase without bound? Or does it approach a finite limit?

**74–82. Variations on the substitution method** *Find the following integrals.*

**74.** $\int \frac{x}{(x-2)^3}\,dx$

**75.** $\int \frac{x}{\sqrt{x-4}}\,dx$

**76.** $\int \frac{y^2}{(y+1)^4}\,dy$

**77.** $\int \frac{x}{\sqrt[3]{x+4}}\,dx$

**78.** $\int \frac{2x}{\sqrt{3x+2}}\,dx$

**79.** $\int x \sqrt[3]{2x+1}\, dx$

**80.** $\int (z+1)\sqrt{3z+2}\, dz$

**81.** $\int x(x+10)^9\, dx$

**82.** $\int x^3\left(x^2+1\right)^{10}\, dx$

**83–90. Integrals with $\sin^2 x$ and $\cos^2 x$** *Evaluate the following integrals.*

**83.** $\int_{-\pi}^{\pi} \cos^2 x\, dx$

**84.** $\int \sin^2 x\, dx$

**85.** $\int \sin^2\left(\theta + \frac{\pi}{6}\right) d\theta$

**86.** $\int_0^{\pi/4} \cos^2 8\theta\, d\theta$

**87.** $\int_{-\pi/4}^{\pi/4} \sin^2 2\theta\, d\theta$

**88.** $\int x\cos^2 x^2\, dx$

**89.** $\int_0^{\pi/6} \frac{\sin 2y}{\left(\sin^2 y+2\right)^2}\, dy$ (*Hint*: $\sin 2y = 2\sin y\cos y$.)

**90.** $\int_0^{\pi/2} \sin^4\theta\, d\theta$

**91. Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample. Assume $f$, $f'$, and $f''$ are continuous functions for all real numbers.

**a.** $\int f(x) f'(x)\, dx = \frac{1}{2}(f(x))^2 + C.$

**b.** $\int (f(x))^n f'(x)\, dx = \frac{1}{n+1}(f(x))^{n+1} + C, \quad n \neq -1.$

**c.** $\int \sin 2x\, dx = 2\int \sin x\, dx.$

**d.** $\int \left(x^2+1\right)^9 dx = \frac{\left(x^2+1\right)^{10}}{10} + C.$

**e.** $\int_a^b f'(x) f''(x)\, dx = f'(b) - f'(a).$

**92–94. Areas of regions** *Find the area of the following regions.*

**92.** The region bounded by the graph of $f(x) = \dfrac{x}{\sqrt{x^2 - 9}}$ and the $x$-axis between $x = 4$ and $x = 5$

**93.** The region bounded by the graph of $f(x) = x \sin x^2$ and the $x$-axis between $x = 0$ and $x = \sqrt{\pi}$

**94.** The region bounded by the graph of $f(x) = (x - 4)^4$ and the $x$-axis between $x = 2$ and $x = 6$

## Explorations and Challenges »

**95. Morphing parabolas** The family of parabolas $y = \dfrac{1}{a} - \dfrac{x^2}{a^3}$, where $a > 0$, has the property that for $x \geq 0$, the $x$-intercept is $(a, 0)$ and the $y$-intercept is $(0, 1/a)$. Let $A(a)$ be the area of the region in the first quadrant bounded by the parabola and the $x$-axis. Find $A(a)$ and determine whether it is an increasing, decreasing, or constant function of $a$.

**96. Substitutions** Suppose $f$ is an even function with $\int_0^8 f(x)\,dx = 9$. Evaluate each integral.

**a.** $\int_{-1}^{1} x\, f\left(x^2\right) dx.$

**b.** $\int_{-2}^{2} x^2\, f\left(x^3\right) dx.$

**97. Substitutions** Suppose $p$ is a nonzero real number and $f$ is an odd function with $\int_0^1 f(x)\,dx = \pi$. Evaluate each integral.

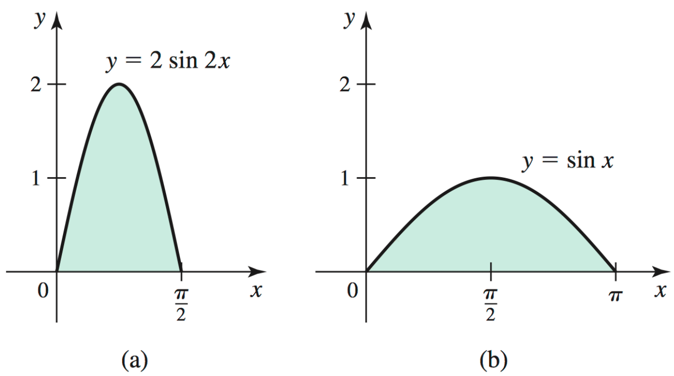
**a.** $\int_0^{\pi/(2p)} (\cos px)\, f(\sin px)\, dx$

**b.** $\int_{-\pi/2}^{\pi/2} (\cos x)\, f(\sin x)\, dx$

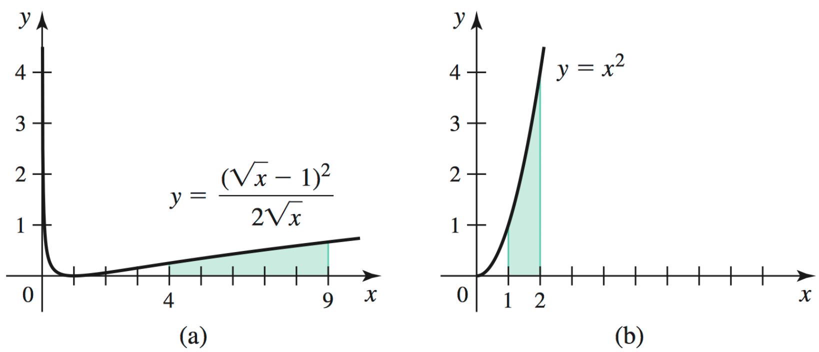
**98. Average distance on a triangle** Consider the right triangle with vertices $(0, 0)$, $(0, b)$, and $(a, 0)$, where $a > 0$ and $b > 0$. Show that the average vertical distance from points on the $x$-axis to the hypotenuse is $b/2$, for all $a > 0$.

T **99. Average value of sine functions** Use a graphing utility to verify that the functions $f(x) = \sin kx$ have a period of $2\pi/k$, where $k = 1, 2, 3, \ldots$. Equivalently, the first "hump" of $f(x) = \sin kx$ occurs on the interval $[0, \pi/k]$. Verify that the average value of the first hump of $f(x) = \sin kx$ is independent of $k$. What is the average value?

**100. Equal areas** The area of the shaded region under the curve $y = 2 \sin 2x$ in part (a) of the figure equals the area of the shaded region under the curve $y = \sin x$ in (b). Explain why this is true without computing areas.




**101. Equal areas** The area of the shaded region under the curve $y = \dfrac{\left(\sqrt{x} - 1\right)^2}{2\sqrt{x}}$ on the interval $[4, 9]$ in part (a) of the following figure equals the area of the shaded region under the curve $y = x^2$ on the interval $[1, 2]$ in part (b) of the figure. Without computing areas, explain why.




**102–104. General results** *Evaluate the following integrals in which the function $f$ is unspecified. Note that $f^{(p)}$ is the pth derivative of $f$ and $f^p$ is the pth power of $f$. Assume $f$ and its derivatives are continuous for all real numbers.*

**102.** $\displaystyle\int \left(5 f^3(x) + 7 f^2(x) + f(x)\right) f'(x)\, dx$

**103.** $\displaystyle\int_1^2 \left(5 f^3(x) + 7 f^2(x) + f(x)\right) f'(x)\, dx$, where $f(1) = 4$, $f(2) = 5$

**104.** $\displaystyle\int \left(f^{(p)}(x)\right)^n f^{(p+1)}(x)\, dx$, where $p$ is a positive integer, $n \neq -1$

**105–107. More than one way** *Occasionally, two different substitutions do the job. Use each substitution to evaluate the following integrals.*

**105.** $\displaystyle\int_0^1 x\sqrt{x + a}\, dx;\ a > 0\ \left(u = \sqrt{x + a}\ \text{and}\ u = x + a\right)$

**106.** $\displaystyle\int_0^1 x\sqrt[p]{x + a}\, dx;\ a > 0\ \left(u = \sqrt[p]{x + a}\ \text{and}\ u = x + a\right)$

**107.** $\int \sec^3 \theta \, \tan \theta \, d\theta$ ($u = \cos \theta$ and $u = \sec \theta$)

**108. $\sin^2 ax$ and $\cos^2 ax$ integrals** Use the Substitution Rule to prove that

$$\int \sin^2 ax\, dx = \frac{x}{2} - \frac{\sin(2ax)}{4a} + C \quad \text{and}$$

$$\int \cos^2 ax\, dx = \frac{x}{2} + \frac{\sin(2ax)}{4a} + C.$$

**109. Integral of $\sin^2 x \cos^2 x$** Consider the integral $I = \int \sin^2 x \cos^2 x \, dx$.

- **a.** Find $I$ using the identity $\sin 2x = 2 \sin x \cos x$.
- **b.** Find $I$ using the identity $\cos^2 x = 1 - \sin^2 x$.
- **c.** Confirm that the results in parts (a) and (b) are consistent and compare the work involved in each method.

**110. Substitution: shift** Perhaps the simplest change of variables is the shift or translation given by $u = x + c$, where $c$ is a real number.

- **a.** Prove that shifting a function does not change the net area under the curve, in the sense that

$$\int_a^b f(x + c)\, dx = \int_{a+c}^{b+c} f(u)\, du.$$

- **b.** Draw a picture to illustrate this change of variables in the case that $f(x) = \sin x$, $a = 0$, $b = \pi$, and $c = \pi/2$.

**111. Substitution: scaling** Another change of variables that can be interpreted geometrically is the scaling $u = cx$, where $c$ is a real number. Prove and interpret the fact that

$$\int_a^b f(cx)\, dx = \frac{1}{c} \int_{ac}^{bc} f(u)\, du.$$

Draw a picture to illustrate this change of variables in the case that $f(x) = \sin x$, $a = 0$, $b = \pi$, and $c = 1/2$.

**112–□. Multiple substitutions** *If necessary, use two or more substitutions to find the following integrals.*

**112.** $\int x \sin^4 x^2 \cos x^2 \, dx$ (*Hint:* Begin with $u = x^2$, and then use $v = \sin u$.)

**113.** $\int \frac{dx}{\sqrt{1 + \sqrt{1 + x}}}$ (*Hint:* Begin with $u = \sqrt{1 + x}$.)

**114.** $\int \tan^{10} 4x \sec^2 4x \, dx$ (*Hint:* Begin with $u = 4x$.)

**115.** $\int_0^{\pi/2} \frac{\cos \theta \sin \theta}{\sqrt{\cos^2 \theta + 16}} \, d\theta$ (*Hint:* Begin with $u = \cos \theta$.)