5.4 Working with Integrals

With the Fundamental Theorem of Calculus in hand, we may begin an investigation of integration and its applications. In this section, we discuss the role of symmetry in integrals, we use the slice-and-sum strategy to define the average value of a function, and we explore a theoretical result called the Mean Value Theorem for Integrals.

Integrating Even and Odd Functions »

Symmetry appears throughout mathematics in many different forms, and its use often leads to insights and efficiencies. Here we use the symmetry of a function to simplify integral calculations.

Section 1.1 introduced the symmetry of even and odd functions. An **even function** satisfies the property $f(-x) = f(x)$, which means that its graph is symmetric about the y-axis (**Figure 5.53a**). Examples of even functions are $f(x) = \cos x$ and $f(x) = x^n$, where n is an even integer. An **odd function** satisfies the property $f(-x) = -f(x)$, which means that its graph is symmetric about the origin (**Figure 5.53b**). Examples of odd functions are $f(x) = \sin x$ and $f(x) = x^n$, where n is an odd integer.

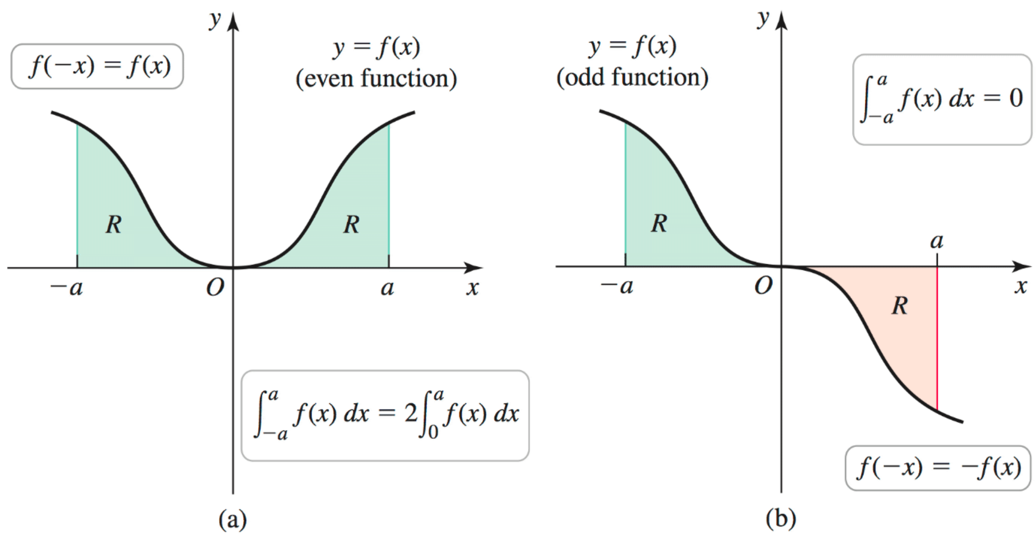


Figure 5.53

Special things happen when we integrate even and odd functions on intervals centered at the origin. First, suppose f is an even function and consider $\int_{-a}^{a} f(x)\,dx$. From Figure 5.53a, we see that the integral of f on $[-a, 0]$ equals the integral of f on $[0, a]$. Therefore, the integral on $[-a, a]$ is twice the integral on $[0, a]$, or

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx.$$

On the other hand, suppose f is an odd function and consider $\int_{-a}^{a} f(x)\,dx$. As shown in Figure 5.53b, the integral on the interval $[-a, 0]$ is the negative of the integral on $[0, a]$. Therefore, the integral on $[-a, a]$ is zero, or

$$\int_{-a}^{a} f(x)\,dx = 0.$$

We summarize these results in the following theorem.

THEOREM 5.4 Integrals of Even and Odd Functions

Let a be a positive real number and let f be an integrable function on the interval $[-a, a]$.

- If f is even, $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.
- If f is odd, $\int_{-a}^{a} f(x)\,dx = 0$.

Quick Check 1 If f and g are both even functions, is the product fg even or odd? Use the facts that $f(-x) = f(x)$ and $g(-x) = g(x)$. ◆

Answer »

The following example shows how symmetry can simplify integration.

EXAMPLE 1 Integrating symmetric functions

Evaluate the following integrals using symmetry arguments.

a. $\int_{-2}^{2} \left(x^4 - 3x^3\right) dx$

b. $\int_{-\pi/2}^{\pi/2} \left(\cos x - 4\sin^3 x\right) dx$

SOLUTION »

a. Note that $x^4 - 3x^3$ is neither odd nor even so Theorem 5.4 cannot be applied directly. However, we can split the integral and then use symmetry:

$$\begin{aligned}
\int_{-2}^{2} \left(x^4 - 3x^3\right) dx &= \int_{-2}^{2} x^4\,dx - 3\underbrace{\int_{-2}^{2} x^3\,dx}_{0} && \text{Properties 3 and 4 of Table 5.4} \\
&= 2\int_{0}^{2} x^4\,dx - 0 && x^4 \text{ is even, } x^3 \text{ is odd.} \\
&= 2\left(\frac{x^5}{5}\right)\Bigg|_{0}^{2} && \text{Fundamental Theorem} \\
&= 2\left(\frac{32}{5}\right) = \frac{64}{5}. && \text{Simplify.}
\end{aligned}$$

Notice how the odd-powered term of the integrand is eliminated by symmetry. Integration of the even-powered term is simplified because the lower limit is zero.

b. The $\cos x$ term is an even function, so it can be integrated on the interval $\left[0, \frac{\pi}{2}\right]$. What about $\sin^3 x$? It is an odd function raised to an odd power, which results in an odd function; its integral on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is zero. Therefore,

$$\int_{-\pi/2}^{\pi/2}\left(\cos x - 4\sin^3 x\right)dx = 2\int_0^{\pi/2}\cos x\,dx - 0 \quad \text{Symmetry}$$
$$= 2\sin x\,\Big|_0^{\pi/2} \quad \text{Fundamental Theorem}$$
$$= 2\,(1-0) = 2. \quad \text{Simplify.}$$

Note »

There are a couple of ways to see that $\sin^3 x$ is an odd function. Its graph is symmetric about the origin, indicating that $\sin^3(-x) = -\sin^3 x$. Or by analogy, take an odd power of x and raise it to an odd power. For example, $\left(x^5\right)^3 = x^{15}$, which is odd. See Exercises 49–52 for direct proofs of symmetry in composite functions.

Related Exercises 15–16 ◆

Average Value of a Function »

If five people weigh 155, 143, 180, 105, and 123 lb, their average (mean) weight is

$$\frac{155 + 143 + 180 + 105 + 123}{5} = 141.2 \text{ lb}.$$

This idea generalizes quite naturally to functions. Consider a function f that is continuous on $[a, b]$. Using a regular partition $x_0 = a, x_1, x_2, \ldots, x_n = b$ with $\Delta x = \dfrac{b-a}{n}$, we now select a point x_k^* in each subinterval and compute $f(x_k^*)$, for $k = 1, \ldots, n$. The values of $f(x_k^*)$ may be viewed as a sampling of f on $[a, b]$. The average of these function values is

$$\frac{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)}{n}.$$

Noting that $n = \dfrac{b-a}{\Delta x}$, we write the average of the n sample values as the Riemann sum

$$\frac{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)}{(b-a)/\Delta x} = \frac{1}{b-a}\sum_{k=1}^{n} f(x_k^*)\,\Delta x.$$

Now suppose we increase n, taking more and more samples of f, while Δx decreases to zero. The limit of this sum is a definite integral that gives the average value $\overline{f}$ on $[a, b]$:

$$\overline{f} = \frac{1}{b-a}\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k^*)\,\Delta x$$
$$= \frac{1}{b-a}\int_a^b f(x)\,dx.$$

This definition of the average value of a function is analogous to the definition of the average of a finite set of numbers.

DEFINITION **Average Value of a Function**

The average value of an integrable function f on the interval $[a, b]$ is

$$\overline{f} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

The average value of a function f on an interval $[a, b]$ has a clear geometrical interpretation. Multiplying both sides of the definition of average value by $(b-a)$, we have

$$\underbrace{(b-a)\,\overline{f}}_{\text{net area of rectangle}} = \underbrace{\int_a^b f(x)\,dx}_{\text{net area of region bounded by curve}}.$$

We see that the average value is the height of the rectangle with base $[a, b]$ that has the same net area as the region bounded by the graph of f on the interval $[a, b]$ (**Figure 5.54**). Note that $\overline{f}$ may be zero or negative.

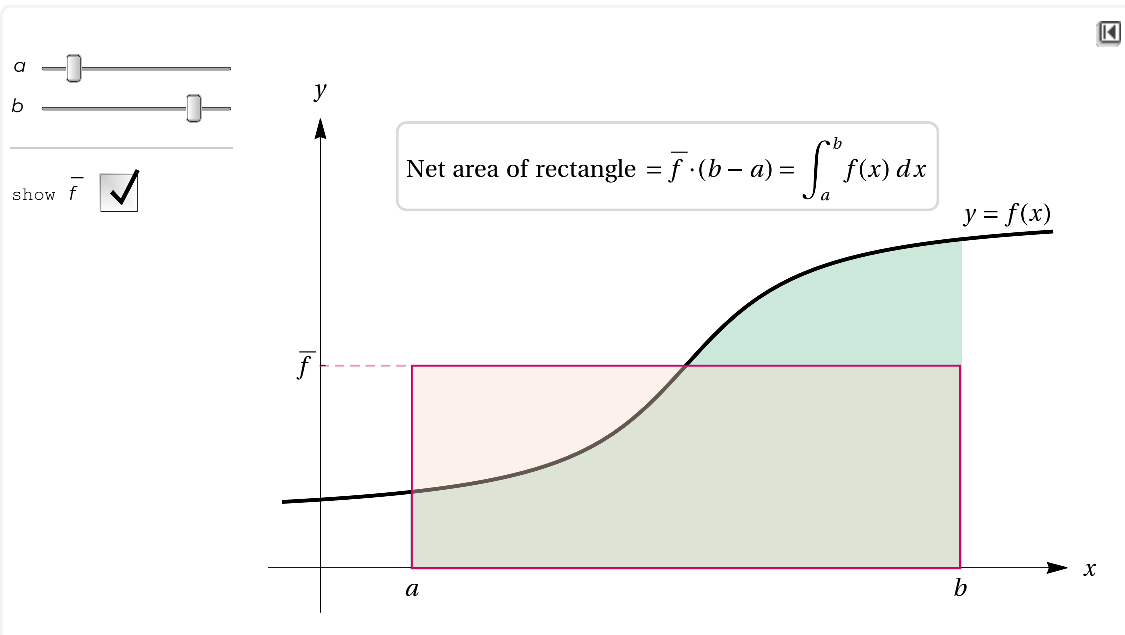


Figure 5.54

Quick Check 2 What is the average value of a constant function on an interval? What is the average value of an odd function on an interval $[-a, a]$? ◆

Answer »

The average value is the constant; the average value is 0.

EXAMPLE 2 Average elevation

A hiking trail has an elevation given by

$$f(x) = 60\,x^3 - 650\,x^2 + 1200\,x + 4500,$$

where f is measured in feet above sea level and x represents horizontal distance along the trail in miles, with $0 \le x \le 5$. What is the average elevation of the trail?

SOLUTION »

The trail ranges between elevations of about 2000 and 5000 ft (**Figure 5.55**). If we let the endpoints of the trail correspond to the horizontal distances $a = 0$ and $b = 5$, the average elevation of the trail in feet is

$$\begin{aligned} \bar{f} &= \frac{1}{5}\int_0^5 \left(60x^3 - 650x^2 + 1200x + 4500\right) dx \\ &= \frac{1}{5}\left(60\frac{x^4}{4} - 650\frac{x^3}{3} + 1200\frac{x^2}{2} + 4500x\right)\Bigg|_0^5 && \text{Fundamental Theorem} \\ &= 3958\frac{1}{3}. && \text{Simplify.} \end{aligned}$$

The average elevation of the trail is slightly less than 3960 ft.

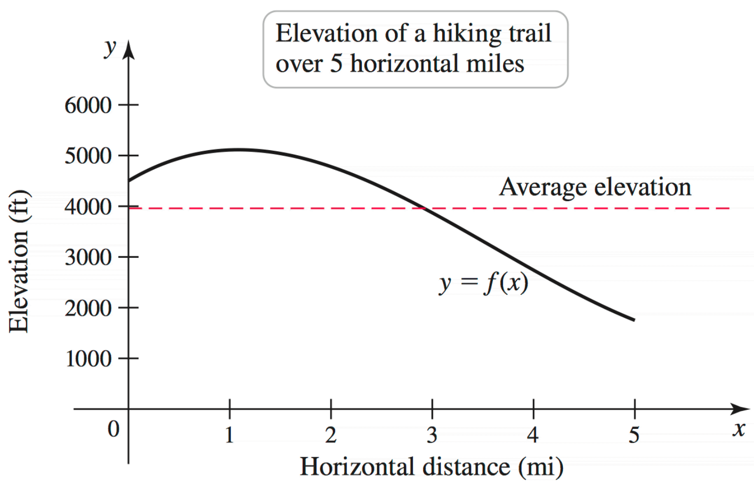


Figure 5.55

Related Exercises 26, 34 ◆

Mean Value Theorem for Integrals »

The average value of a function brings us close to an important theoretical result. The Mean Value Theorem for Integrals says that if f is continuous on $[a, b]$, then there is at least one point c in the interval $[a, b]$ such that $f(c)$ equals the average value of f on $[a, b]$. In other words, the horizontal line $y = \bar{f}$ intersects the graph of f for some point c in $[a, b]$ (**Figure 5.56**). If f were not continuous, such a point might not exist.

Note »

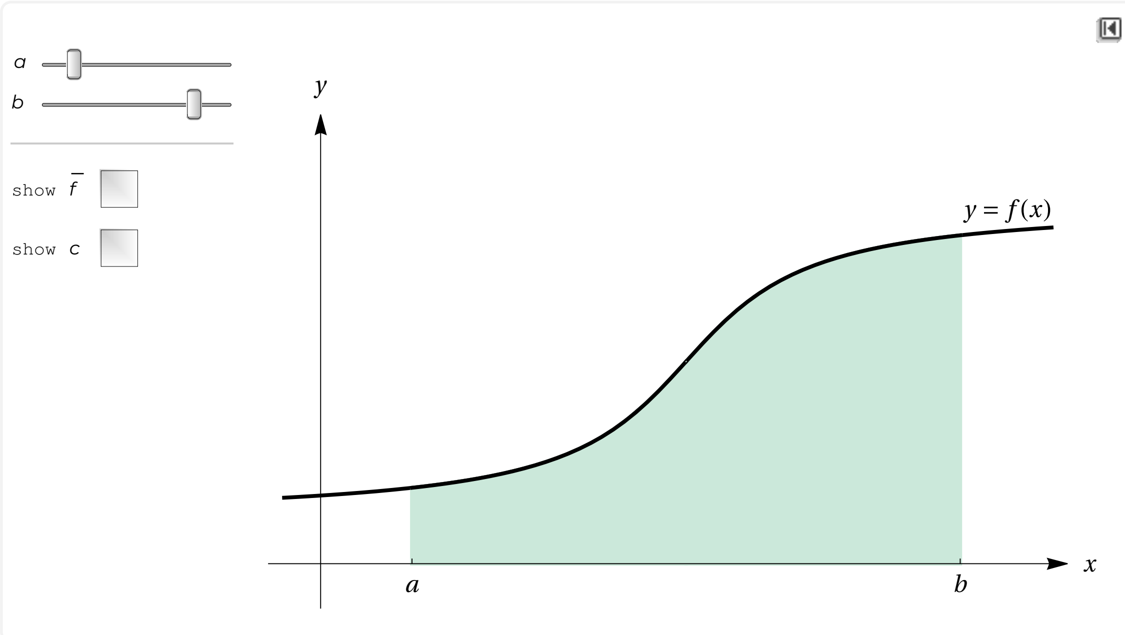


Figure 5.56

THEOREM 5.5 Mean Value Theorem for Integrals

Let f be continuous on the interval $[a, b]$. There exists a point c in $[a, b]$ such that

$$f(c) = \bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Note »

Theorem 5.5 guarantees a point c in the open interval (a, b) at which f equals its average value. However, f may also equal its average value at an endpoint of that interval.

Proof: We begin by letting $F(x) = \int_a^x f(t)\,dt$ and noting that F is continuous on $[a, b]$ and differentiable on (a, b) (by Theorem 5.3, Part 1). We now apply the Mean Value Theorem for derivatives (Theorem 4.4) to F and conclude that there exists at least one point c in (a, b) such that

$$\underbrace{F'(c)}_{f(c)} = \frac{F(b) - F(a)}{b-a}.$$

By Theorem 5.3, Part 1, we know that $F'(c) = f(c)$ and by Theorem 5.3, Part 2, we know that

$$F(b) - F(a) = \int_a^b f(t)\,dt.$$

Combining these observations, we have

$$f(c) = \frac{1}{b-a}\int_a^b f(t)\,dt,$$

where c is a point in (a, b). ◆

Note »

A more general form of the Mean Value Theorem for Integrals states that if f and g are continuous on $[a, b]$ with $g(x) \geq 0$ on $[a, b]$, then there exists a number c in $[a, b]$ such that $\int_a^b f(x)\, g(x)\, dx = f(c) \int_a^b g(x)\, dx$.

Quick Check 3 Explain why $f(x) = 0$ for at least one point of $[a, b]$ if f is continuous and $\int_a^b f(x)\, dx = 0$. ◆

Answer »

The average value is zero on the interval; by the Mean Value Theorem for Integrals, $f(x) = 0$ at some point on the interval.

EXAMPLE 3 Average value equals function value

Find the point(s) on the interval $(0, 1)$ at which $f(x) = 2x(1 - x)$ equals its average value on $[0, 1]$.

SOLUTION »

The average value of f on $[0, 1]$ is

$$\bar{f} = \frac{1}{1-0} \int_0^1 2x(1-x)\, dx = \left(x^2 - \frac{2}{3}x^3\right)\Bigg|_0^1 = \frac{1}{3}.$$

We must find the points on $[0, 1]$ at which $f(x) = \frac{1}{3}$ (**Figure 5.57**).

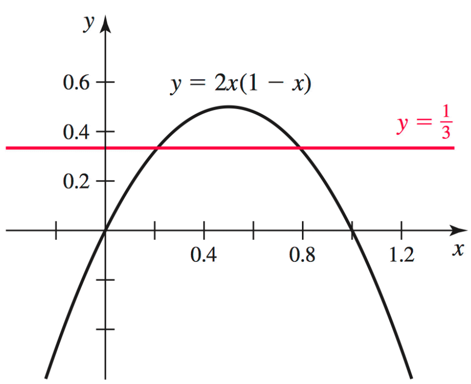


Figure 5.57

Using the quadratic formula, the two solutions of $f(x) = 2x(1 - x) = \frac{1}{3}$ are

$$\frac{1 - \sqrt{1/3}}{2} \approx 0.211 \quad \text{and} \quad \frac{1 + \sqrt{1/3}}{2} \approx 0.789.$$

These two points are located symmetrically on either side of $x = \frac{1}{2}$. The two solutions, 0.211 and 0.789, are the same for $f(x) = a\,x(1 - x)$ for any nonzero value of a (Exercise 53).

Related Exercises 41–42 ◆

Exercises »

Getting Started »

Practice Exercises »

11–24. Symmetry in integrals *Use symmetry to evaluate the following integrals.*

11. $\int_{-2}^{2} x^9\, dx$

12. $\int_{-200}^{200} 2x^5\, dx$

13. $\int_{-2}^{2} \left(3x^8 - 2\right) dx$

14. $\int_{-\pi/4}^{\pi/4} \cos x\, dx$

15. $\int_{-2}^{2} \left(x^2 + x^3\right) dx$

16. $\int_{-\pi}^{\pi} t^2 \sin t\, dt$

17. $\int_{-2}^{2} \left(x^9 - 3x^5 + 2x^2 - 10\right) dx$

18. $\int_{-\pi/2}^{\pi/2} 5 \sin \theta\, d\theta$

19. $\int_{-\pi/4}^{\pi/4} \sin^5 t\, dt$

20. $\int_{-1}^{1} (1 - |x|)\, dx$

21. $\int_{-\pi/4}^{\pi/4} \sec^2 x\, dx$

22. $\int_{-\pi/4}^{\pi/4} \tan \theta\, d\theta$

23. $\int_{-2}^{2} \frac{x^3 - 4x}{x^2 + 1}\, dx$

24. $\int_{-2}^{2} \left(1 - |x|^3\right) dx$

25–32. Average values *Find the average value of the following functions on the given interval. Draw a graph of the function and indicate the average value.*

25. $f(x) = x^3$ on $[-1, 1]$

26. $f(x) = x^2 + 1$ on $[-2, 2]$

27. $f(x) = \dfrac{1}{\sqrt{x}}$ on $[1, 4]$

28. $f(x) = \dfrac{1}{x^3}$ on $[1, 2]$

29. $f(x) = \cos x$ on $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

30. $f(x) = x(1 - x)$ on $[0, 1]$

31. $f(x) = x^n$ on $[0, 1]$, for any positive integer n

32. $f(x) = x^{1/n}$ on $[0, 1]$, for any positive integer n

33. **Average distance on a parabola** What is the average distance between the parabola $y = 30x(20 - x)$ and the x-axis on the interval $[0, 20]$?

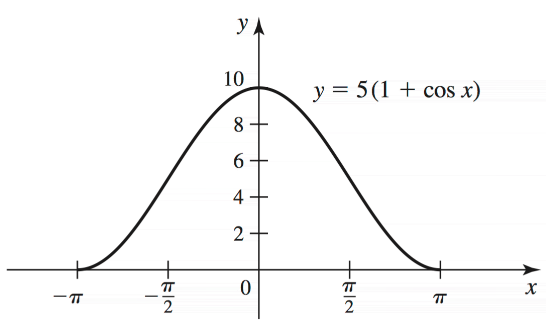
T **34.** **Average elevation** The elevation of a path is given by $f(x) = x^3 - 5x^2 + 30$, where x measures horizontal distance. Draw a graph of the elevation function and find its average value, for $0 \le x \le 4$.

35. **Average velocity** The velocity in m/s of an object moving along a line over the time interval from $[0, 6]$ is $v(t) = t^2 + 3t$. Find the average velocity of the object over this time interval.

36. **Average velocity** A rock is launched vertically upward from the ground with a speed of 64 ft/s. The height of the rock (in ft) above the ground after t seconds is given by the function $s(t) = -16t^2 + 64t$. Find its average velocity during its flight.

37. **Average height of an arch** The height of an arch above the ground is given by the function $y = 10 \sin x$, for $0 \le x \le \pi$. What is the average height of the arch above the ground?

38. **Average height of a wave** The surface of a water wave is described by $y = 5(1 + \cos x)$, for $-\pi \le x \le \pi$, where $y = 0$ corresponds to a trough of the wave (see figure). Find the average height of the wave above the trough on $[-\pi, \pi]$.



39–44. Mean Value Theorem for Integrals *Find or approximate all points at which the given function equals its average value on the given interval.*

39. $f(x) = 8 - 2x$ on $[0, 4]$

T **40.** $f(x) = \cos x$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

41. $f(x) = 1 - \frac{x^2}{a^2}$ on $[0, a]$, where a is a positive real number

T **42.** $f(x) = \frac{\pi}{4} \sin x$ on $[0, \pi]$

43. $f(x) = 1 - |x|$ on $[-1, 1]$

44. $f(x) = \frac{1}{x^2}$ on $[1, 4]$

45. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.

a. If f is symmetric about the line $x = 2$, then $\int_0^4 f(x)\,dx = 2\int_0^2 f(x)\,dx$.

b. If f has the property $f(a + x) = -f(a - x)$, for all x, where a is constant, then $\int_{a-2}^{a+2} f(x)\,dx = 0$.

c. The average value of a linear function on an interval $[a, b]$ is the function value at the midpoint of $[a, b]$.

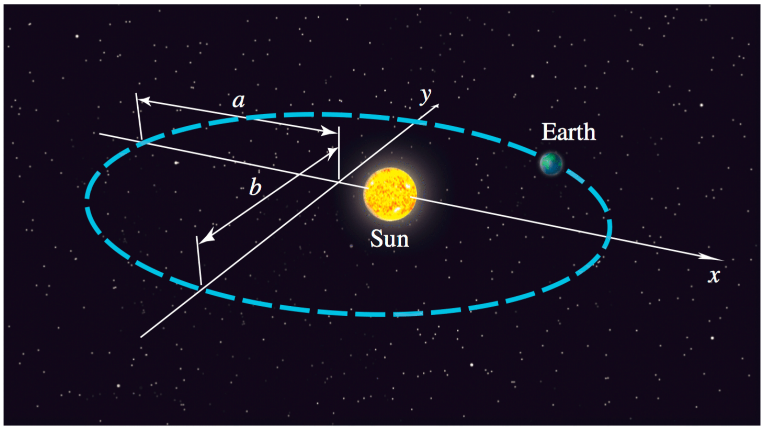
d. Consider the function $f(x) = x(a - x)$ on the interval $[0, a]$, for $a > 0$. Its average value on $[0, a]$ is $\frac{1}{2}$ of its maximum value.

46. **Planetary orbits** The planets orbit the Sun in elliptical orbits with the Sun at one focus (see Section 12.4 for more on ellipses). The equation of an ellipse whose dimensions are $2a$ in the x-direction and $2b$ in the y-direction is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

a. Let d^2 denote the square of the distance from a planet to the center of the ellipse at $(0, 0)$. Integrate over the interval $[-a, a]$ to show that the average value of d^2 is $\frac{a^2 + 2b^2}{3}$.

b. Show that in the case of a circle $(a = b = R)$, the average value in part (a) is R^2.

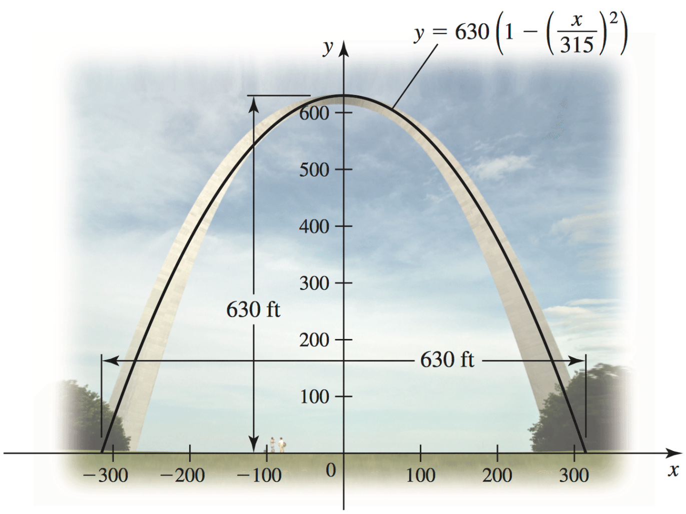
c. Assuming $0 < b < a$, the coordinates of the Sun are $\left(\sqrt{a^2 - b^2}, 0\right)$. Let D^2 denote the square of the distance from the planet to the Sun. Integrate over the interval $[-a, a]$ to show that the average value of D^2 is $\frac{4a^2 - b^2}{3}$.



47. **Gateway Arch** The Gateway Arch in St. Louis is 630 ft high and has a 630-ft base. Its shape can be modeled by the parabola

$$y = 630\left(1 - \left(\frac{x}{315}\right)^2\right).$$

Find the average height of the arch above the ground.



T **48.** **Comparing a sine and a quadratic function** Consider the functions $f(x) = \sin x$ and $g(x) = \frac{4}{\pi^2}x(\pi - x)$.

a. Carefully graph f and g on the same set of axes. Verify that both functions have a single local maximum on the interval $[0, \pi]$ and that they have the same maximum value on $[0, \pi]$.

b. On the interval $[0, \pi]$, which is true: $f(x) \geq g(x)$, $g(x) \geq f(x)$, or neither?

c. Compute and compare the average values of f and g on $[0, \pi]$.

49–52. Symmetry of composite functions *Prove that the integrand is either even or odd. Then give the value of the integral or show how it can be simplified. Assume f and g are even functions and p and q are odd functions.*

49. $\int_{-a}^{a} f(g(x))\,dx$

50. $\int_{-a}^{a} f(p(x))\,dx$

51. $\int_{-a}^{a} p(g(x))\,dx$

52. $\int_{-a}^{a} p(q(x))\,dx$

53. **Average value with a parameter** Consider the function $f(x) = a\,x\,(1 - x)$ on the interval $[0,\ 1]$, where a is a positive real number.

a. Find the average value of f as a function of a.

b. Find the points at which the value of f equals its average value and prove that they are independent of a.

Explorations and Challenges »

54. **Alternative definitions of means** Consider the function

$$f(t) = \frac{\int_a^b x^{t+1}\,dx}{\int_a^b x^t\,dx}.$$

Show that the following means can be defined in terms of f.

a. Arithmetic mean: $f(0) = \dfrac{a+b}{2}$

b. Geometric mean: $f\left(-\dfrac{3}{2}\right) = \sqrt{a\,b}$

c. Harmonic mean: $f(-3) = \dfrac{2\,a\,b}{a+b}$

(*Source: Mathematics Magazine* 78, 5, Dec 2005)

55. **Problems of antiquity** Several calculus problems were solved by Greek mathematicians long before the discovery of calculus. The following problems were solved by Archimedes using methods that predated calculus by 2000 years.

a. Show that the area of a segment of a parabola is $4/3$ that of its inscribed triangle of greatest area. In other words, the area bounded by the parabola $y = a^2 - x^2$ and the x-axis is $4/3$ the area of the triangle with vertices $(\pm a,\ 0)$ and $\left(0,\ a^2\right)$. Assume $a > 0$ is unspecified.

b. Show that the area bounded by the parabola $y = a^2 - x^2$ and the x-axis is $2/3$ the area of the rectangle with vertices $(\pm a,\ 0)$ and $\left(\pm a,\ a^2\right)$. Assume $a > 0$ is unspecified.

56. **Average value of the derivative** Suppose that f' is a continuous function for all real numbers. Show that the average value of the derivative on an interval $[a,\ b]$ is $\overline{f}\,' = \dfrac{f(b) - f(a)}{b - a}$. Interpret this result in terms of secant lines.

57. Symmetry of powers Fill in the following table with either **even** or **odd**, and prove each result. Assume n is a nonnegative integer and f^n means the nth power of f.

	***f* is even**	***f* is odd**
***n* is even**	f^n is ___	f^n is ___
***n* is odd**	f^n is ___	f^n is ___

58. Bounds on an integral Suppose f is continuous on $[a, b]$ with $f''(x) > 0$ on the interval. It can be shown that

$$(b-a)\,f\left(\frac{a+b}{2}\right) \le \int_a^b f(x)\,dx \le (b-a)\,\frac{f(a)+f(b)}{2}.$$

a. Assuming f is nonnegative on $[a, b]$, draw a figure to illustrate the geometric meaning of these inequalities. Discuss your conclusions.

b. Divide these inequalities by $(b-a)$ and interpret the resulting inequalities in terms of the average value of f on $[a, b]$.

59. Generalizing the Mean Value Theorem for Integrals Suppose f and g are continuous on $[a, b]$ and let

$$h(x) = (x-b)\int_a^x f(t)\,dt + (x-a)\int_x^b g(t)\,dt.$$

a. Use Rolle's Theorem to show that there is a number c in (a, b) such that

$$\int_a^c f(t)\,dt + \int_c^b g(t)\,dt = f(c)\,(b-c) + g(c)\,(c-a),$$

which is a generalization of the Mean Value Theorem for Integrals.

b. Show that there is a number c in (a, b) such that $\int_a^c f(t)\,dt = f(c)\,(b-c)$.

c. Use a sketch to interpret part (b) geometrically.

d. Use the result of part (a) to give an alternative proof of the Mean Value Theorem for Integrals.

(*Source: The College Mathematics Journal*, 33, 5, Nov 2002)

60. Evaluating a sine integral by Riemann sums Consider the integral $I = \int_0^{\pi/2} \sin x\,dx$.

a. Write the left Riemann sum for I with n subintervals.

b. Show that $\lim_{\theta \to 0} \theta\left(\frac{\cos\theta + \sin\theta - 1}{2\,(1-\cos\theta)}\right) = 1$.

c. It is a fact that $\sum_{k=0}^{n-1} \sin\left(\frac{\pi k}{2n}\right) = \frac{\cos\left(\frac{\pi}{2n}\right) + \sin\left(\frac{\pi}{2n}\right) - 1}{2\left(1-\cos\left(\frac{\pi}{2n}\right)\right)}$. Use this fact and part (b) to evaluate I by taking the limit of the Riemann sum as $n \to \infty$.