5.3 Fundamental Theorem of Calculus

Evaluating definite integrals using limits of Riemann sums, as described in Section 5.2, is usually not possible or practical. Fortunately, there is a powerful and practical method for evaluating definite integrals, which is developed in this section. Along the way, we discover the inverse relationship between differentiation and integration, expressed in the most important result of calculus, The Fundamental Theorem of Calculus. The first step in this process is to introduce *area functions* (first seen in Section 1.2).

Area Functions »

The concept of an area function is crucial to the discussion about the connection between derivatives and integrals. We start with a continuous function $y = f(t)$ defined for $t \geq a$, where a is a fixed number. The *area function* for f with left endpoint a is denoted $A(x)$; it gives the net area of the region bounded by the graph of f and the t-axis between $t = a$ and $t = x$ (**Figure 5.35**).

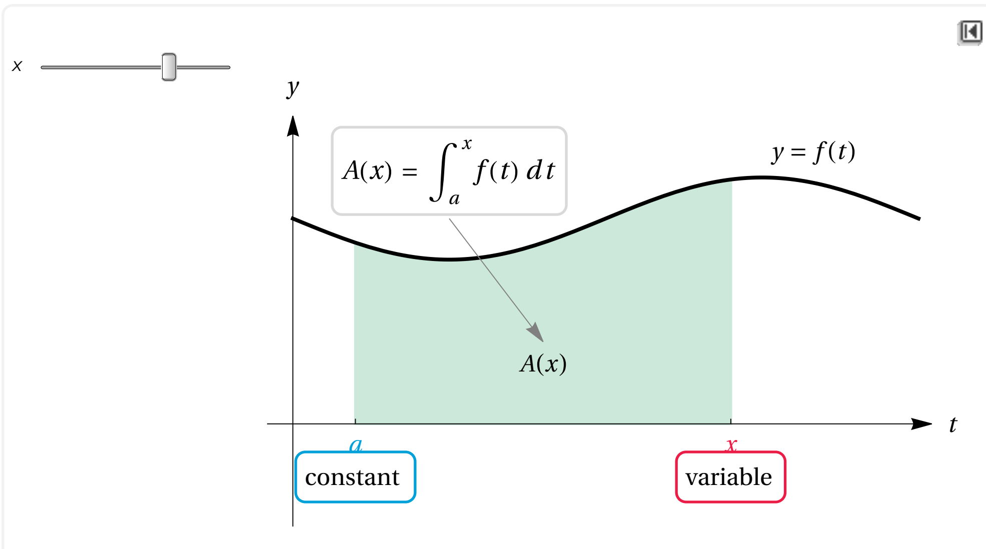
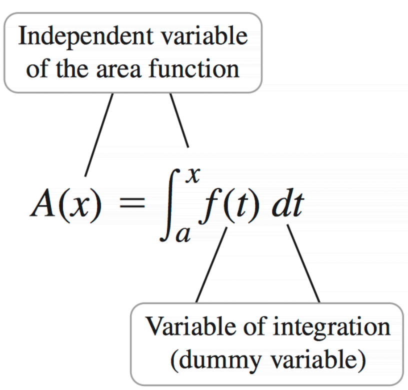


Figure 5.35

The net area of this region is also given by the definite integral



Notice that x is the upper limit of the integral *and* the independent variable of the area function: As x changes, so does the net area under the curve. Because the symbol x is already in use as the independent variable for A, we must choose another symbol for the variable of integration. Any symbol—except x—can be used because it is a *dummy variable*; we have chosen t as the integration variable.

Note »

Figure 5.36 gives a general view of how an area function is generated. Suppose f is a continuous function and a is a fixed number. Now choose a point $b > a$. The net area of the region between the graph of f and the t-axis on the interval $[a, b]$ is $A(b)$. Moving the right endpoint to $(c, 0)$ or $(d, 0)$ produces different regions with net areas $A(c)$ and $A(d)$, respectively. In general, if $x > a$ is a variable point, then $A(x) = \int_a^x f(t)\,dt$ is the net area of the region between the graph of f and the t-axis on the interval $[a, x]$.

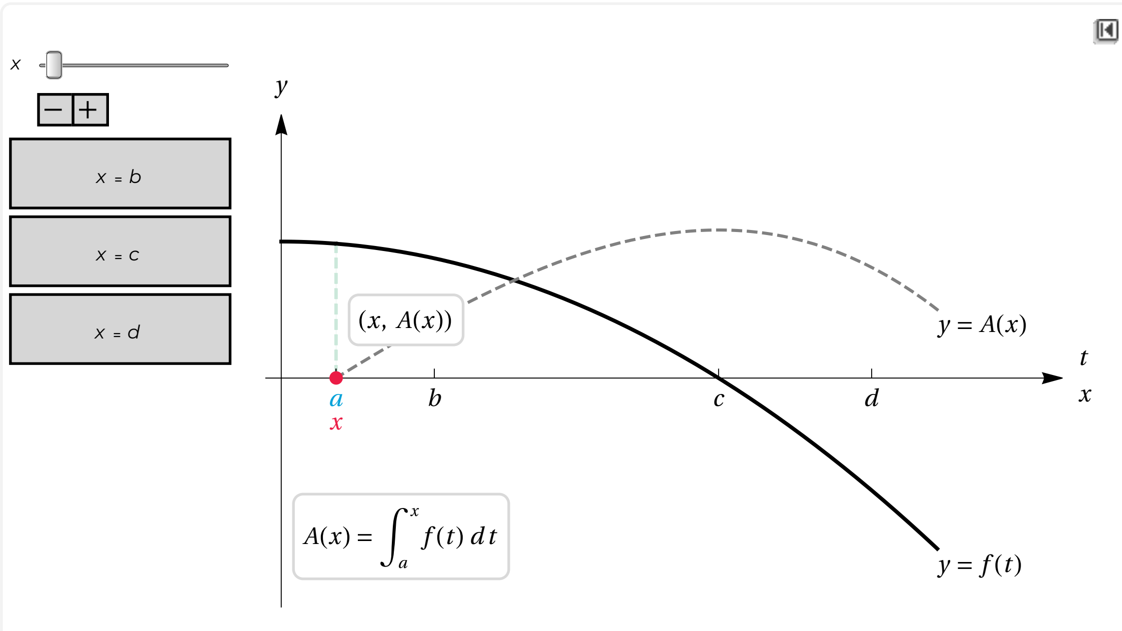


Figure 5.36

Note »

Figure 5.36 shows how $A(x)$ varies with respect to x. Notice that $A(a) = \int_a^a f(t)\,dt = 0$. Then, for $x > a$ the net area increases for $x < c$, at which point $f(c) = 0$. For $x > c$, the function f is negative, which produces a negative contribution to the area function. As a result, the area function decreases for $x > c$.

DEFINITION **Area Function**

Let f be a continuous function, for $t \ge a$. The **area function for f with left endpoint a** is

$$A(x) = \int_a^x f(t)\,dt,$$

where $x \ge a$. The area function gives the net area of the region bounded by the graph of f and the t-axis on the interval $[a, x]$.

The following two examples illustrate the idea of area functions.

EXAMPLE 1 Comparing area functions

The graph of f is shown in **Figure 5.37** with areas of various regions marked. Let $A(x) = \int_{-1}^x f(t)\,dt$ and

$F(x) = \int_3^x f(t)\,dt$ be two area functions for f (note the different left endpoints). Evaluate the following area functions.

a. $A(3)$ and $F(3)$

b. $A(5)$ and $F(5)$

c. $A(9)$ and $F(9)$

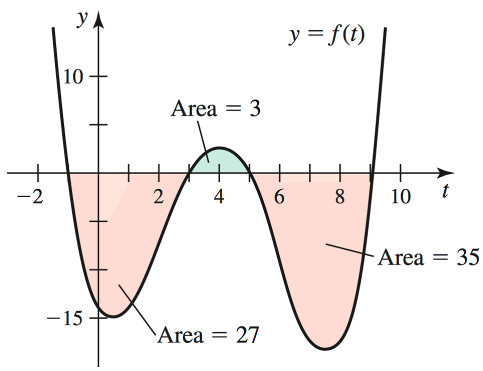


Figure 5.37

SOLUTION »

a. The value of $A(3) = \int_{-1}^{3} f(t)\,dt$ is the net area of the region bounded by the graph of f and the t-axis on the interval $[-1, 3]$. Using the graph of f, we see that $A(3) = -27$ (because this region has an area of 27 and lies below the t-axis). On the other hand, $F(3) = \int_3^3 f(t)\,dt = 0$ by Property 1 of Table 5.4. Notice that $A(3) - F(3) = -27$.

b. The value of $A(5) = \int_{-1}^{5} f(t)\,dt$ is found by subtracting the area of the region that lies below the t-axis on $[-1, 3]$ from the area of the region that lies above the t-axis on $[3, 5]$. Therefore, $A(5) = 3 - 27 = -24$. Similarly, $F(5)$ is the net area of the region bounded by the graph of f and the t-axis on the interval $[3, 5]$; therefore, $F(5) = 3$. Notice that $A(5) - F(5) = -27$.

c. Reasoning as in parts (a) and (b), we see that $A(9) = -27 + 3 - 35 = -59$ and $F(9) = 3 - 35 = -32$. As before, observe that $A(9) - F(9) = -27$.

Related Exercises 13–14 ◆

Example 1 illustrates the important fact (to be explained shortly) that two area functions of the same function differ by a constant; in Example 1, the constant is -27.

Quick Check 1 In Example 1, let $B(x)$ be the area function for f with left endpoint 5. Evaluate $B(5)$ and $B(9)$. ◆

Answer »

0, -35

EXAMPLE 2 Area of a trapezoid

Consider the trapezoid bounded by the line $f(t) = 2t + 3$ and the t-axis from $t = 2$ to $t = x$ (**Figure 5.38**). The area function $A(x) = \int_2^x f(t)\,dt$ gives the area of a trapezoid, for $x \geq 2$.

a. Evaluate $A(2)$.

b. Evaluate $A(5)$.

c. Find and graph the area function $y = A(x)$, for $x \geq 2$.

d. Compare the derivative of A to f.

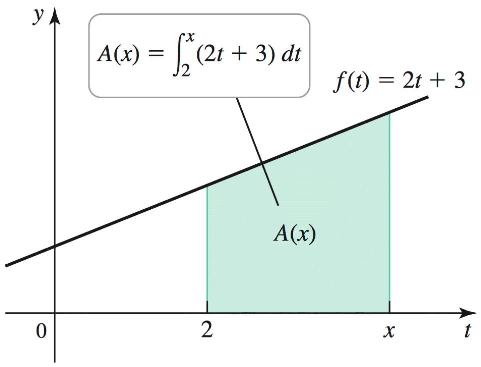


Figure 5.38

SOLUTION »

a. By Property 1 of Table 5.4, $A(2) = \int_2^2 (2t + 3)\,dt = 0$.

b. Notice that $A(5)$ is the area of the trapezoid (Figure 5.38) bounded by the line $y = 2t + 3$ and the t-axis on the interval $[2, 5]$. Using the area formula for a trapezoid (**Figure 5.39**), we find that

$$A(5) = \int_2^5 (2t + 3)\,dt = \frac{1}{2}\underbrace{(5 - 2)}_{\text{distance between parallel sides}} \cdot \underbrace{(f(2) + f(5))}_{\text{sum of parallel side lengths}} = \frac{1}{2} \cdot 3\,(7 + 13) = 30.$$

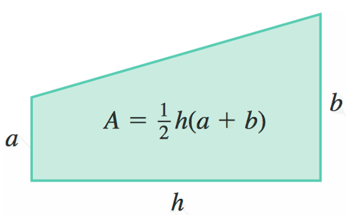


Figure 5.39

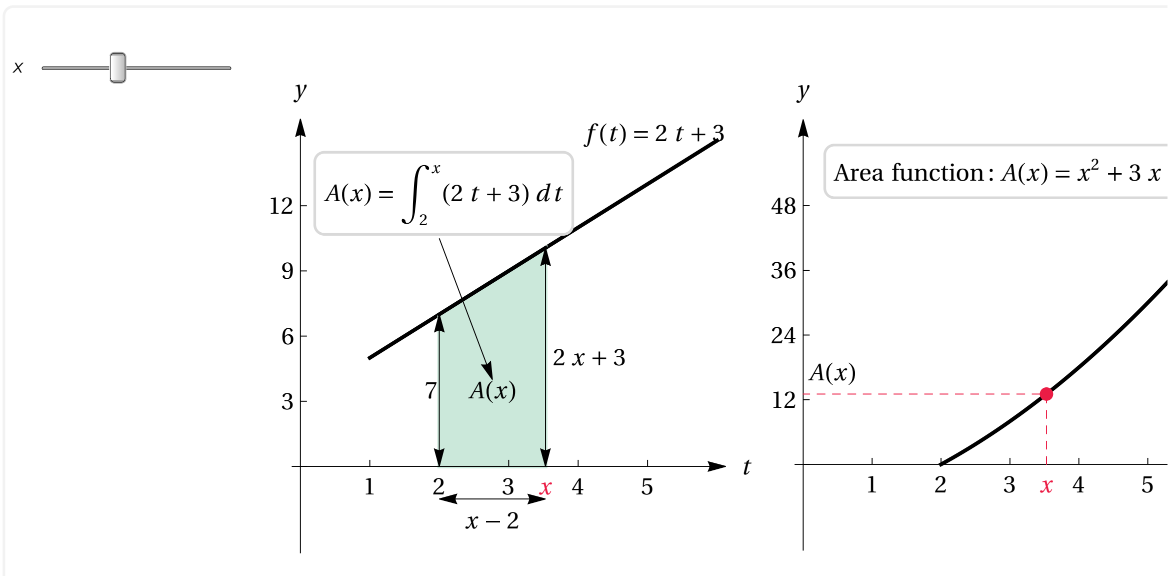
c. Now the right endpoint of the base is a variable $x \geq 2$ (**Figure 5.40**). The distance between the parallel sides of the trapezoid is $x - 2$. By the area formula for a trapezoid, the area of this trapezoid, for any $x \geq 2$, is

$$A(x) = \frac{1}{2} \underbrace{(x-2)}_{\substack{\text{distance between} \\ \text{parallel sides}}} \cdot \underbrace{(f(2)+f(x))}_{\substack{\text{sum of parallel} \\ \text{side lengths}}}$$
$$= \frac{1}{2}(x-2)(7+2x+3)$$
$$= (x-2)(x+5)$$
$$= x^2 + 3x - 10.$$

Expressing the area function in terms of an integral with a variable upper limit, we have

$$A(x) = \int_2^x (2t+3)\,dt = x^2 + 3x - 10, \text{ for } x \geq 2.$$

Because the line $f(t) = 2t + 3$ is above the t-axis for $t \geq 2$, the area function $A(x) = x^2 + 3x - 10$ is an increasing function of x with $A(2) = 0$ (**Figure 5.41**).



Figures 5.40 and 5.41

d. Differentiating the area function, we find that

$$A'(x) = \frac{d}{dx}\left(x^2 + 3x - 10\right) = 2x + 3 = f(x).$$

Therefore, $A'(x) = f(x)$, or equivalently, the area function A is an antiderivative of f. We soon show this relationship is not an accident; it is the first part of the Fundamental Theorem of Calculus.

Note »

Recall that if $A'(x) = f(x)$, then f is the derivative of A; equivalently, A is an antiderivative of f.

Related Exercises 18–19 ◆

Quick Check 2 Verify that the area function in Example 2c gives the correct area when $x = 6$ and $x = 10$.

◆
Answer »

Fundamental Theorem of Calculus »

Example 2 suggests that the area function A for a linear function f is an antiderivative of f; that is, $A'(x) = f(x)$. Our goal is to show that this conjecture is true for more general functions. Let's start with an intuitive argument; a formal proof is given at the end of the section.

Assume f is a continuous function defined on an interval $[a, b]$. As before, $A(x) = \int_a^x f(t)\,dt$ is the area function for f with a left endpoint a: It gives the net area of the region bounded by the graph of f and the t-axis on the interval $[a, x]$, for $x \geq a$. **Figure 5.42** is the key to the argument.

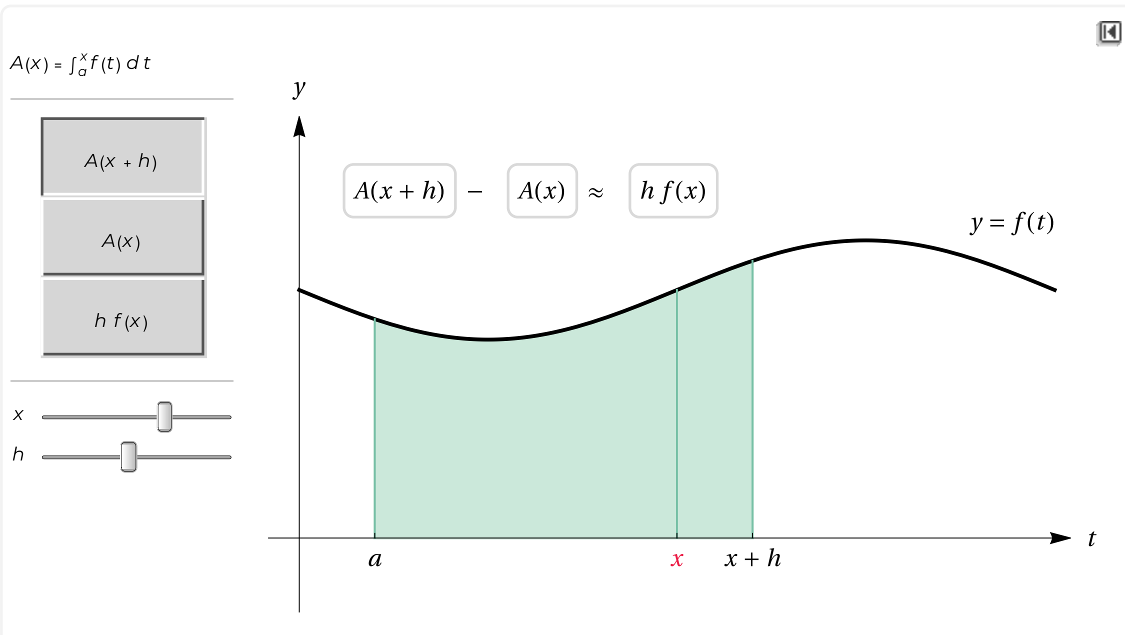


Figure 5.42

Note that with $h > 0$, $A(x + h)$ is the area of the region whose base is the interval $[a, x + h]$, while $A(x)$ is the area of the region whose base is the interval $[a, x]$. So the difference $A(x + h) - A(x)$ is the area of the region whose base is the interval $[x, x + h]$. If h is small, this region is nearly rectangular with a base of length h and a height $f(x)$. Therefore, the area of this region is

$$A(x + h) - A(x) \approx h\,f(x).$$

Dividing by h, we have

$$\frac{A(x + h) - A(x)}{h} \approx f(x).$$

An analogous argument can be made with $h < 0$. Now observe that as h tends to zero, this approximation improves. In the limit as $h \to 0$, we have

$$\underbrace{\lim_{h\to 0}\frac{A(x+h)-A(x)}{h}}_{A'(x)} = \underbrace{\lim_{h\to 0} f(x)}_{f(x)}.$$

Note »

Recall that

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

If the function f is replaced by A, then

$$A'(x) = \lim_{h\to 0}\frac{A(x+h)-A(x)}{h}.$$

We see that indeed $A'(x) = f(x)$. Because $A(x) = \int_a^x f(t)\,dt$, the result can also be written

$$A'(x) = \frac{d}{dx}\underbrace{\int_a^x f(t)\,dt}_{A(x)} = f(x),$$

which says that the derivative of the integral of f is f. This conclusion is the first part of the Fundamental Theorem of Calculus.

THEOREM 5.3 (PART 1) Fundamental Theorem of Calculus

If f is continuous on $[a, b]$, then the area function

$$A(x) = \int_a^x f(t)\,dt, \quad \text{for } a \le x \le b,$$

is continuous on $[a, b]$ and differentiable on (a, b). The area function satisfies $A'(x) = f(x)$. Equivalently,

$$A'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x),$$

which means that the area function of f is an antiderivative of f on $[a, b]$.

Given that A is an antiderivative of f, it is one short step to a powerful method for evaluating definite integrals. Remember (Section 4.9) that any two antiderivatives of f differ by a constant. Assuming F is any other antiderivative of f, we have

$$F(x) = A(x) + C, \quad \text{for } a \le x \le b.$$

Noting that $A(a) = 0$, it follows that

$$F(b) - F(a) = (A(b) + C) - (A(a) + C) = A(b).$$

Writing $A(b)$ in terms of a definite integral leads to the remarkable result

$$A(b) = \int_a^b f(x)\,dx = F(b) - F(a).$$

We have shown that to evaluate a definite integral of f, we

- find any antiderivative of f, which we call F; and
- compute $F(b) - F(a)$, the difference in the values of F between the upper and lower limits of integration.

This process is the essence of the second part of the Fundamental Theorem of Calculus.

THEOREM 5.3 (PART 2) Fundamental Theorem of Calculus

If f is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

It is customary and convenient to denote the difference $F(b) - F(a)$ by $F(x)\big|_a^b$. Using this shorthand, Part 2 of the Fundamental Theorem is summarized in **Figure 5.43**.

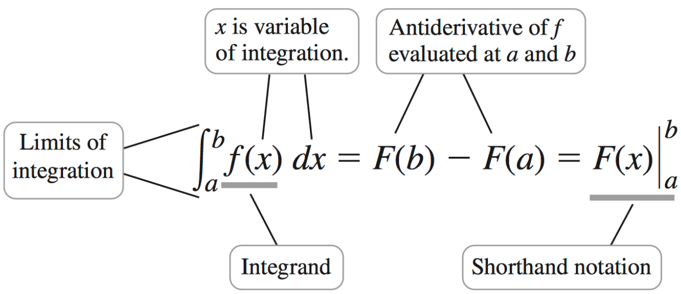


Figure 5.43

Quick Check 3 Evaluate $\left(\frac{x}{x+1}\right)\Big|_1^2$. ◆

Answer »

$\frac{2}{3} - \frac{1}{2} = \frac{1}{6}$

The Inverse Relationship Between Differentiation and Integration

It is worth pausing to observe that the two parts of the Fundamental Theorem express the inverse relationship between differentiation and integration. Part 1 of the Fundamental Theorem says

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x),$$

or the derivative of the integral of f is f itself.

Noting that f is an antiderivative of f', Part 2 of the Fundamental Theorem says

$$\int_a^b f'(x)\,dx = f(b) - f(a),$$

or the definite integral of the derivative of f is given in terms of f evaluated at two points. In other words, the integral "undoes" the derivative.

This last relationship is important because it expresses the integral as an *accumulation* operation. Suppose we know the rate of change of f (which is f') on an interval $[a, b]$. The Fundamental Theorem says that we can integrate (that is, sum or accumulate) the rate of change over that interval and the result is simply the difference in f evaluated at the endpoints. You will see this accumulation property used in many ways in the next chapter. Now let's use the Fundamental Theorem to evaluate definite integrals.

Quick Check 4 Explain why f is an antiderivative of f'. ◆

Answer »

EXAMPLE 3 Evaluating definite integrals

Evaluate the following definite integrals using the Fundamental Theorem of Calculus, Part 2. Interpret each result geometrically.

a. $\int_0^{10} (60x - 6x^2)\,dx$

b. $\int_4^{16} 3\sqrt{x}\,dx$

c. $\int_0^{2\pi} 3\sin x\,dx$

d. $\int_{1/16}^{1/4} \frac{\sqrt{t} - 2t}{t}\,dt$

SOLUTION »

a. Using the antiderivative rules of Section 4.9, an antiderivative of $f(x) = 60x - 6x^2$ is $30x^2 - 2x^3$. By the Fundamental Theorem, the value of the definite integral is

$$\int_0^{10} (60x - 6x^2)\,dx = \underbrace{(30x^2 - 2x^3)}_{F(x)}\Big|_0^{10} \quad \text{Fundamental Theorem}$$

$$= \underbrace{(30 \cdot 10^2 - 2 \cdot 10^3)}_{F(10)} - \underbrace{(30 \cdot 0^2 - 2 \cdot 0^3)}_{F(0)} \quad \text{Evaluate at } x = 10 \text{ and } x = 0.$$

$$= (3000 - 2000) - 0$$

$$= 1000. \quad \text{Simplify.}$$

Because $f(x) = 60x - x^2$ is positive on $[0, 10]$, the definite integral $\int_0^{10} (60x - 6x^2)\,dx$ is the area of the region between the graph of f and the x-axis on the interval $[0, 10]$ (**Figure 5.44**).

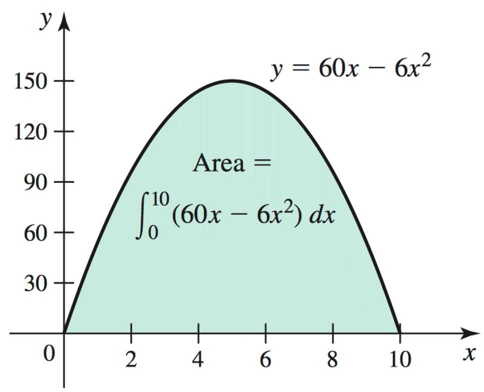


Figure 5.44

b. Because $f(x) = 3\sqrt{x}$ is positive on [4, 16] (**Figure 5.45**), the definite integral $\int_4^{16} 3\sqrt{x}\,dx$ equals the area of the region under the graph of f on [4, 16]. In Example 3 of Section 5.1, we used a midpoint Riemann sum to show that the area is approximately 112.062. Using the Fundamental Theorem, we can compute the exact area. Noting that an antiderivative of $x^{1/2}$ is $\frac{2}{3}x^{3/2}$, we have

$$\begin{aligned}
\int_4^{16} 3\sqrt{x}\,dx &= 3\int_4^{16} x^{1/2}\,dx && \text{Property 4, Table 5.4}\\
&= 3\cdot\frac{2}{3}x^{3/2}\Big|_4^{16} && \text{Fundamental Theorem}\\
&= 2\left(16^{3/2} - 4^{3/2}\right) && \text{Evaluate at } x = 16 \text{ and } x = 4.\\
&= 2(64 - 8) = 112. && \text{Simplify.}
\end{aligned}$$

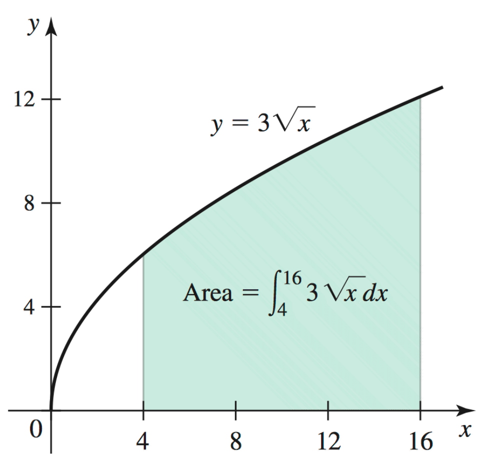


Figure 5.45

c. As shown in **Figure 5.46**, the region bounded by the graph of $f(x) = 3\sin x$ and the x-axis on $[0, 2\pi]$ consists of two parts, one above the x-axis and one below the x-axis. By the symmetry of f, these two regions have the same area, so the definite integral over $[0, 2\pi]$ is zero. Let's confirm this fact. An antiderivative of $f(x) = 3\sin x$ is $-3\cos x$. Therefore, the value of the definite integral is

$$\int_0^{2\pi} 3\sin x\,dx = -3\cos x\Big|_0^{2\pi} \qquad \text{Fundamental Theorem}$$
$$= -3(\cos 2\pi - \cos 0) \qquad \text{Substitute.}$$
$$= -3(1-1) = 0. \qquad \text{Simplify.}$$

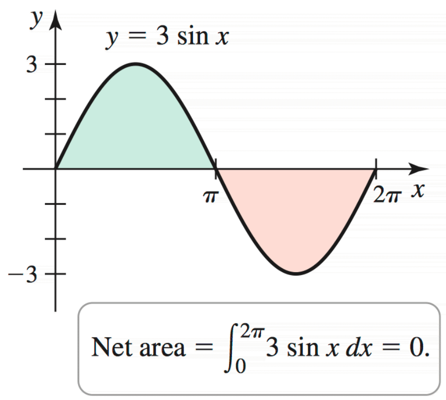


Figure 5.46

Note »

When evaluating definite integrals, factor out multiplicative constants when possible, as we did in parts (b) and (c) of Example 3. To illustrate: It is better to write

$$-3\cos x\Big|_0^{2\pi} = -3(\cos 2\pi - \cos 0)$$

than

$$-3\cos x\Big|_0^{2\pi} = (-3\cos 2\pi) - (-3\cos 0).$$

d. Although the variable of integration is t, rather than x, we proceed as before after simplifying the integrand:

$$\frac{\sqrt{t}-2t}{t} = \frac{1}{\sqrt{t}} - 2.$$

Finding antiderivatives with respect to t and applying the Fundamental Theorem, we have

$$\int_{1/16}^{1/4} \frac{\sqrt{t}-2t}{t}\,dt = \int_{1/16}^{1/4} \left(t^{-1/2}-2\right)dt \qquad \text{Simplify the integrand.}$$
$$= \left(2t^{1/2}-2t\right)\Big|_{1/16}^{1/4} \qquad \text{Fundamental Theorem}$$
$$= \left(2\left(\frac{1}{4}\right)^{1/2} - \frac{1}{2}\right) - \left(2\left(\frac{1}{16}\right)^{1/2} - \frac{1}{8}\right) \qquad \text{Evaluate.}$$
$$= 1 - \frac{1}{2} - \frac{1}{2} + \frac{1}{8} \qquad \text{Simplify.}$$
$$= \frac{1}{8}.$$

Note »

We know that

$$\frac{d}{dt}\left(t^{1/2}\right) = \frac{1}{2}\,t^{-1/2}.$$

Therefore,

$$\int \frac{1}{2}\,t^{-1/2}\,dt = t^{1/2} + C$$

and

$$\int \frac{dt}{\sqrt{t}} = \int t^{-1/2}\,dt = 2\,t^{1/2} + C.$$

The definite integral is positive because the graph of f lies above the t-axis on the interval of integration $\left[\frac{1}{16}, \frac{1}{4}\right]$ (**Figure 5.47**).

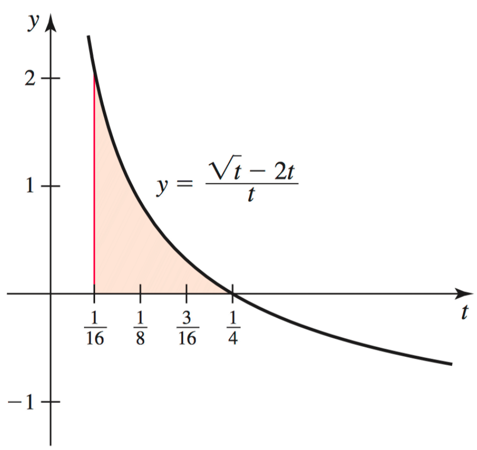


Figure 5.47

Related Exercises 23–24, 26, 36 ◆

EXAMPLE 4 Net areas and definite integrals

The graph of $f(x) = 6\,x\,(x + 1)\,(x - 2)$ is shown in **Figure 5.48**. The region R_1 is bounded by the curve and the x-axis on the interval $[-1, 0]$, and R_2 is bounded by the curve and the x-axis on the interval $[0, 2]$.

a. Find the *net area* of the region bounded by the curve and the x-axis on the interval $[-1, 2]$.

b. Find the *area* of the region bounded by the curve and the x-axis on the interval $[-1, 2]$.

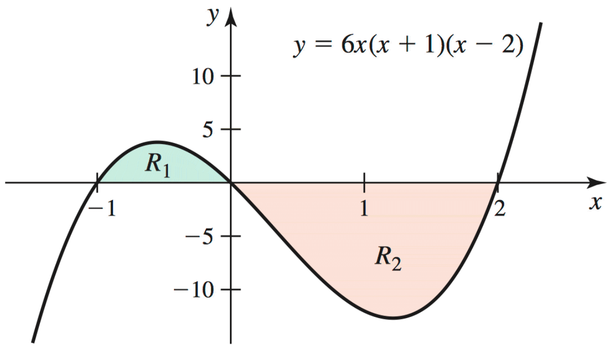


Figure 5.48

SOLUTION »

a. The net area of the region is given by a definite integral. The integrand f is first expanded in order to find an antiderivative:

$$\begin{aligned}\int_{-1}^{2} f(x)\,dx &= \int_{-1}^{2}\left(6x^3 - 6x^2 - 12x\right)dx && \text{Expand } f.\\ &= \left(\frac{3}{2}x^4 - 2x^3 - 6x^2\right)\Big|_{-1}^{2} && \text{Fundamental Theorem}\\ &= -\frac{27}{2}. && \text{Simplify.}\end{aligned}$$

The net area of the region between the curve and the x-axis on $[-1, 2]$ is $-\frac{27}{2}$, which is the area of R_1 *minus* the area of R_2 (Figure 5.48). Because R_2 has a larger area than R_1, the net area is negative.

b. The region R_1 lies above the x-axis, so its area is

$$\int_{-1}^{0}\left(6x^3 - 6x^2 - 12x\right)dx = \left(\frac{3}{2}x^4 - 2x^3 - 6x^2\right)\Big|_{-1}^{0} = \frac{5}{2}.$$

The region R_2 lies below the x-axis, so its net area is negative:

$$\int_{0}^{2}\left(6x^3 - 6x^2 - 12x\right)dx = \left(\frac{3}{2}x^4 - 2x^3 - 6x^2\right)\Big|_{0}^{2} = -16.$$

Therefore, the *area* of R_2 is $-(-16) = 16$. The combined area of R_1 and R_2 is $\frac{5}{2} + 16 = \frac{37}{2}$. We could also find the area of this region directly by evaluating $\int_{-1}^{2} |f(x)|\,dx$.

Related Exercises 66, 68ß ◆

Examples 3 and 4 make use of Part 2 of the Fundamental Theorem, which is the most potent tool for evaluating definite integrals. The remaining examples illustrate the use of the equally important Part 1 of the Fundamental Theorem.

EXAMPLE 5 **Derivatives of integrals**

Use Part 1 of the Fundamental Theorem to simplify the following expressions.

a. $\dfrac{d}{dx}\displaystyle\int_1^x \sin^2 t\, dt$

b. $\dfrac{d}{dx}\displaystyle\int_x^5 \sqrt{t^2+1}\, dt$

c. $\dfrac{d}{dx}\displaystyle\int_0^{x^2} \cos t^2\, dt$

SOLUTION »

a. Using Part 1 of the Fundamental Theorem, we see that

$$\frac{d}{dx}\int_1^x \sin^2 t\, dt = \sin^2 x.$$

b. To apply Part 1 of the Fundamental Theorem, the variable must appear in the upper limit. Therefore, we use the fact that $\int_a^b f(t)\, dt = -\int_b^a f(t)\, dt$ and then apply the Fundamental Theorem:

$$\frac{d}{dx}\int_x^5 \sqrt{t^2+1}\, dt = -\frac{d}{dx}\int_5^x \sqrt{t^2+1}\, dt = -\sqrt{x^2+1}.$$

c. The upper limit of the integral is not x, but a function of x. Therefore, the function to be differentiated is a composite function, which requires the Chain Rule. We let $u = x^2$ to produce

$$y = g(u) = \int_0^u \cos t^2\, dt.$$

By the Chain Rule,

$$\begin{aligned}
\frac{d}{dx}\int_0^{x^2} \cos t^2\, dt &= \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} && \text{Chain Rule}\\
&= \left(\frac{d}{du}\int_0^u \cos t^2\, dt\right)(2x) && \text{Substitute for } y\text{; note } u'(x) = 2x.\\
&= \left(\cos u^2\right)(2x) && \text{Fundamental Theorem}\\
&= 2x\cos x^4. && \text{Substitute } u = x^2.
\end{aligned}$$

Note »

Example 5c illustrates one case of Leibniz's Rule:

$$\frac{d}{dx}\int_a^{g(x)} f(t)\, dt = f(g(x))\, g'(x).$$

Related Exercises 71, 73, 76 ◆

EXAMPLE 6 **Working with area functions**

Consider the function f shown in **Figure 5.49** and its area function $A(x) = \int_0^x f(t)\,dt$, for $0 \le x \le 17$. Assume the four regions R_1, R_2, R_3, and R_4 have the same area. Based on the graph of f, do the following.

a. Find the zeros of A on $[0, 17]$.

b. Find the points on $[0, 17]$ at which A has a local maximum or a local minimum.

c. Sketch a graph of A, for $0 \le x \le 17$.

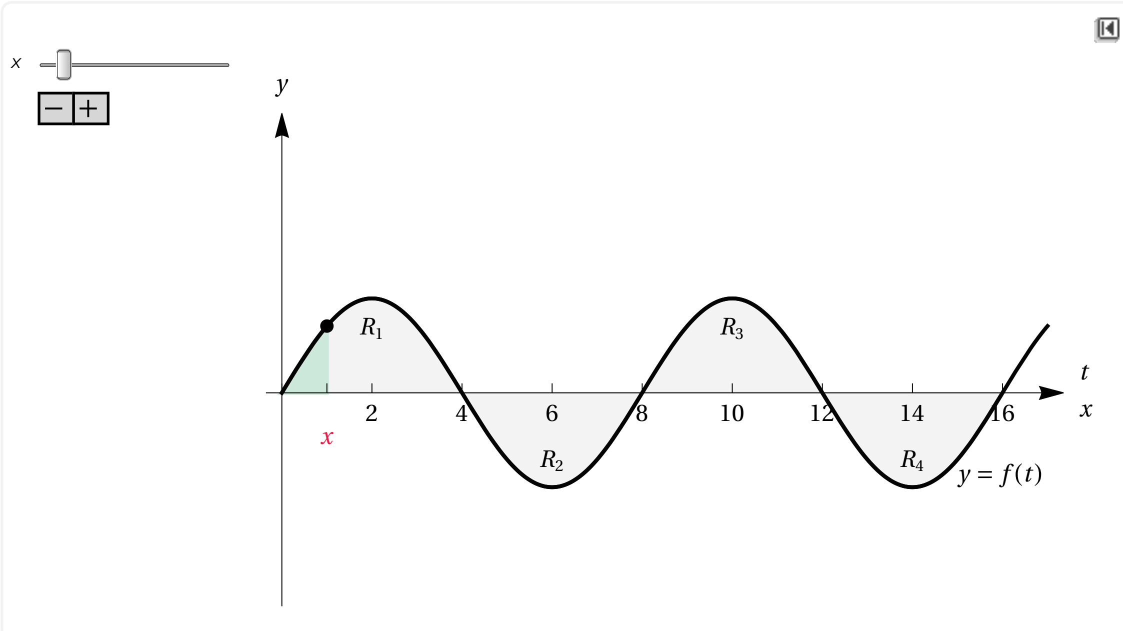


Figure 5.49

SOLUTION »

a. The area function $A(x) = \int_0^x f(t)\,dt$ gives the net area bounded by the graph of f and the t-axis on the interval $[0, x]$ (**Figure 5.50**). Therefore, $A(0) = \int_0^0 f(t)\,dt = 0$. Because R_1 and R_2 have the same area but lie on opposite sides of the t-axis, it follows that $A(8) = \int_0^8 f(t)\,dt = 0$. Similarly, $A(16) = \int_0^{16} f(t)\,dt = 0$. Therefore, the zeros of A are $x = 0, 8$, and 16.

b. Observe that the function f is positive for $0 < t < 4$, which implies that $A(x)$ increases as x increases from 0 to 4 (Figure 5.50). Then, as x increases from 4 to 8, $A(x)$ decreases because f is negative for $4 < t < 8$. Similarly, $A(x)$ increases as x increases from $x = 8$ to $x = 12$ and decreases from $x = 12$ to $x = 16$. By the First Derivative Test, A has local minima at $x = 8$ and $x = 16$ and local maxima at $x = 4$ and $x = 12$.

c. Combining the observations in parts (a) and (b) leads to a qualitative sketch of A (Figure 5.50). Note that $A(x) \ge 0$ for all $x \ge 0$. It is not possible to determine function values (y-coordinates) on the graph of A.

Note »

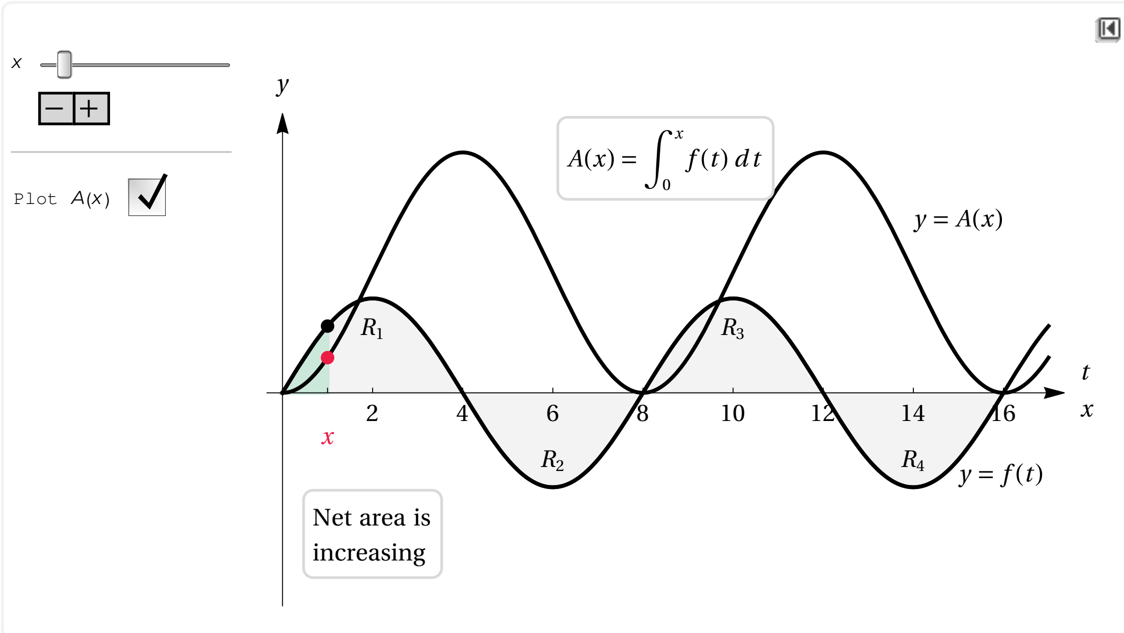


Figure 5.50

Related Exercises 69–70 ◆

EXAMPLE 7 The sine integral function

Let

$$g(t) = \begin{cases} \dfrac{\sin t}{t} & \text{if } t > 0 \\ 1 & \text{if } t = 0. \end{cases}$$

Graph the *sine integral function* $S(x) = \int_0^x g(t)\,dt$, for $x \geq 0$.

SOLUTION »

Notice that S is an area function for g. The independent variable of S is x, while t has been chosen as the (dummy) variable of integration. A good way to start is by graphing the integrand g (**Figure 5.51**). The function oscillates with a decreasing amplitude with $g(0) = 1$. Beginning with $S(0) = 0$, the area function S increases until $x = \pi$ because g is positive on $(0, \pi)$. However, on $(\pi, 2\pi)$, g is negative and the net area decreases. On $(2\pi, 3\pi)$, g is positive again, so S again increases. Therefore, the graph of S has alternating local maxima and minima. Because the amplitude of g decreases, each maximum of S is less than the previous maximum and each minimum of S is greater than the previous minimum (Figure 5.51). Determining the exact value of S at these maxima and minima is difficult.

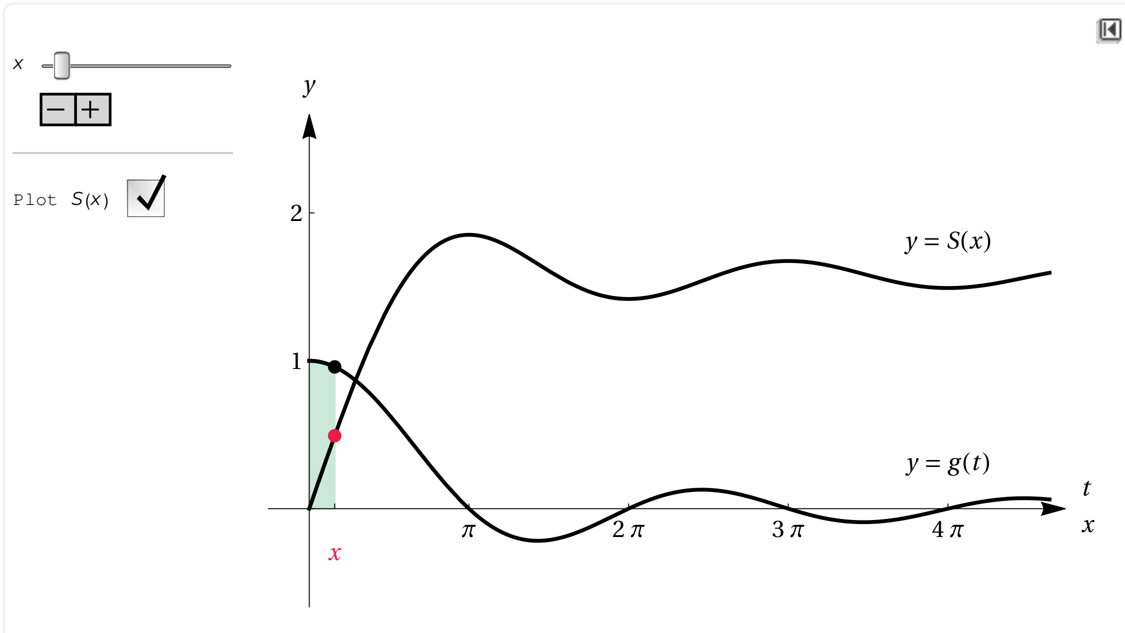


Figure 5.51

Appealing to Part 1 of the Fundamental Theorem, we find that

$$S'(x) = \frac{d}{dx} \int_0^x \frac{\sin t}{t}\, dt = \frac{\sin x}{x}, \quad \text{for } x > 0.$$

As anticipated, the derivative of S changes sign at integer multiples of π. Specifically, S' is positive and S increases on the intervals $(0, \pi)$, $(2\pi, 3\pi)$, … , $(2n\pi, (2n+1)\pi)$, … , while S' is negative and S decreases on the remaining intervals. Clearly, S has local maxima at $x = \pi, 3\pi, 5\pi, \ldots$, and it has local minima at $x = 2\pi, 4\pi, 6\pi, \ldots$.

Note »

Note that $\lim_{x\to\infty} S'(x) = \lim_{x\to\infty} g(x) = 0$.

One more observation is helpful. It can be shown that, while S oscillates for increasing x, its graph gradually flattens out and approaches a horizontal asymptote. (Finding the exact value of this horizontal asymptote is challenging; see Exercise 113.) Assembling all these observations, the graph of the sine integral function emerges (Figure 5.51).

Related Exercises 99–100 ◆

We conclude this section with a formal proof of the Fundamental Theorem of Calculus.

Proof of the Fundamental Theorem: Let f be continuous on $[a, b]$ and let A be the area function for f with left endpoint a. The first step is to prove that $A'(x) = f(x)$, which is Part 1 of the Fundamental Theorem. The proof of Part 2 then follows.

Step 1. We assume $a < x < b$ and use the definition of the derivative,

$$A'(x) = \lim_{h\to 0} \frac{A(x+h) - A(x)}{h}.$$

First assume $h > 0$. Using **Figure 5.52** and Property 5 of Table 5.4, we have

$$A(x+h) - A(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt = \int_x^{x+h} f(t)\,dt.$$

That is, $A(x+h) - A(x)$ is the net area of the region bounded by the curve on the interval $[x, x+h]$.

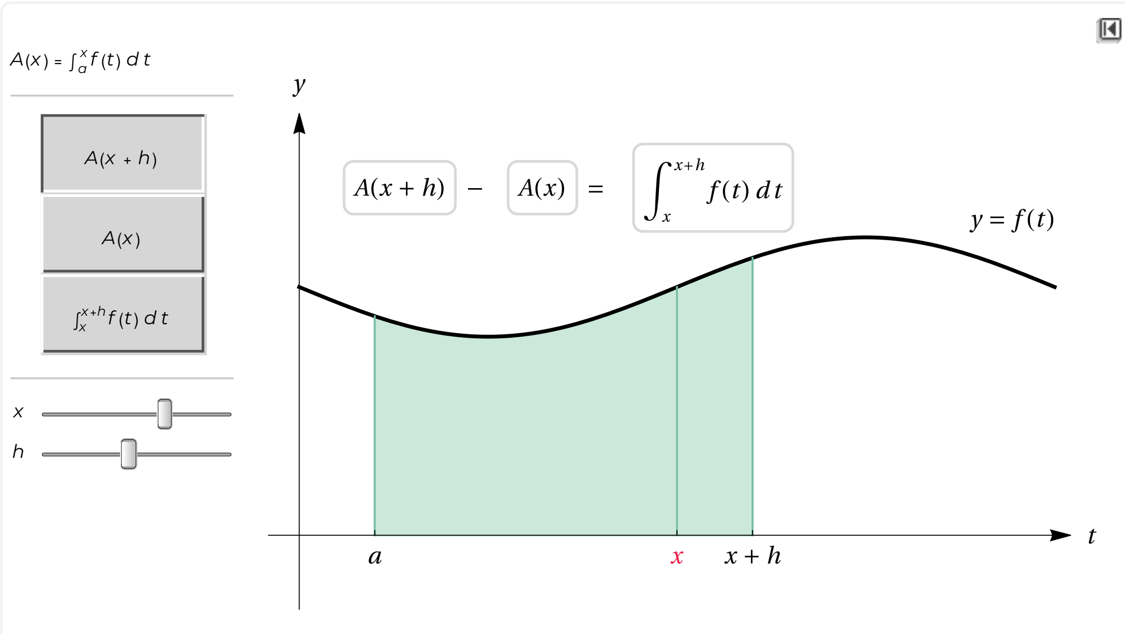


Figure 5.52

Let m and M be the minimum and maximum values of f on $[x, x+h]$, respectively, which exist by the continuity of f. Note that $m \le f(t) \le M$ on $[x, x+h]$ (which is an interval of length h). It follows, by property 9 of Table 5.5, that

$$mh \le \underbrace{\int_x^{x+h} f(t)\,dt}_{A(x+h)\,-\,A(x)} \le Mh.$$

Substituting for the integral, we find that

$$mh \le A(x+h) - A(x) \le Mh.$$ **Figure 5.52 b**

Note »

> The quantities m and M exist for any $h > 0$; however, they also depend on h.

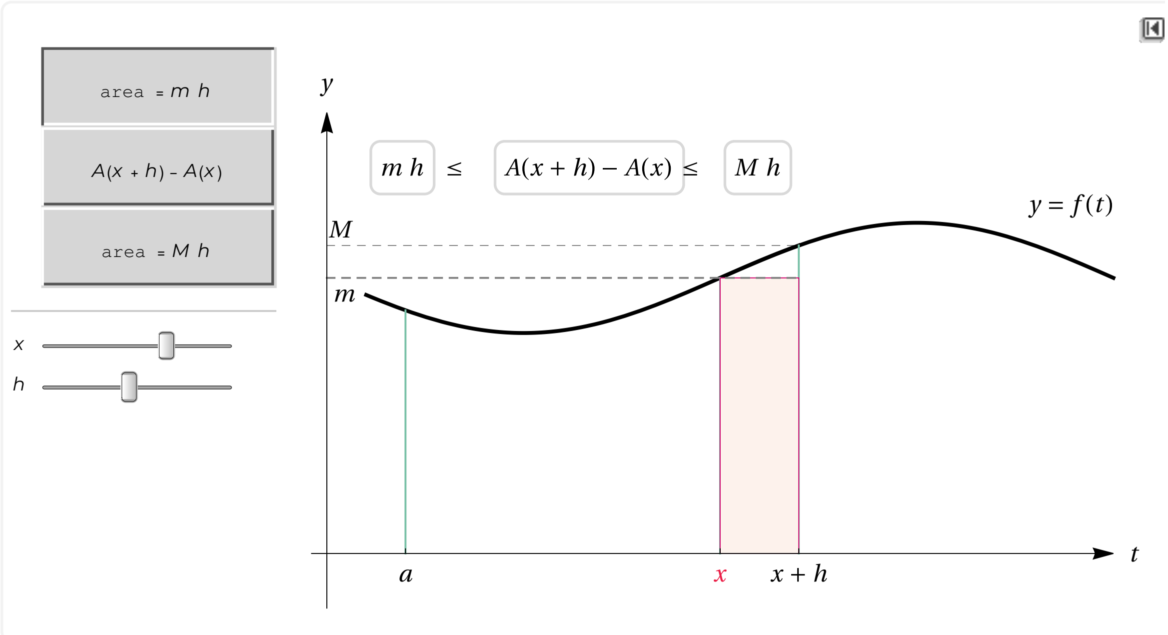


Figure 5.52 b

Dividing these inequalities by $h > 0$, we have

$$m \le \frac{A(x+h) - A(x)}{h} \le M.$$

The case $h < 0$ is handled similarly and leads to the same conclusion.

We now take the limit as $h \to 0$ across these inequalities. As $h \to 0$, m and M approach $f(x)$, because f is continuous at x. At the same time, as $h \to 0$, the quotient that is sandwiched between m and M approaches $A'(x)$:

$$\underbrace{\lim_{h \to 0} m}_{f(x)} \le \underbrace{\lim_{h \to 0} \frac{A(x+h) - A(x)}{h}}_{A'(x)} \le \underbrace{\lim_{h \to 0} M}_{f(x)}.$$

By the Squeeze Theorem (Theorem 2.5), we conclude that $A'(x)$ exists and A is differentiable for $a < x < b$. Furthermore, $A'(x) = f(x)$. Finally, because A is differentiable on (a, b), A is continuous on (a, b) by Theorem 3.1. Exercise 116 shows that A is also right and left-continuous at the endpoints a and b, respectively.

Step 2. Having established that the area function A is an antiderivative of f, we know that $F(x) = A(x) + C$, where F is any antiderivative of f and C is a constant. Noting that $A(a) = 0$, it follows that

$$F(b) - F(a) = (A(b) + C) - (A(a) + C) = A(b).$$

Note »

> Once again we use an important fact: Two antiderivatives of the same function differ by a constant.

Writing $A(b)$ in terms of a definite integral, we have

$$A(b) = \int_a^b f(x)\,dx = F(b) - F(a),$$

which is Part 2 of the Fundamental Theorem. ◆

Exercises »

Getting Started »

Practice Exercises »

13. Area functions The graph of f is shown in the figure. Let $A(x) = \int_{-2}^{x} f(t)\,dt$ and $F(x) = \int_{4}^{x} f(t)\,dt$ be two area functions for f. Evaluate the following area functions.

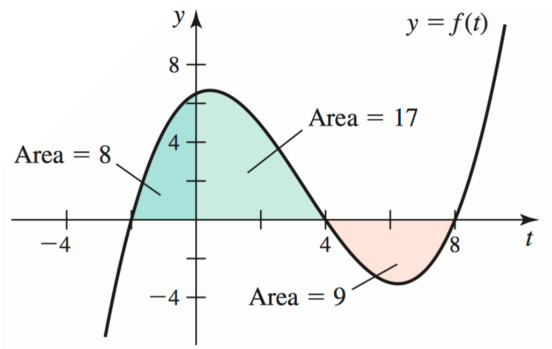
a. $A(-2)$

b. $F(8)$

c. $A(4)$

d. $F(4)$

e. $A(8)$



14. Area functions The graph of f is shown in the figure. Let $A(x) = \int_{0}^{x} f(t)\,dt$ and $F(x) = \int_{2}^{x} f(t)\,dt$ be two area functions for f. Evaluate the following area functions.

a. $A(2)$

b. $F(5)$

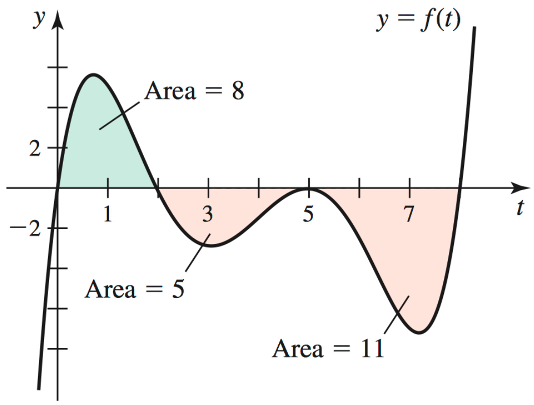
c. $A(0)$

d. $F(8)$

e. $A(8)$

f. $A(5)$

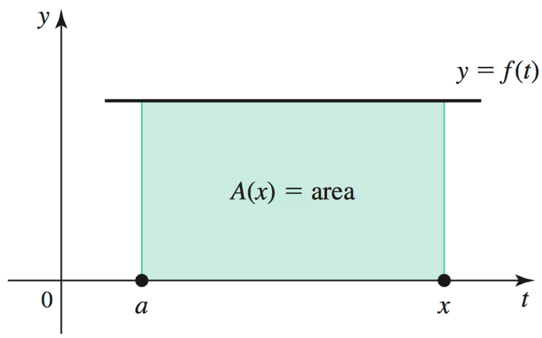
g. $F(2)$



15–16. Area functions for constant functions *Consider the following functions f and real numbers a (see figure).*

a. *Find and graph the area function* $A(x) = \int_a^x f(t)\,dt$ *for* f.

b. *Verify that* $A'(x) = f(x)$.



15. $f(t) = 5, \quad a = 0$

16. $f(t) = 5, \quad a = -5$

17. Area functions for the same linear function Let $f(t) = t$ and consider the two area functions $A(x) = \int_0^x f(t)\,dt$ and $F(x) = \int_2^x f(t)\,dt$.

a. Evaluate $A(2)$ and $A(4)$. Then use geometry to find an expression for $A(x)$, for $x \ge 0$.

b. Evaluate $F(4)$ and $F(6)$. Then use geometry to find an expression for $F(x)$, for $x \ge 2$.

c. Show that $A(x) - F(x)$ is a constant and that $A'(x) = F'(x) = f(x)$.

18. Area functions for the same linear function Let $f(t) = 2t - 2$ and consider the two area functions $A(x) = \int_1^x f(t)\,dt$ and $F(x) = \int_4^x f(t)\,dt$.

a. Evaluate $A(2)$ and $A(3)$. Then use geometry to find an expression for $A(x)$, for $x \ge 1$.

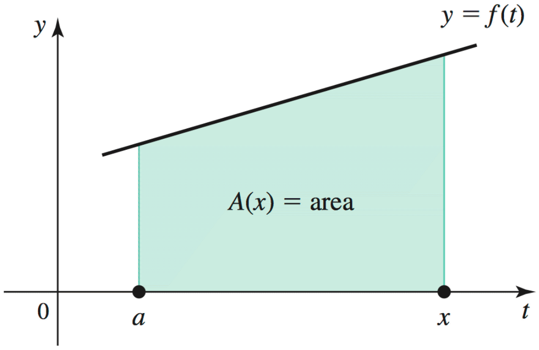
b. Evaluate $F(5)$ and $F(6)$. Then use geometry to find an expression for $F(x)$, for $x \ge 4$.

c. Show that $A(x) - F(x)$ is a constant and that $A'(x) = F'(x) = f(x)$.

19–22. Area functions for linear functions *Consider the following functions f and real numbers a (see figure).*

a. *Find and graph the area function* $A(x) = \int_a^x f(t)\,dt$.

b. *Verify that* $A'(x) = f(x)$.



19. $f(t) = t + 5, \quad a = -5$

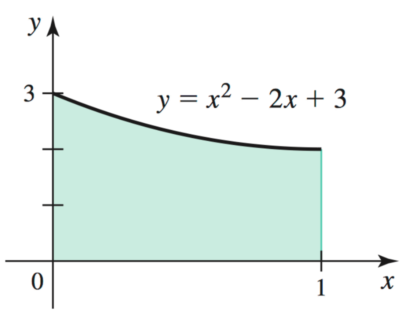
20. $f(t) = 2t + 5, \quad a = 0$

21. $f(t) = 3t + 1, \quad a = 2$

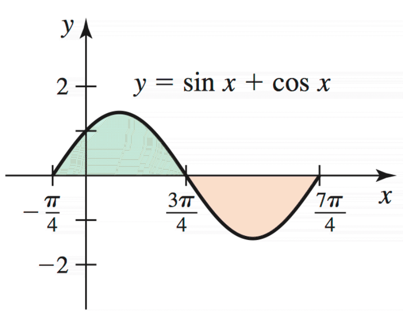
22. $f(t) = 4t + 2, \quad a = 0$

23–24. Definite integrals *Evaluate the following integrals using the Fundamental Theorem of Calculus. Explain why your result is consistent with the figure.*

23. $\int_0^1 \left(x^2 - 2x + 3\right) dx$



24. $\int_{-\pi/4}^{7\pi/4} (\sin x + \cos x)\, dx$



25–28. Definite integrals *Evaluate the following integrals using the Fundamental Theorem of Calculus. Sketch the graph of the integrand and shade the region whose net area you have found.*

25. $\int_{-2}^{3} \left(x^2 - x - 6\right) dx$

26. $\int_{0}^{1} \left(x - \sqrt{x}\right) dx$

27. $\int_{0}^{5} \left(x^2 - 9\right) dx$

28. $\int_{1/2}^{2} \left(1 - \frac{1}{x^2}\right) dx$

29–60. Definite integrals *Evaluate the following integrals using the Fundamental Theorem of Calculus.*

29. $\int_{0}^{2} 4\, x^3\, dx$

30. $\int_{0}^{2} \left(3\, x^2 + 2\, x\right) dx$

31. $\int_{1}^{8} 8\, x^{1/3}\, dx$

32. $\int_{1}^{16} x^{-5/4}\, dx$

33. $\int_{0}^{1} \left(x + \sqrt{x}\right) dx$

34. $\int_{0}^{\pi/4} 2 \cos x\, dx$

35. $\int_{1}^{9} \frac{2}{\sqrt{x}}\, dx$

36. $\int_{4}^{9} \frac{2 + \sqrt{t}}{\sqrt{t}}\, dt$

37. $\int_{-2}^{2} \left(x^2 - 4\right) dx$

38. $\int_{0}^{\pi/4} (\sin x + \cos x)\, dx$

39. $\int_{1/2}^{1} \left(t^{-3} - 8\right) dt$

40. $\int_{0}^{4} t\,(t - 2)\,(t - 4)\, dt$

41. $\int_{1}^{4} (1-x)(x-4)\,dx$

42. $\int_{0}^{2} t(t+1)\,dt$

43. $\int_{-2}^{-1} x^{-3}\,dx$

44. $\int_{0}^{\pi} (1-\sin x)\,dx$

45. $\int_{0}^{\pi/4} \sec^2\theta\,d\theta$

46. $\int_{-\pi/2}^{\pi/2} (\cos x - 1)\,dx$

47. $\int_{1}^{2} \frac{3}{w^2}\,dw$

48. $\int_{4}^{9} \frac{x-\sqrt{x}}{x^3}\,dx$

49. $\int_{1}^{8} \sqrt[3]{y}\,dy$

50. $\frac{1}{2}\int_{1}^{4} \frac{x^2-1}{x^2}\,dx$

51. $\int_{1}^{4} \frac{x-2}{\sqrt{x}}\,dx$

52. $\int_{1}^{2} \frac{2s-4}{s^3}\,ds$

53. $\int_{0}^{\pi/3} \sec x \tan x\,dx$

54. $\int_{\pi/4}^{\pi/2} \csc^2\theta\,d\theta$

55. $\int_{\pi/4}^{3\pi/4} \left(\cot^2 x + 1\right)dx$

56. $\int_{1}^{2} \frac{x^2+6x+8}{x^4+2x^3}\,dx$

57. $\int_{\pi/4}^{\pi/2} \csc x \cot x\,dx$

58. $\int_0^{\pi/4} \sec x\,(\sec x + \cos x)\,dx$

59. $\int_0^{\pi} f(x)\,dx$, where $f(x) = \begin{cases} \sin x + 1 & \text{if } x \le \pi/2 \\ 2\cos x + 2 & \text{if } x > \pi/2 \end{cases}$

60. $\int_1^{3} g(x)\,dx$, where $g(x) = \begin{cases} 3x^2 + 4x + 1 & \text{if } x \le 2 \\ 2x + 5 & \text{if } x > 2 \end{cases}$

61–64. Area *Find (i) the net area and (ii) the area of the following regions. Graph the function and indicate the region in question.*

61. The region bounded by $y = x^{1/2}$ and the x-axis between $x = 1$ and $x = 4$

62. The region above the x-axis bounded by $y = 4 - x^2$

63. The region below the x-axis bounded by $y = x^4 - 16$

64. The region bounded by $y = 6\cos x$ and the x-axis between $x = -\pi/2$ and $x = \pi$

65–70. Areas of regions *Find the area of the region bounded by the graph of f and the x-axis on the given interval.*

65. $f(x) = x^2 - 25$ on $[2, 4]$

66. $f(x) = x^3 - 1$ on $[-1, 2]$

67. $f(x) = \dfrac{1}{x^5}$ on $[-2, -1]$

68. $f(x) = x\,(x+1)\,(x-2)$ on $[-1, 2]$

69. $f(x) = \sin x$ on $\left[-\dfrac{\pi}{4}, \dfrac{3\pi}{4}\right]$

70. $f(x) = \cos x$ on $\left[\dfrac{\pi}{2}, \pi\right]$

71–84. Derivatives of integrals *Simplify the following expressions.*

71. $\dfrac{d}{dx}\int_3^{x} \left(t^2 + t + 1\right) dt$

72. $\dfrac{d}{dx}\int_0^{x} \sin^2 t\,dt$

73. $\dfrac{d}{dx}\int_x^{1} \sqrt{t^4 + 1}\,dt$

74. $\dfrac{d}{dx}\int_x^{0} \dfrac{dp}{p^2 + 1}$

75. $\frac{d}{dx}\int_2^{x^3}\frac{dp}{p^2}$

76. $\frac{d}{dx}\int_0^{x^2}\frac{dt}{t^2+4}$

77. $\frac{d}{dx}\int_0^{\cos x}\left(t^4+6\right)dt$

78. $\frac{d}{dx}\int_x^1\cos^3 t\,dt$

79. $\frac{d}{dz}\int_{\sin z}^{10}\frac{dt}{t^4+1}$

80. $\frac{d}{dy}\int_{y^3}^{10}\sqrt{x^6+1}\,dx$

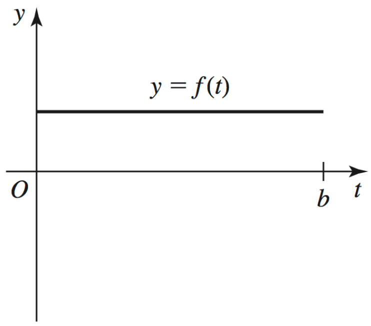
81. $\frac{d}{dt}\left(\int_1^t\frac{3}{x}\,dx-\int_{t^2}^1\frac{3}{x}\,dx\right)$

82. $\frac{d}{dt}\left(\int_0^t\frac{dx}{1+x^2}+\int_1^{1/t}\frac{dx}{1+x^2}\right)$

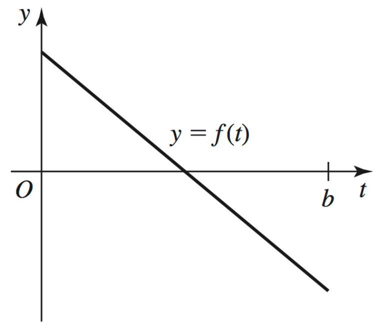
83. $\frac{d}{dx}\int_{-x}^x\sqrt{1+t^2}\,dt$ (*Hint*: $\int_{-x}^x\sqrt{1+t^2}\,dt=\int_{-x}^0\sqrt{1+t^2}\,dt+\int_0^x\sqrt{1+t^2}\,dt$.)

84. $\frac{d}{dx}\int_x^{x^2}\sin t^2\,dt$

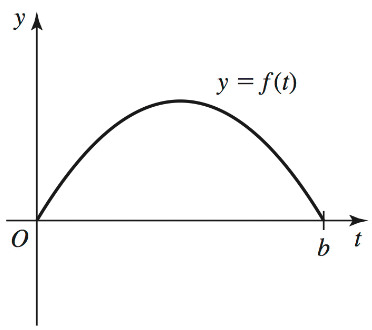
85. **Matching functions with area functions** Match the functions f, whose graphs are given in a–d, with the area functions $A(x)=\int_0^x f(t)\,dt$, whose graphs are given in A–D.



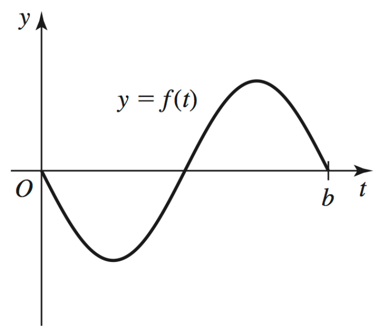
(a)



(b)



(c)



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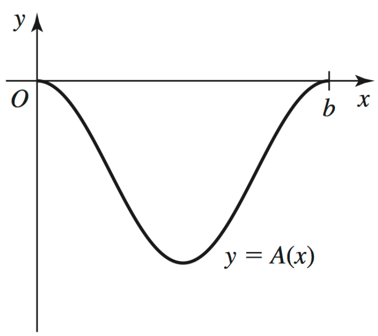
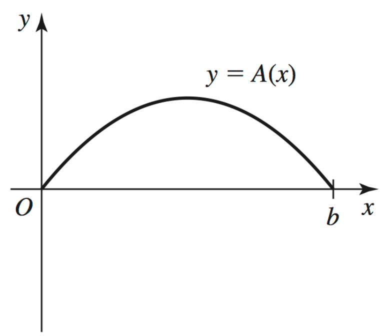
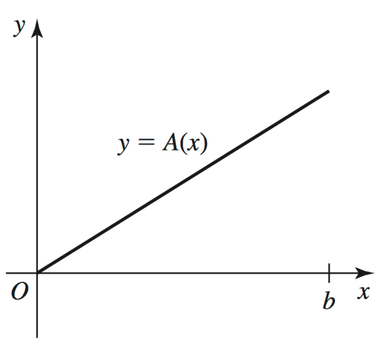


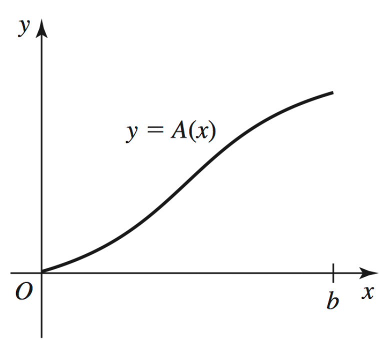
(A)



(B)



(C)



(D)

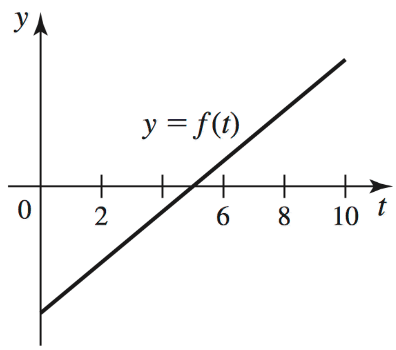
86–89. Working with area functions *Consider the function f and its graph.*

a. *Estimate the zeros of the area function* $A(x) = \int_0^x f(t)\,dt$, *for* $0 \le x \le 10$.

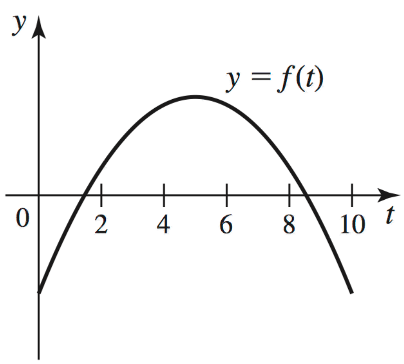
b. *Estimate the points (if any) at which A has a local maximum or minimum.*

c. *Sketch a graph of A, for* $0 \le x \le 10$, *without a scale on the y-axis.*

86.



87.



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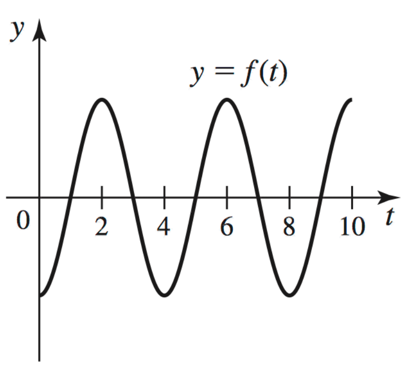
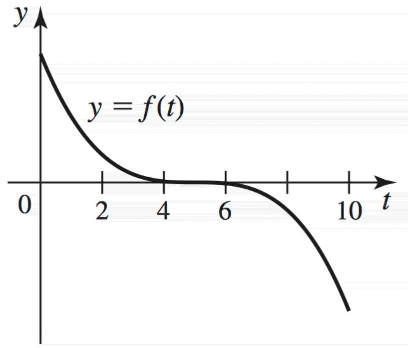
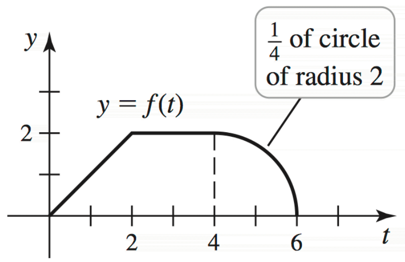


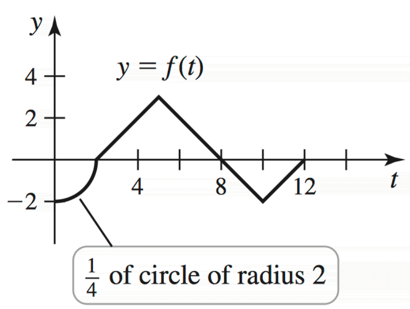
89.



90. **Area functions from graphs** The graph of f is given in the figure. Let $A(x) = \int_0^x f(t)\,dt$ and evaluate $A(1)$, $A(2)$, $A(4)$, and $A(6)$.



91. **Area functions from graphs** The graph of f is given in the figure. Let $A(x) = \int_0^x f(t)\,dt$ and evaluate $A(2)$, $A(5)$, $A(8)$, and $A(12)$.



92–96. Working with area functions *Consider the function f and the points a, b, and c.*

- ***a.*** *Find the area function $A(x) = \int_a^x f(t)\,dt$ using the Fundamental Theorem.*
- ***b.*** *Graph f and A.*
- ***c.*** *Evaluate $A(b)$ and $A(c)$. Interpret the results using the graphs of part (b).*

92. $f(x) = \sin x$; $a = 0$, $b = \frac{\pi}{2}$, $c = \pi$

93. $f(x) = \cos \pi x$; $a = 0$, $b = \frac{1}{2}$, $c = 1$

94. $f(x) = -12\,x\,(x-1)\,(x-2)$; $a = 0$, $b = 1$, $c = 2$

95. $f(x) = \cos x;\ a = 0,\ b = \dfrac{\pi}{2},\ c = \pi$

96. $f(x) = \dfrac{1}{x^2};\ a = 1,\ b = 2,\ c = 4$

97. Find the critical points of the function

$$f(x) = \int_{-1}^{x} t^2 (t-3)(t-4)\,dt,$$

and determine the intervals on which f is increasing and decreasing.

98. Determine the intervals on which the function $g(x) = \displaystyle\int_{x}^{0} \frac{t}{t^2+1}\,dt$ is concave up or concave down.

T **99–100. Functions defined by integrals** *Consider the function g, which is given in terms of a definite integral with a variable upper limit.*

- ***a.*** *Graph the integrand.*
- ***b.*** *Calculate $g'(x)$.*
- ***c.*** *Graph g, showing all your work and reasoning.*

99. $g(x) = \displaystyle\int_{0}^{x} \sin^2 t\,dt$

100. $g(x) = \displaystyle\int_{0}^{x} \sin\left(\pi t^2\right)dt$ (a Fresnel integral)

T **101–104. Areas of regions** *Find the area of the region R bounded by the graph of f and the x-axis on the given interval. Graph f and show the region R.*

101. $f(x) = 2 - |x|$ on $[-2, 4]$

102. $f(x) = 16 - x^4$ on $[-2, 2]$

103. $f(x) = x^4 - 4$ on $[1, 4]$

104. $f(x) = x^2(x-2)$ on $[-1, 3]$

105. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

- **a.** Suppose f is a positive decreasing function, for $x > 0$. Then the area function $A(x) = \displaystyle\int_{0}^{x} f(t)\,dt$ is an increasing function of x.
- **b.** Suppose f is a negative increasing function, for $x > 0$. Then the area function $A(x) = \displaystyle\int_{0}^{x} f(t)\,dt$ is a decreasing function of x.
- **c.** The functions $p(x) = \sin 3x$ and $q(x) = 4\sin 3x$ are antiderivatives of the same function.
- **d.** If $A(x) = 3x^2 - x - 3$ is an area function for f, then $B(x) = 3x^2 - x$ is also an area function for f.

Explorations and Challenges »

106. Evaluate $\lim_{x \to 2} \frac{\int_2^x \sqrt{t^2 + t + 3}\, dt}{x^2 - 4}$.

107. Maximum net area What value of $b > -1$ maximizes the integral $\int_{-1}^{b} x^2(3 - x)\, dx$?

108. Maximum net area Graph the function $f(x) = 8 + 2x - x^2$ and determine the values of a and b that maximize the value of the integral $\int_a^b f(x)\, dx$.

109. Zero net area Consider the function $f(x) = x^2 - 4x$.

a. Graph f on the interval $x \geq 0$.

b. For what value of $b > 0$ is $\int_0^b f(x)\, dx = 0$?

c. In general, for the function $f(x) = x^2 - ax$, where $a > 0$, for what value of $b > 0$ (as a function of a) is $\int_0^b f(x)\, dx = 0$?

110. Cubic zero net area Consider the graph of the cubic $y = x(x - a)(x - b)$, where $0 < a < b$. Verify that the graph bounds a region above the x-axis, for $0 < x < a$, and bounds a region below the x-axis, for $a < x < b$. What is the relationship between a and b if the areas of these two regions are equal?

111. An integral equation Use the Fundamental Theorem of Calculus, Part 1, to find the function f that satisfies the equation

$$\int_0^x f(t)\, dt = 2\cos x + 3x - 2.$$

Verify the result by substitution into the equation.

112. Max/min of area functions Suppose f is continuous on $[0, \infty)$ and $A(x)$ is the net area of the region bounded by the graph of f and the t-axis on $[0, x]$. Show that the local maxima and minima of A occur at the zeros of f. Verify this fact with the function $f(x) = x^2 - 10x$.

T **113. Asymptote of sine integral** Use a calculator to approximate

$$\lim_{x \to \infty} S(x) = \lim_{x \to \infty} \int_0^x \frac{\sin t}{t}\, dt,$$

where S is the sine integral function (see Example 7). Explain your reasoning.

114. Sine integral Show that the sine integral $S(x) = \int_0^x \frac{\sin t}{t}\, dt$ satisfies the (differential) equation $x S'(x) + 2S''(x) + x S'''(x) = 0$.

115. Fresnel integral Show that the Fresnel integral $S(x) = \int_0^x \sin t^2\, dt$ satisfies the (differential) equation $(S'(x))^2 + \left(\frac{S''(x)}{2x}\right)^2 = 1$.

116. Continuity at the endpoints Assume f is continuous on $[a, b]$ and let A be the area function for f with left endpoint a. Let m^* and M^* be the absolute minimum and maximum values of f on $[a, b]$, respectively.

a. Prove that $m^*(x-a) \le A(x) \le M^*(x-a)$, for all x in $[a, b]$. Use this result and the Squeeze Theorem to show that A is continuous from the right at $x = a$.

b. Prove that $m^*(b-x) \le A(b) - A(x) \le M^*(b-x)$, for all x in $[a, b]$. Use this result to show that A is continuous from the left at $x = b$.

117. Discrete version of the Fundamental Theorem In this exercise, we work with a discrete problem and show why the relationship $\int_a^b f'(x)\,dx = f(b) - f(a)$ makes sense. Suppose we have a set of equally spaced grid points

$$\{a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b\},$$

where the distance between any two grid points is Δx. Suppose also that at each grid point x_k, a function value $f(x_k)$ is defined, for $k = 0, \ldots, n$.

a. We now replace the integral with a sum and replace the derivative with a difference quotient. Explain why $\int_a^b f'(x)\,dx$ is analogous to $\sum_{k=1}^{n} \underbrace{\frac{f(x_k) - f(x_{k-1})}{\Delta x}}_{\approx f'(x_k)} \Delta x$.

b. Simplify the sum in part (a) and show that it is equal to $f(b) - f(a)$.

c. Explain the correspondence between the integral relationship and the summation relationship.