5.2 Definite Integrals

We introduced Riemann sums in Section 5.1 as a way to approximate the area of a region bounded by a curve $y = f(x)$ and the x-axis on an interval $[a, b]$. In that discussion, we assumed f to be nonnegative on the interval. Our next task is to discover the geometric meaning of Riemann sums when f is negative on some or all of $[a, b]$. Once this matter is settled, we proceed to the main event of this section, which is to define the *definite integral*. With definite integrals, the approximations given by Riemann sums become exact.

Net Area »

How do we interpret Riemann sums when f is negative at some or all points of $[a, b]$? The answer follows directly from the Riemann sum definition.

EXAMPLE 1 Interpreting Riemann Sums

Evaluate and interpret the following Riemann sums for $f(x) = 1 - x^2$ on the interval $[a, b]$ with n equally spaced subintervals.

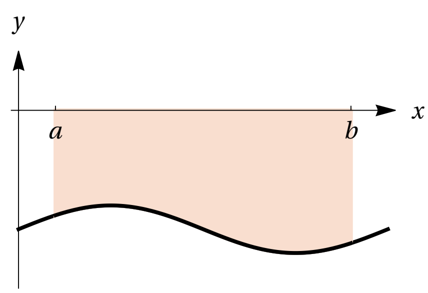
a. A midpoint Riemann sum with $[a, b] = [1, 3]$ and $n = 4$

b. A left Riemann sum with $[a, b] = [0, 3]$ and $n = 6$

SOLUTION »

Let's recap what was learned in Example 1. On intervals where $f(x) < 0$, Riemann sums approximate the *negative* of the area of the region bounded by the curve (**Figure 5.18**).

The Riemann sum $\sum_{k=1}^{n} f(x_k^*)\,\Delta x$ approximates the negative of the area of the region bounded by the x-axis and the curve.



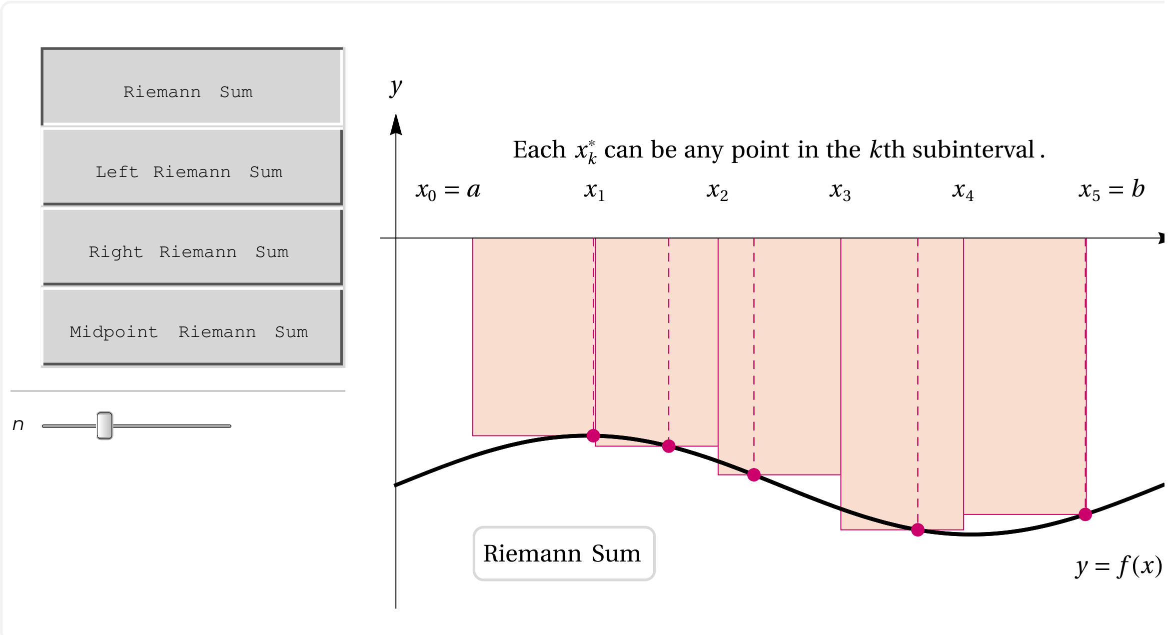


Figure 5.18

In the more general case that f is positive on part of $[a, b]$, we get positive contributions to the sum where f is positive and negative contributions to the sum where f is negative. In this case, Riemann sums approximate the area of the regions that lie above the x-axis *minus* the area of the regions that lie *below* the x-axis (**Figure 5.19**). This difference between the positive and negative contributions is called the *net area*; it can be positive, negative, or zero.

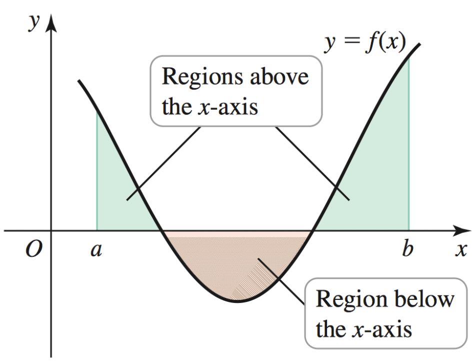


Figure 5.19

Quick Check 1 Suppose $f(x) = -5$. What is the net area of the region bounded by the graph of f and the x-axis on the interval $[1, 5]$? Make a sketch of the function and the region. ◆

Answer »

−20

DEFINITION **Net Area**

Consider the region R bounded by the graph of a continuous function f and the x-axis between $x = a$ and $x = b$. The **net area** of R is the sum of the areas of the parts of R that lie above the x-axis *minus* the area of the parts of R that lie below the x-axis on $[a, b]$.

Note »

Net area suggests the difference between positive and negative contributions much like net change or net profit. Some texts use the term **signed area** for net area.

Quick Check 2 Sketch a function f that is continuous and positive over the interval $[0, 1)$ and negative over the interval $(1, 2]$, such that the net area of the region bounded by the graph of f and the x-axis on $[0, 2]$ is zero. ◆

Answer »

$f(x) = 1 - x$ is one possibility.

The Definite Integral »

Riemann sums for f on $[a, b]$ give *approximations* to the net area of the region bounded by the graph of f and the x-axis between $x = a$ and $x = b$. How can we make these approximations exact? If f is continuous on $[a, b]$, it is reasonable to expect the Riemann sum approximations to approach the exact value of the net area as the number of subintervals $n \to \infty$ and as the length of the subintervals $\Delta x \to 0$ (**Figure 5.20**). In terms of limits, we write

$$\text{net area} = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\,\Delta x.$$

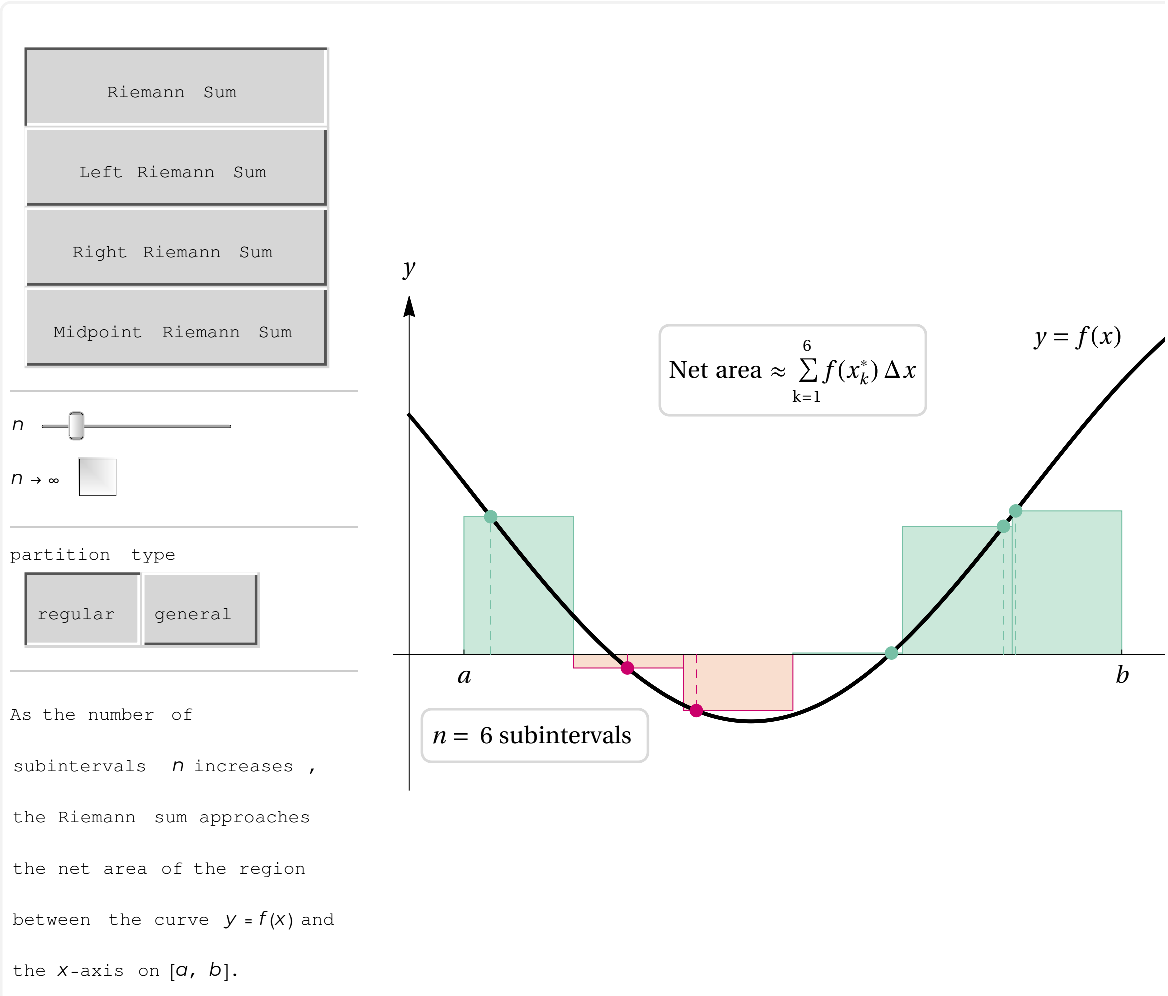


Figure 5.20

The Riemann sums we have used so far involve regular partitions in which the subintervals have the same length Δx. We now introduce partitions of $[a, b]$ in which the lengths of the subintervals are not necessarily equal. A **general partition** of $[a, b]$ consists of the n subintervals

$$[x_0, x_1],\ [x_1, x_2],\ \ldots,\ [x_{n-1}, x_n],$$

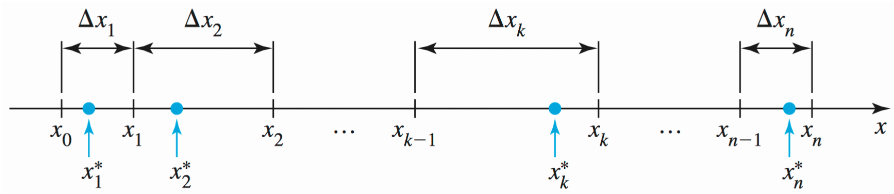
where $x_0 = a$ and $x_n = b$. The length of the kth subinterval is $\Delta x_k = x_k - x_{k-1}$, for $k = 1, \ldots, n$. We let x_k^* be any point in the subinterval $[x_{k-1}, x_k]$. This general partition is used to define the *general Riemann sum*.

DEFINITION **General Riemann Sum**

Suppose $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$ are subintervals of $[a, b]$ with

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

Let Δx_k be the length of the subinterval $[x_{k-1}, x_k]$ and let x_k^* be any point in $[x_{k-1}, x_k]$, for $k = 1, 2, \ldots, n$.



If f is defined on $[a, b]$, the sum

$$\sum_{k=1}^{n} f(x_k^*)\,\Delta x_k = f(x_1^*)\,\Delta x_1 + f(x_2^*)\,\Delta x_2 + \cdots + f(x_n^*)\,\Delta x_n$$

is called a **general Riemann sum for f on $[a, b]$**.

As was the case for regular Riemann sums, if we choose x_k^* to be the left endpoint of $[x_{k-1}, x_k]$, for $k = 1, 2, \ldots, n$, then the general Riemann sum is a left Riemann sum. Similarly, if we choose x_k^* to be the right endpoint of $[x_{k-1}, x_k]$, for $k = 1, 2, \ldots, n$, then the general Riemann sum is a right Riemann sum, and if we choose x_k^* to be the midpoint of the interval $[x_{k-1}, x_k]$, for $k = 1, 2, \ldots, n$, then the general Riemann sum is a midpoint Riemann sum.

Now consider the limit of $\sum_{k=1}^{n} f(x_k^*)\,\Delta x_k$ as $n \to \infty$ and as *all* of the $\Delta x_k \to 0$. We let Δ denote the largest value of Δx_k; that is, $\Delta = \max\{\Delta x_1, \Delta x_2, \ldots, \Delta x_n\}$. Observe that if $\Delta \to 0$, then $\Delta x_k \to 0$ for $k = 1, 2, \ldots, n$. In order for the limit $\lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*)\,\Delta x_k$ to exist, it must have the same value over all general partitions of $[a, b]$ and for all choices of x_k^* on a partition.

Note »

DEFINITION **Definite Integral**

A function f defined on $[a, b]$ is **integrable** on $[a, b]$ if $\lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*)\,\Delta x_k$ exists and is unique over all partitions of $[a, b]$ and all choices of x_k^* on a partition. This limit is the **definite integral of f from a to b**, which we write

$$\int_a^b f(x)\,dx = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*)\,\Delta x_k.$$

When the limit defining the definite integral of f exists, it equals the net area of the region bounded by the graph of f and the x-axis on $[a, b]$. It is imperative to remember that the indefinite integral $\int f(x)\,dx$ is a family of functions of x (the antiderivatives of f) and that the definite integral $\int_a^b f(x)\,dx$ is a real number (the net area of a region).

Notation

The notation for the definite integral requires some explanation. There is a direct match between the notation on either side of the equation in the definition (**Figure 5.21**).

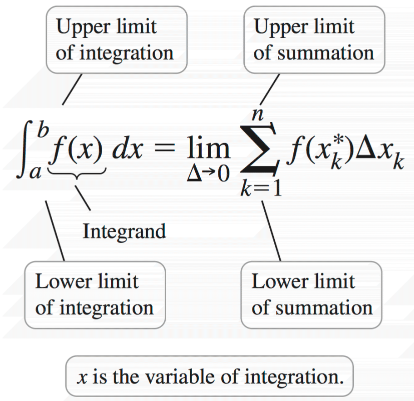


Figure 5.21

In the limit as $\Delta \to 0$, the finite sum, denoted $\sum$, becomes a sum with an infinite number of terms, denoted $\int$. The integral sign $\int$ is an elongated S for sum. In this limit, the lengths of the subintervals Δx_k are replaced by dx. The **limits of integration**, a and b, and the limits of summation also match: The lower limit in the sum, $k = 1$, corresponds to the left endpoint of the interval, $x = a$, and the upper limit in the sum, $k = n$, corresponds to the right endpoint of the interval, $x = b$. The function under the integral sign is called the **integrand**. Finally, the differential dx in the integral (which corresponds to Δx_k in the sum) is an essential part of the notation; it tells us that the **variable of integration** is x.

Note »

The variable of integration is a dummy variable that is completely internal to the integral. It does not matter what the variable of integration is called, as long as it does not conflict with other variables that are in use. Therefore, the integrals in **Figure 5.22** all have the same meaning.

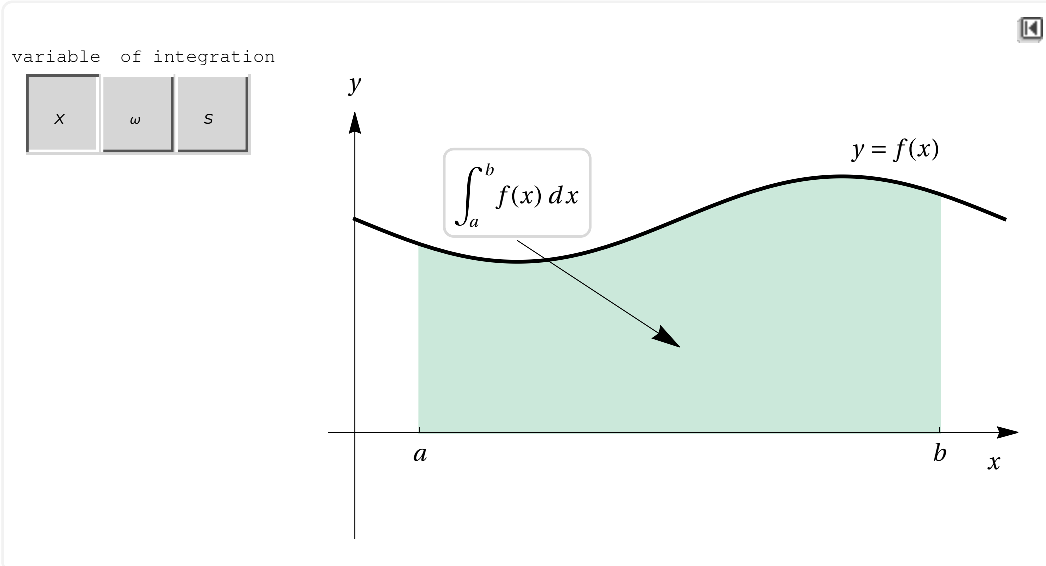


Figure 5.22

The strategy of slicing a region into smaller parts, summing the results from the parts, and taking a limit is used repeatedly in calculus and its applications. We call this strategy the **slice-and-sum method**. It often results in a Riemann sum whose limit is a definite integral.

Evaluating Definite Integrals »

Most of the functions encountered in this text are integrable on some interval (see Exercise 91 for an exception). In fact, if f is continuous on $[a, b]$ or if f is bounded on $[a, b]$ with a finite number of discontinuities, then f is integrable on $[a, b]$. The proof of this result goes beyond the scope of this text.

Note »

> A function f is bounded on an interval I if there is a number M such that $|f(x)| < M$ for all x in I.

THEOREM 5.2 Integrable Functions

If f is continuous on $[a, b]$ or bounded on $[a, b]$ with a finite number of discontinuities, then f is integrable on $[a, b]$.

When f is continuous on $[a, b]$, we have seen that the definite integral $\int_a^b f(x)\,dx$ is the net area of the region bounded by the graph of f and the x-axis on $[a, b]$. **Figure 5.23** illustrates how the idea of net area carries over to piecewise continuous functions (Exercises 84–88ß).

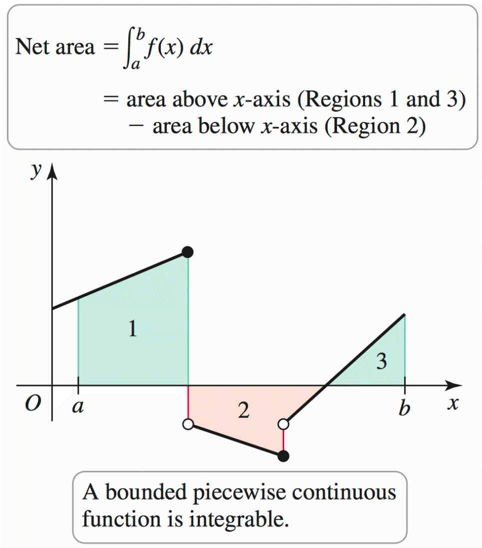


Figure 5.23

Quick Check 3 Graph $f(x) = x$ and use geometry to evaluate $\int_{-1}^{1} x\,dx$. ◆

Answer »

0

EXAMPLE 2 Identifying the limit of a sum

Assume

$$\lim_{\Delta \to 0} \sum_{k=1}^{n} \left(3\,x_k^{*2} + 2\,x_k^{*} + 1\right) \Delta x_k$$

is the limit of a Riemann sum for a function f on [1, 3]. Identify the function f and express the limit as a definite integral. What does the definite integral represent geometrically?

SOLUTION »

By comparing the sum $\sum_{k=1}^{n} \left(3\,x_k^{*2} + 2\,x_k^{*} + 1\right) \Delta x_k$ to the general Riemann sum $\sum_{k=1}^{n} f(x_k^{*})\,\Delta x_k$, we see that $f(x) = 3\,x^2 + 2\,x + 1$. Because f is a polynomial, it is continuous on [1, 3] and is, therefore, integrable on [1, 3]. It follows that

$$\lim_{\Delta \to 0} \sum_{k=1}^{n} \left(3\,x_k^{*2} + 2\,x_k^{*} + 1\right) \Delta x_k = \int_1^3 \left(3\,x^2 + 2\,x + 1\right) dx.$$

Because f is positive on [1, 3], the definite integral $\int_1^3 \left(3\,x^2 + 2\,x + 1\right) dx$ is the area of the region bounded by the curve $y = 3\,x^2 + 2\,x + 1$ and the x-axis on [1, 3] (**Figure 5.24**).

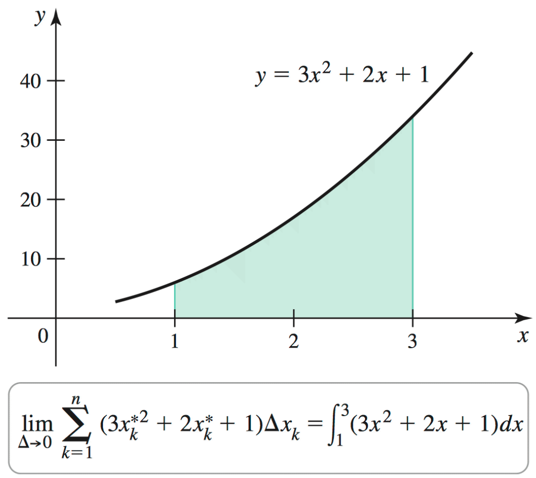


Figure 5.24

Related Exercises 33–34 ◆

EXAMPLE 3 Evaluating definite integrals using geometry

Use familiar area formulas to evaluate the following definite integrals.

a. $\int_2^4 (2x + 3)\, dx$

b. $\int_1^6 (2x - 6)\, dx$

c. $\int_3^4 \sqrt{1 - (x - 3)^2}\, dx$

SOLUTION »

To evaluate these definite integrals geometrically, a sketch of the corresponding region is essential.

a. The definite integral $\int_2^4 (2x + 3)\, dx$ is the area of the trapezoid bounded by the x-axis and the line $y = 2x + 3$ from $x = 2$ to $x = 4$ (**Figure 5.25**). The width of its base is 2 and the lengths of its two parallel sides are $f(2) = 7$ and $f(4) = 11$. Using the area formula for a trapezoid we have

$$\int_2^4 (2x + 3)\, dx = \frac{1}{2} \cdot 2\,(11 + 7) = 18.$$

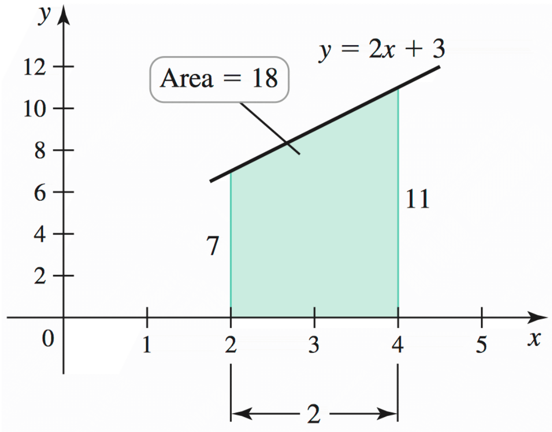


Figure 5.25

Note »

b. A sketch shows that the regions bounded by the line $y = 2x - 6$ and the x-axis are triangles (**Figure 5.26**). The area of the triangle on the interval [1, 3] is $\frac{1}{2} \cdot 2 \cdot 4 = 4$. Similarly, the area of the triangle on [3, 6] is $\frac{1}{2} \cdot 3 \cdot 6 = 9$. The definite integral is the net area of the entire region, which is the area of the triangle above the x-axis minus the area of the triangle below the x-axis:

$$\int_1^6 (2x - 6)\, dx = \text{net area} = 9 - 4 = 5.$$

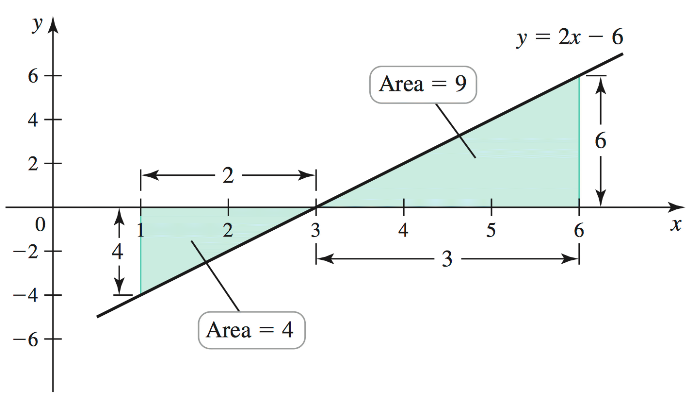


Figure 5.26

c. We first let $y = \sqrt{1 - (x-3)^2}$ and observe that $y \geq 0$ when $2 \leq x \leq 4$. Squaring both sides leads to the equation $(x-3)^2 + y^2 = 1$, whose graph is a circle of radius 1 centered at (3, 0). Because $y \geq 0$, the graph of $y = \sqrt{1 - (x-3)^2}$ is the upper half of the circle. It follows that the integral $\int_3^4 \sqrt{1 - (x-3)^2}\, dx$ is the area of a quarter circle of radius 1 (**Figure 5.27**). Therefore,

$$\int_3^4 \sqrt{1 - (x-3)^2}\, dx = \frac{1}{4} \pi (1)^2 = \frac{\pi}{4}.$$

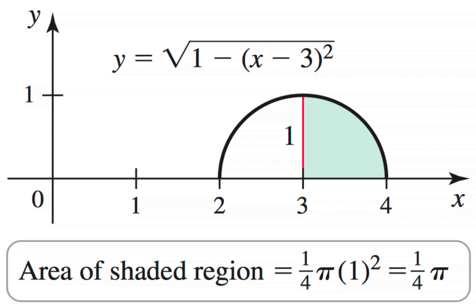


Figure 5.27

Related Exercises 37–38, 41 ◆

Quick Check 4 Let $f(x) = 5$ and use geometry to evaluate $\int_1^3 f(x)\,dx$. What is the value of $\int_a^b c\,dx$, where c is a real number? ◆

Answer »

10; $c(b - a)$

EXAMPLE 4 Definite integrals from graphs

Figure 5.28 shows the graph of a function f with the areas of the regions bounded by its graph and the x-axis given. Find the values of the following definite integrals.

a. $\int_a^b f(x)\,dx$

b. $\int_b^c f(x)\,dx$

c. $\int_a^c f(x)\,dx$

d. $\int_b^d f(x)\,dx$

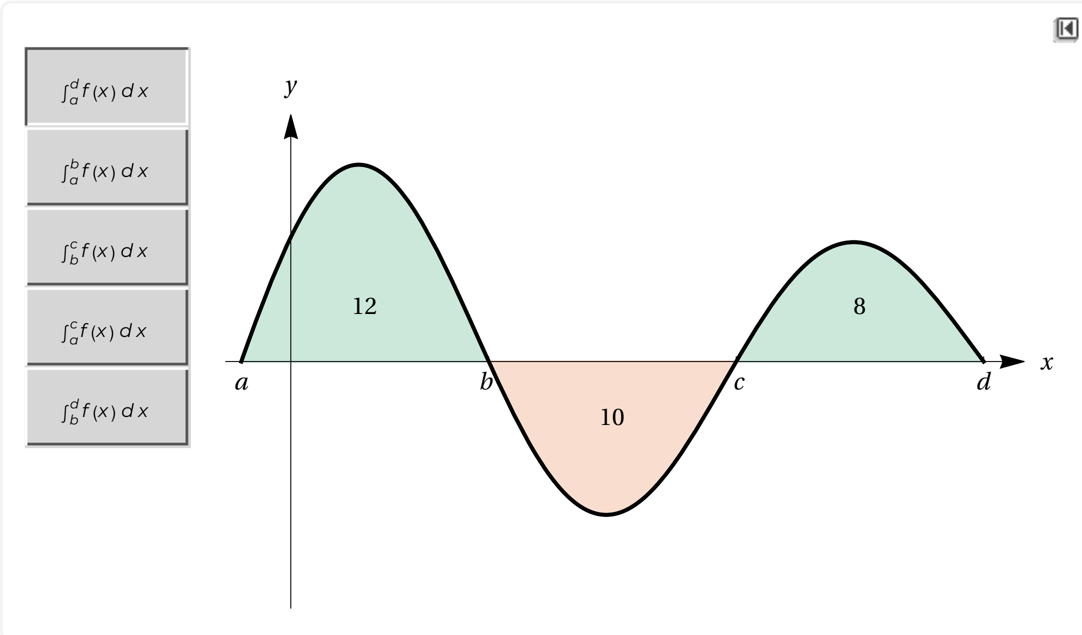


Figure 5.28

SOLUTION »

a. Because $f(x) \ge 0$ on $[a, b]$, the value of the definite integral is the area of the region between the graph and the x-axis on $[a, b]$; that is, $\int_a^b f(x)\,dx = 12$.

b. Because $f(x) \le 0$ on $[b, c]$, the value of the definite integral is the negative of the area of the corresponding region; that is, $\int_b^c f(x)\,dx = -10$.

c. The value of the definite integral is the area of the region on $[a, b]$ (where $f(x) \ge 0$) minus the area of the region on $[b, c]$ (where $f(x) \le 0$). Therefore, $\int_a^c f(x)\,dx = 12 - 10 = 2$.

d. Reasoning as in part (c), we have $\int_b^d f(x)\,dx = -10 + 8 = -2$.

Related Exercises 57–60 ◆

Properties of Definite Integrals »

Recall that the definite integral $\int_a^b f(x)\,dx$ was defined assuming $a < b$. There are, however, occasions when it is necessary to reverse the limits of integration. If f is integrable on $[a, b]$, we define

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

In other words, reversing the limits of integration changes the sign of the integral.

Another fundamental property of integrals is that if we integrate from a point to itself, then the length of the interval of integration is zero, which means the definite integral is also zero.

DEFINITION **Reversing Limits and Identical Limits**

Suppose f is integrable on $[a, b]$.

1. $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

2. $\int_a^a f(x)\,dx = 0$

Quick Check 5 Evaluate $\int_a^b f(x)\,dx + \int_b^a f(x)\,dx$ assuming that f is integrable on $[a, b]$. ◆

Answer »

0

Integral of a Sum

Definite integrals possess other properties that often simplify their evaluation. Assume f and g are integrable on $[a, b]$. The first property states that their sum $f + g$ is integrable on $[a, b]$ and the integral of their sum is the sum of their integrals:

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

We prove this property assuming f and g are continuous. In this case, $f + g$ is continuous and, therefore integrable. We then have

$$\begin{aligned}
\int_a^b (f(x) + g(x))\,dx &= \lim_{\Delta \to 0} \sum_{k=1}^{n} [f(x_k^*) + g(x_k^*)]\,\Delta x_k && \text{Definition of definite integral} \\
&= \lim_{\Delta \to 0} \left[\sum_{k=1}^{n} f(x_k^*)\,\Delta x_k + \sum_{k=1}^{n} g(x_k^*)\,\Delta x_k\right] && \text{Addition rule for finite sums} \\
&= \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*)\,\Delta x_k + \lim_{\Delta \to 0} \sum_{k=1}^{n} g(x_k^*)\,\Delta x_k && \text{Limit of a sum} \\
&= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx. && \text{Definition of definite integral}
\end{aligned}$$

Constants in Integrals

Another property of definite integrals is that constants can be factored out of definite integrals. If f is integrable on $[a, b]$ and c is a constant, then $c\,f$ is integrable on $[a, b]$ and

$$\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx.$$

The justification (Exercise 89) is based on the fact that for finite sums,

$$\sum_{k=1}^{n} c\, f(x_k^*)\, \Delta x_k = c \sum_{k=1}^{n} f(x_k^*)\, \Delta x_k.$$

Integrals over Subintervals

If the point p lies between a and b, then the integral on $[a, b]$ may be split into two integrals. As shown in **Figure 5.29**, we have the property

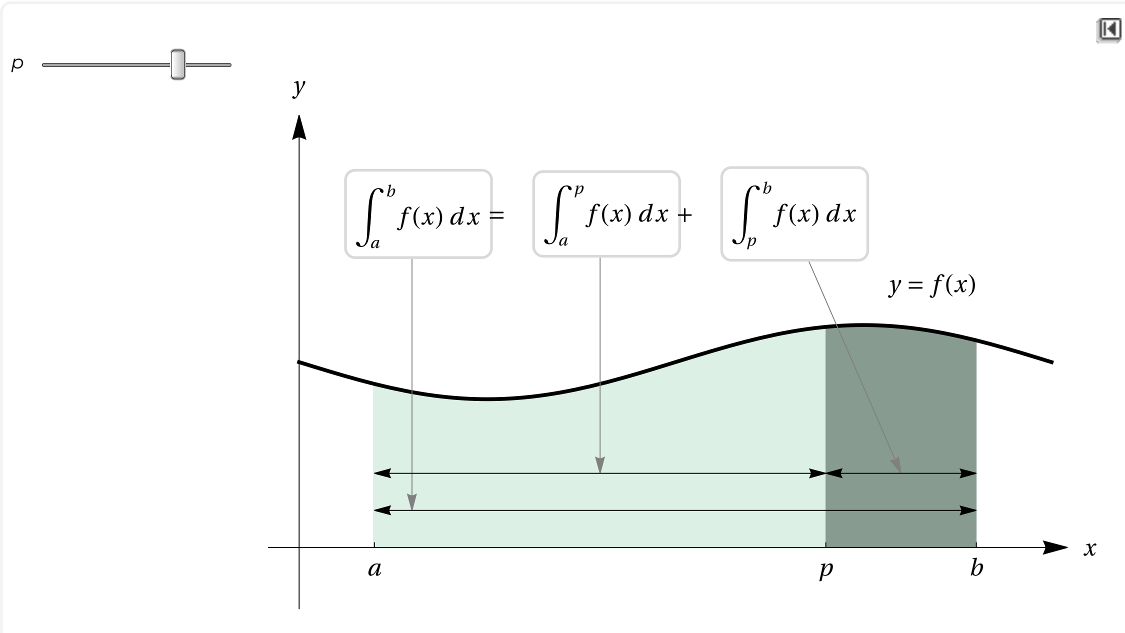


Figure 5.29

It is surprising that this same property also holds when p lies outside the interval $[a, b]$. For example, if $a < b < p$ and f is integrable on $[a, p]$, then it follows (**Figure 5.30**) that

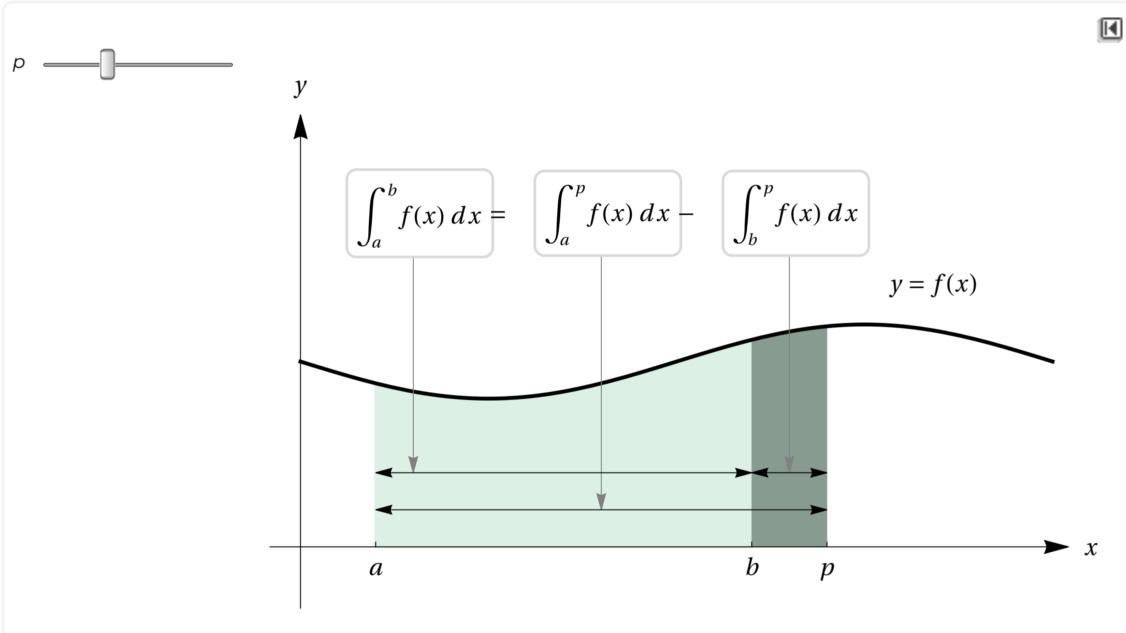


Figure 5.30

Because $\int_p^b f(x)\,dx = -\int_b^p f(x)\,dx$, we have the original property

$$\int_a^b f(x)\,dx = \int_a^p f(x)\,dx + \int_p^b f(x)\,dx.$$

Integrals of Absolute Values

Finally, how do we interpret $\int_a^b |f(x)|\,dx$, the integral of the absolute value of a function? The graphs f and $|f|$ are shown in **Figure 5.31** .

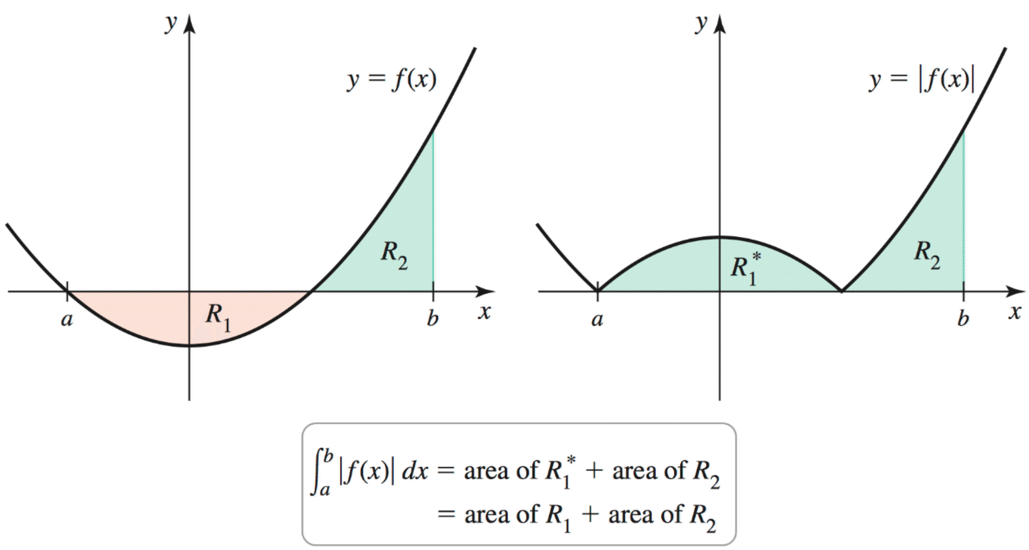


Figure 5.31

The integral $\int_a^b |f(x)|\,dx$ gives the area of regions R_1^* and R_2. But R_1 and R_1^* have the same area; therefore, $\int_a^b |f(x)|\,dx$ also gives the area of R_1 and R_2. The conclusion is that $\int_a^b |f(x)|\,dx$ is the area of the entire region (above and below the x-axis) that lies between the graph of f and the x-axis on $[a, b]$.

All these properties will be used frequently in upcoming work. It's worth collecting them in one table (Table 5.4).

Table 5.4 Properties of definite integrals

Let f and g be integrable functions on an interval that contains a, b, and p.

1. $\int_a^a f(x)\,dx = 0$ Definition
2. $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ Definition
3. $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
4. $\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$, for any constant c
5. $\int_a^b f(x)\,dx = \int_a^p f(x)\,dx + \int_p^b f(x)\,dx$
6. The function $|f|$ is integrable on $[a, b]$, and $\int_a^b |f(x)|\,dx$ is the sum of the areas of the regions bounded by the graph of f and the x-axis on $[a, b]$.

EXAMPLE 5 Properties of integrals

Assume $\int_0^5 f(x)\,dx = 3$ and $\int_0^7 f(x)\,dx = -10$. Evaluate the following integrals, if possible.

a. $\int_0^7 2\,f(x)\,dx$

b. $\int_5^7 f(x)\,dx$

c. $\int_5^0 f(x)\,dx$

d. $\int_7^0 6\,f(x)\,dx$

e. $\int_0^7 |f(x)|\,dx$

SOLUTION »

a. By Property 4 of Table 5.4,

$$\int_0^7 2\,f(x)\,dx = 2\int_0^7 f(x)\,dx = 2\cdot(-10) = -20.$$

b. By Property 5 of Table 5.4, $\int_0^7 f(x)\,dx = \int_0^5 f(x)\,dx + \int_5^7 f(x)\,dx$. Therefore,

$$\int_5^7 f(x)\,dx = \int_0^7 f(x)\,dx - \int_0^5 f(x)\,dx = -10 - 3 = -13.$$

c. By Property 2 of Table 5.4,

$$\int_5^0 f(x)\,dx = -\int_0^5 f(x)\,dx = -3.$$

d. Using Properties 2 and 4 of Table 5.4, we have

$$\int_7^0 6\,f(x)\,dx = -\int_0^7 6\,f(x)\,dx = -6\int_0^7 f(x)\,dx = (-6)(-10) = 60.$$

e. This integral cannot be evaluated without knowing the intervals on which f is positive and negative. It could have any value greater than or equal to 10.

Related Exercises 49–50 ◆

Quick Check 6 Evaluate $\int_{-1}^2 x\,dx$ and $\int_{-1}^2 |x|\,dx$ using geometry. ◆

Answer »

$\frac{3}{2}$; $\frac{5}{2}$

Bounds on Definite Integrals

We conclude our discussion of properties of definite integrals with three results that are helpful for upcoming theoretical work. We assume f and g are continuous on $[a, b]$, where $b > a$.

Nonnegative Integrand

If $f(x) \geq 0$ on $[a, b]$, it is geometrically apparent that $\int_a^b f(x)\,dx \geq 0$ (**Figure 5.32**).

$\int_a^b f(x)\,dx$ = net area under the curve $y = f(x)$;
when $f(x) \geq 0$, net area ≥ 0.

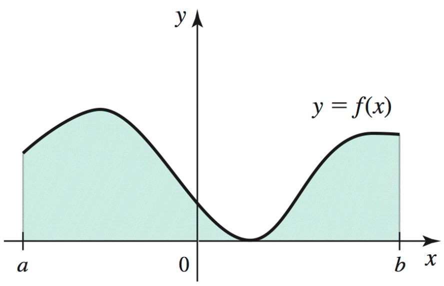


Figure 5.32

To prove this result, suppose m is the absolute minimum value of f on $[a, b]$, guaranteed to exist by Theorem 4.1 (note that $m \geq 0$). Working with a general partition of $[a, b]$, observe that

$$\begin{aligned} 0 \leq m(b-a) &= m\sum_{k=1}^{n} \Delta x_k && \sum_{k=1}^{n} \Delta x_k = (b-a) \\ &= \sum_{k=1}^{n} m\,\Delta x_k && \text{Property of finite sums, Section 5.1} \\ &\leq \underbrace{\sum_{k=1}^{n} f(x_k^*)\,\Delta x_k}_{\text{General Riemann sum for } f \text{ on } [a,b].} && m \leq f(x_k^*) \text{ for all } x_k^* \text{ in } [a, b] \end{aligned}$$

We have shown that for any general partition of $[a, b]$ and for every choice of x_k^*, the general Riemann sum for f is nonnegative. Taking the limit as $\Delta \to 0$, where $\Delta = \max\{\Delta x_1, \Delta x_2, \ldots, \Delta x_n\}$, we have

$$\int_a^b f(x)\,dx \geq 0.$$

Comparing Definite Integrals

A related property of definite integrals says that if $f(x) \geq g(x)$ on $[a, b]$ (**Figure 5.33**), then

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx.$$

This property follows from the previous result and Property 3 of Table 5.4 (Exercise 94).

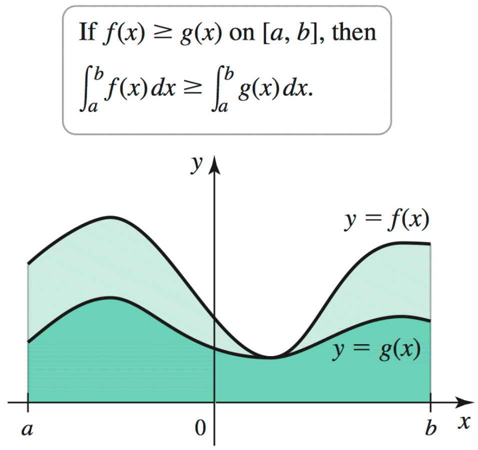


Figure 5.33

Lower and Upper Bounds

Because f is continuous on $[a, b]$, it attains an absolute minimum value m and an absolute maximum value M on $[a, b]$ (Theorem 4.1). Our final property (**Figure 5.34**) claims that

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$

To prove this result, once again working with a general partition of $[a, b]$, notice that

$$\begin{aligned}
m(b-a) &= m\sum_{k=1}^{n}\Delta x_k && \sum_{k=1}^{n}\Delta x_k = b-a\\
&= \sum_{k=1}^{n} m\,\Delta x_k && \text{Property of finite sums}\\
&\leq \sum_{k=1}^{n} f(x_k^*)\,\Delta x_k && m \leq f(x_k^*) \text{ for all } x_k^* \text{ in } [a, b]\\
&\leq \sum_{k=1}^{n} M\,\Delta x_k && f(x_k^*) \leq M \text{ for all } x_k^* \text{ in } [a, b]\\
&= M\sum_{k=1}^{n}\Delta x_k && \text{Property of finite sums}\\
&= M(b-a). && \sum_{k=1}^{n}\Delta x_k = b-a
\end{aligned}$$

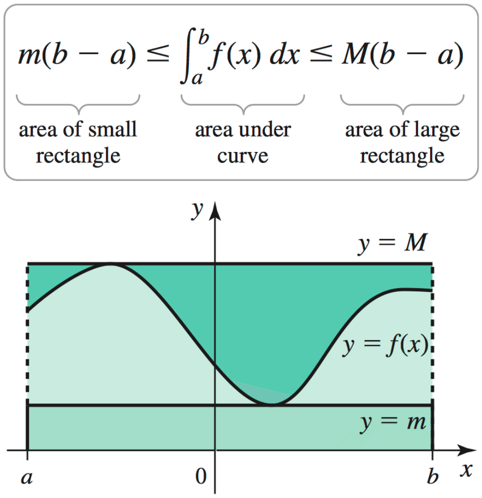


Figure 5.34

We have shown that $m\,(b-a) \le \sum_{k=1}^{n} f(x_k^*)\,\Delta x_k \le M\,(b-a)$ for any partition of $[a,\ b]$ and for every choice of x_k^*. Letting $\Delta \to 0$, we obtain

$$m\,(b-a) = \int_a^b f(x)\,dx < M\,(b-a).$$

Table 5.5 summarizes our findings.

Table 5.5 Additional properties of definite integrals

Let f and g be integrable functions on $[a,\ b]$, where $b > a$.

7. If $m \le f(x) \le M$, then $m\,(b-a) \le \int_a^b f(x)\,dx \le M\,(b-a)$.

8. If $f(x) \ge g(x)$ on $[a,\ b]$, then $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$.

9. If $m \le f(x) \le M$, then $m\,(b-a) \le \int_a^b f(x)\,dx \le M\,(b-a)$.

Note »

Although we proved Properties 3, 4, 7, 8, and 9 of Tables 5.4 and 5.5 for continuous functions f and g, these properties hold when f and g are integrable

In the next section, Property 9 is used to prove the central result of this chapter, the Fundamental Theorem of Calculus.

Evaluating Definite Integrals Using Limits »

In Example 3 we used area formulas for trapezoids, triangles, and circles to evaluate definite integrals. Regions bounded by more general functions have curved boundaries for which conventional geometrical methods do not work. At the moment the only way to handle such integrals is to appeal to the definition of the definite integral and the summation formulas given in Theorem 5.1.

We know that if f is integrable on $[a,\ b]$, then $\int_a^b f(x)\,dx = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*)\,\Delta x_k$ for any partition of $[a,\ b]$

and any points x_k^*. To simplify these calculations, we use equally spaced grid points and right Riemann sums. That is, for any value of n we let $\Delta x_k = \Delta x = \dfrac{b-a}{n}$ and $x_k^* = a + k\,\Delta x$, for $k = 1, 2, \ldots, n$. Then, as $n \to \infty$ and $\Delta \to 0$,

$$\int_a^b f(x)\,dx = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*)\,\Delta x_k = \lim_{n \to \infty} \sum_{k=1}^{n} f(a + k\,\Delta x)\,\Delta x.$$

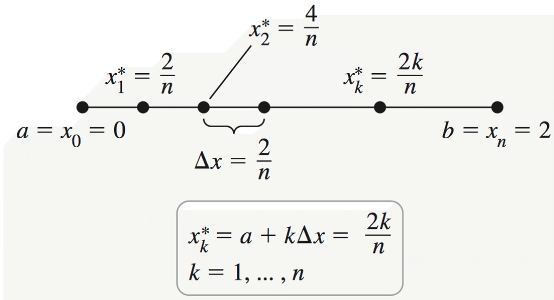
EXAMPLE 6 **Evaluating definite integrals**

Find the value of $\int_0^2 \left(x^3 + 1\right) dx$ by evaluating a right Riemann sum and letting $n \to \infty$.

SOLUTION »

Based on approximations found in Example 5, Section 5.1, we conjectured that the value of this integral is 6. To verify this conjecture, we now evaluate the integral exactly. The interval $[a, b] = [0, 2]$ is divided into n subintervals of length $\Delta x = \dfrac{b-a}{n} = \dfrac{2}{n}$, which produces the grid points

$$x_k^* = a + k\,\Delta x = 0 + k \cdot \frac{2}{n} = \frac{2\,k}{n}, \text{ for } k = 1, 2, \ldots, n.$$



Letting $f(x) = x^3 + 1$, the right Riemann sum is

$$\begin{aligned}
\sum_{k=1}^{n} f(x_k^*)\,\Delta x_k &= \sum_{k=1}^{n} \left(\left(\frac{2\,k}{n}\right)^3 + 1\right) \frac{2}{n} \\
&= \frac{2}{n} \sum_{k=1}^{n} \left(\frac{8\,k^3}{n^3} + 1\right) && \sum_{k=1}^{n} c\,a_k = c \sum_{k=1}^{n} a_k \\
&= \frac{2}{n} \left(\frac{8}{n^3} \sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} 1\right) && \sum_{k=1}^{n} (c\,a_k + b_k) = c \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k \\
&= \frac{2}{n} \left(\frac{8}{n^3} \left(\frac{n^2 (n+1)^2}{4}\right) + n\right) && \sum_{k=1}^{n} k^3 = \frac{n^2 (n+1)^2}{4} \text{ and } \sum_{k=1}^{n} 1 = n; \text{ Theorem 5.1} \\
&= \frac{4\left(n^2 + 2\,n + 1\right)}{n^2} + 2. && \text{Simplify.}
\end{aligned}$$

Note »

Now we evaluate $\int_0^2 (x^3 + 1)\, dx$ by letting $n \to \infty$ in the Riemann sum:

$$\begin{aligned}\int_0^2 (x^3+1)\,dx &= \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\,\Delta x \\ &= \lim_{n\to\infty} \left(\frac{4\left(n^2 + 2n + 1\right)}{n^2} + 2 \right) \\ &= 4 \underbrace{\lim_{n\to\infty} \left(\frac{n^2+2n+1}{n^2} \right)}_{1} + \lim_{n\to\infty} 2 \\ &= 4(1) + 2 = 6.\end{aligned}$$

Therefore, $\int_0^2 (x^3 + 1)\, dx = 6$, confirming our conjecture in Example 5, Section 5.1.

Related Exercises 78, 80 ◆

The Riemann sum calculations in Example 6 are tedious even if f is a simple function. For polynomials of degree 4 and higher, the calculations are much more challenging, and for rational and transcendental functions, advanced mathematical results are needed. The next section introduces more efficient methods for evaluating definite integrals.

Exercises »

Getting Started »

Practice Exercises »

17–20. Approximating net area *The following functions are negative on the given interval.*

- ***a.*** *Sketch the function on the interval.*
- ***b.*** *Approximate the net area bounded by the graph of f and the x-axis on the interval using a left, right, and midpoint Riemann sum with $n = 4$.*

17. $f(x) = -2x - 1$ on $[0, 4]$

T **18.** $f(x) = -4 - x^3$ on $[3, 7]$

T **19.** $f(x) = \sin 2x$ on $\left[\frac{\pi}{2}, \pi\right]$

T **20.** $f(x) = x^3 - 1$ on $[-2, 0]$

T **21–24. Approximating net area** *The following functions are positive and negative on the given interval.*

- ***a.*** *Sketch the function on the interval.*
- ***b.*** *Approximate the net area bounded by the graph of f and the x-axis on the interval using a left, right, and midpoint Riemann sum with $n = 4$.*
- ***c.*** *Use the sketch in part (a) to show which intervals of $[a, b]$ make positive and negative contributions to the net area.*

21. $f(x) = 4 - 2x$ on $[0, 4]$

22. $f(x) = 8 - 2x^2$ on $[0, 4]$

23. $f(x) = \sin 2x$ on $\left[0, \frac{3\pi}{4}\right]$

24. $f(x) = x^3$ on $[-1, 2]$

25–28. Area versus net area *Graph the following functions. Then use geometry (not Riemann sums) to find the area and the net area of the region described.*

25. The region between the graph of $y = -3x$ and the x-axis, for $-2 \le x \le 2$

26. The region between the graph of $y = 4x - 8$ and the x-axis, for $-4 \le x \le 8$

27. The region between the graph of $y = 1 - |x|$ and the x-axis, for $-2 \le x \le 2$

28. The region between the graph of $y = 3x - 6$ and the x-axis, for $0 \le x \le 6$

T **29–32. Approximating definite integrals** *Complete the following steps for the given integral and the given value of n.*

- ***a.*** *Sketch the graph of the integrand on the interval of integration.*
- ***b.*** *Calculate Δx and the grid points $x_0, x_1, \ldots, x_n$, assuming a regular partition.*
- ***c.*** *Calculate the left and right Riemann sums for the given value of n.*
- ***d.*** *Determine which Riemann sum (left or right) underestimates the value of the definite integral and which overestimates the value of the definite integral.*

29. $\int_3^6 (1 - 2x)\, dx;\ n = 6$

30. $\int_0^2 \left(x^2 - 2\right) dx;\ n = 4$

31. $\int_1^7 \frac{1}{x}\, dx;\ n = 6$

32. $\int_0^{\pi/2} \cos x\, dx;\ n = 4$

33–36. Identifying definite integrals as limits of sums *Consider the following limits of Riemann sums for a function f on $[a, b]$. Identify f and express the limit as a definite integral.*

33. $\lim_{\Delta \to 0} \sum_{k=1}^{n} \left(x_k^{*2} + 1\right) \Delta x_k$ on $[0, 2]$

34. $\lim_{\Delta \to 0} \sum_{k=1}^{n} \left(4 - x_k^{*2}\right) \Delta x_k$ on $[-2, 2]$

35. $\lim_{\Delta \to 0} \sum_{k=1}^{n} x_k^* (\cos x_k^*)\, \Delta x_k$ on $[1, 2]$

36. $\lim_{\Delta \to 0} \sum_{k=1}^{n} \left| x_k^{*2} - 1 \right| \Delta x_k$ on $[-2, 2]$

37–44. Definite integrals *Use geometry (not Riemann sums) to evaluate the following definite integrals. Sketch a graph of the integrand, show the region in question, and interpret your result.*

37. $\int_0^4 (8 - 2x)\, dx$

38. $\int_{-4}^2 (2x + 4)\, dx$

39. $\int_{-1}^2 (-|x|)\, dx$

40. $\int_0^2 (1 - x)\, dx$

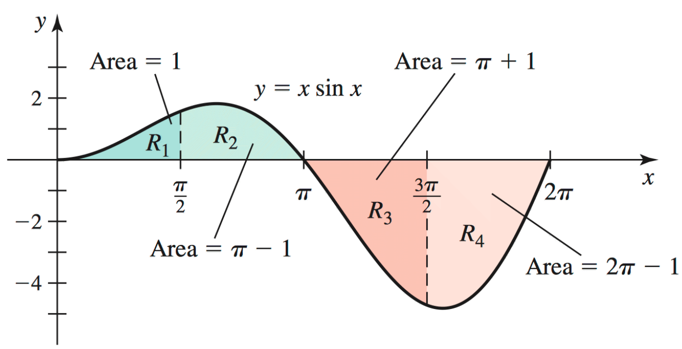
41. $\int_0^4 \sqrt{16 - x^2}\, dx$

42. $\int_{-1}^3 \sqrt{4 - (x-1)^2}\, dx$

43. $\int_0^4 f(x)\, dx$, where $f(x) = \begin{cases} 5 & \text{if } x \le 2 \\ 3x - 1 & \text{if } x > 2 \end{cases}$

44. $\int_1^{10} g(x)\, dx$, where $g(x) = \begin{cases} 4x & \text{if } 0 \le x \le 2 \\ -8x + 16 & \text{if } 2 < x \le 3 \\ -8 & \text{if } x > 3 \end{cases}$

45–48. *The accompanying figure shows four regions bounded by the graph of $y = x \sin x$: R_1, R_2, R_3, and R_4, whose areas are $1, \pi - 1, \pi + 1$, and $2\pi - 1$, respectively. (We verify these results later in the text.) Use this information to evaluate the following integrals.*



45. $\int_0^{\pi} x \sin x\, dx$

46. $\int_0^{3\pi/2} x \sin x\, dx$

47. $\int_0^{2\pi} x \sin x\, dx$

48. $\int_{\pi/2}^{2\pi} x \sin x\, dx$

49. Properties of integrals Use only the fact that $\int_0^4 3x(4-x)\, dx = 32$, and the definitions and properties of integrals, to evaluate the following integrals, if possible.

a. $\int_4^0 3x(4-x)\, dx$

b. $\int_0^4 x(x-4)\, dx$

c. $\int_4^0 6x(4-x)\, dx$

d. $\int_0^8 3x(4-x)\, dx$

50. Properties of integrals Suppose $\int_1^4 f(x)\, dx = 8$ and $\int_1^6 f(x)\, dx = 5$. Evaluate the following integrals.

a. $\int_1^4 (-3f(x))\, dx$

b. $\int_1^4 3f(x)\, dx$

c. $\int_6^4 12f(x)\, dx$

d. $\int_4^6 3f(x)\, dx$

51. Properties of integrals Suppose $\int_0^3 f(x)\, dx = 2$, $\int_3^6 f(x)\, dx = -5$, and $\int_3^6 g(x)\, dx = 1$. Evaluate the following integrals.

a. $\int_0^3 5f(x)\, dx$

b. $\int_3^6 (-3g(x))\, dx$

c. $\int_3^6 (3f(x) - g(x))\, dx$

d. $\int_6^3 (f(x) + 2g(x))\, dx$

52. Properties of integrals Suppose $f(x) \geq 0$ on $[0, 2]$, $f(x) \leq 0$ on $[2, 5]$, $\int_0^2 f(x)\, dx = 6$, and $\int_2^5 f(x)\, dx = -8$. Evaluate the following integrals.

a. $\int_0^5 f(x)\,dx$

b. $\int_0^5 |f(x)|\,dx$

c. $\int_2^5 4\,|f(x)|\,dx$

d. $\int_0^5 (f(x) + |f(x)|)\,dx$

53. Properties of integrals Consider two functions f and g on $[1, 6]$ such that $\int_1^6 f(x)\,dx = 10$, $\int_1^6 g(x)\,dx = 5$, $\int_4^6 f(x)\,dx = 5$, and $\int_1^4 g(x)\,dx = 2$. Evaluate the following integrals.

a. $\int_1^4 3\,f(x)\,dx$

b. $\int_1^6 (f(x) - g(x))\,dx$

c. $\int_1^4 (f(x) - g(x))\,dx$

d. $\int_4^6 (g(x) - f(x))\,dx$

e. $\int_4^6 8\,g(x)\,dx$

f. $\int_4^1 2\,f(x)\,dx$

54. Suppose f is continuous on $[1, 5]$ and $2 \le f(x) \le 3$, for all x in $[1, 5]$. Find lower and upper bounds for $\int_1^5 f(x)\,dx$.

55–56. Using properties of integrals *Use the value of the first integral I to evaluate the two given integrals.*

55. $I = \int_0^1 \left(x^3 - 2\,x\right)dx = -\frac{3}{4}$

a. $\int_0^1 \left(4\,x - 2\,x^3\right)dx$

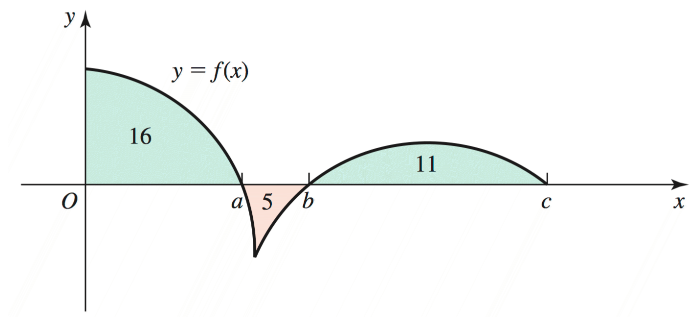
b. $\int_1^0 \left(2\,x - x^3\right)dx$

56. $I = \int_0^{\pi/2} (\cos\theta - 2\sin\theta)\,d\theta = -1$

a. $\int_0^{\pi/2} (2\sin\theta - \cos\theta)\,d\theta$

b. $\int_{\pi/2}^0 (4\cos\theta - 8\sin\theta)\,d\theta$

57–64. Definite integrals from graphs *The figure shows the areas of regions bounded by the graph of f and the x-axis. Evaluate the following integrals.*



57. $\int_0^a f(x)\,dx$

58. $\int_0^b f(x)\,dx$

59. $\int_a^c f(x)\,dx$

60. $\int_0^c f(x)\,dx$

61. $\int_0^c |f(x)|\,dx$

62. $\int_0^c (2\,|f(x)| + 3\,f(x))\,dx$

63. $\int_a^0 f(x)\,dx$

64. $\int_c^0 |f(x)|\,dx$

65. Use geometry and properties of integrals to evaluate $\int_0^1 \left(2\,x + \sqrt{1 - x^2} + 1\right) dx$.

66. Use geometry and properties of integrals to evaluate $\int_1^5 \left(|x - 2| + \sqrt{-x^2 + 6\,x - 5}\right) dx$.

67. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.

a. If f is a constant function on the interval $[a,\, b]$, then the right and left Riemann sums give the exact value of $\int_a^b f(x)\,dx$, for any positive integer n.

b. If f is a linear function on the interval $[a,\, b]$, then a midpoint Riemann sum gives the exact value of $\int_a^b f(x)\,dx$, for any positive integer n.

c. $\int_0^{2\pi/a} \sin ax\, dx = \int_0^{2\pi/a} \cos ax\, dx = 0$ (*Hint:* Graph the functions and use properties of trigonometric functions.)

d. If $\int_a^b f(x)\, dx = \int_b^a f(x)\, dx$, then f is a constant function.

e. Property 4 of Table 5.4 implies that $\int_a^b x\, f(x)\, dx = x \int_a^b f(x)\, dx$.

T **68–70. Approximating definite integrals with a calculator** *Consider the following definite integrals.*

a. *Write the left and right Riemann sums in sigma notation for an arbitrary value of n.*

b. *Evaluate each sum using a calculator with* $n = 20, 50,$ *and* 100*. Use these values to estimate the value of the integral.*

68. $\int_4^9 3\sqrt{x}\, dx$

69. $\int_0^1 \left(x^2 + 1\right) dx$

70. $\int_{-1}^1 \pi \cos\left(\frac{\pi x}{2}\right) dx$

T **71–74. Midpoint Riemann sums with a calculator** *Consider the following definite integrals.*

a. *Write the midpoint Riemann sum in sigma notation for an arbitrary value of n.*

b. *Evaluate each sum using a calculator with* $n = 20, 50,$ *and* 100*. Use these values to estimate the value of the integral.*

71. $\int_1^4 2\sqrt{x}\, dx$

72. $\int_{-1}^2 \sin\left(\frac{\pi x}{4}\right) dx$

73. $\int_0^4 \left(4x - x^2\right) dx$

74. $\int_0^3 \frac{2}{\sqrt{x + 1}}\, dx$

75–81. Limits of sums *Use the definition of the definite integral to evaluate the following definite integrals. Use right Riemann sums and Theorem 5.1.*

75. $\int_0^2 (2x + 1)\, dx$

76. $\int_1^5 (1 - x)\, dx$

77. $\int_3^7 (4x + 6)\, dx$

78. $\int_0^2 \left(x^2 - 1\right) dx$

79. $\int_1^4 \left(x^2 - 1\right) dx$

80. $\int_0^2 \left(x^3 + x + 1\right) dx$

81. $\int_0^1 \left(4\,x^3 + 3\,x^2\right) dx$

Explorations and Challenges »

82–83. Area by geometry *Use geometry to evaluate the following integrals.*

82. $\int_1^6 |2\,x - 4|\,dx$

83. $\int_{-6}^4 \sqrt{24 - 2\,x - x^2}\,dx$

84. Integrating piecewise continuous functions Suppose f is continuous on the intervals $[a, p]$ and $(p, b]$, where $a < p < b$, with a finite jump at p. Form a uniform partition on the interval $[a, p]$ with n grid points and another uniform partition on the interval $[p, b]$ with m grid points, where p is a grid point of both partitions. Write a Riemann sum for $\int_a^b f(x)\,dx$ and separate it into two pieces for $[a, p]$ and $[p, b]$. Explain why $\int_a^b f(x)\,dx = \int_a^p f(x)\,dx + \int_p^b f(x)\,dx$.

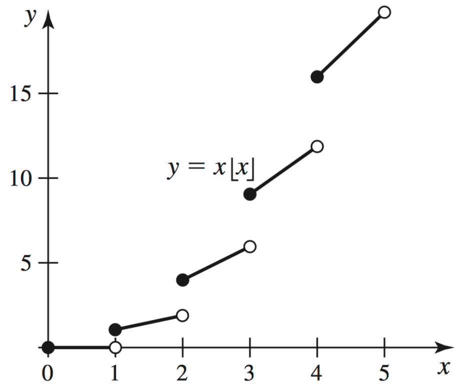
85–86. Integrating piecewise continuous functions *Use geometry and the result of Exercise 84 to evaluate the following integrals.*

85. $\int_0^{10} f(x)\,dx$, where $f(x) = \begin{cases} 2 & \text{if } 0 \le x \le 5 \\ 3 & \text{if } 5 < x \le 10 \end{cases}$

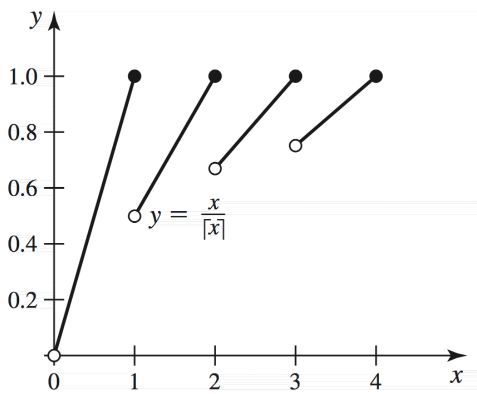
86. $\int_1^6 f(x)\,dx$, where $f(x) = \begin{cases} 2\,x & \text{if } 1 \le x < 4 \\ 10 - 2\,x & \text{if } 4 \le x \le 6 \end{cases}$

87–88. Integrating piecewise continuous functions *Recall that the* floor function $\lfloor x \rfloor$ *is the greatest integer less than or equal to x and that the* ceiling function $\lceil x \rceil$ *is the least integer greater than or equal to x. Use the result of Exercise 84 and the graphs to evaluate the following integrals.*

87. $\int_1^5 x\,\lfloor x \rfloor\,dx$



88. $\displaystyle\int_0^4 \frac{x}{\lceil x \rceil}\,dx$



89. Constants in integrals Use the definition of the definite integral to justify the property $\displaystyle\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$, where f is continuous and c is a real number.

90. Zero net area Assuming $0 < c < d$, find the value of b (in terms of c and d) for which $\displaystyle\int_c^d (x + b)\,dx = 0$.

91. A nonintegrable function Consider the function defined on $[0, 1]$ such that $f(x) = 1$ if x is a rational number and $f(x) = 0$ if x is irrational. This function has an infinite number of discontinuities, and the integral $\displaystyle\int_0^1 f(x)\,dx$ does not exist. Show that the right, left, and midpoint Riemann sums on *regular* partitions with n subintervals equal 1 for all n. (*Hint:* Between any two real numbers lie a rational and an irrational number.)

92. Powers of *x* by Riemann sums Consider the integral $I(p) = \displaystyle\int_0^1 x^p\,dx$, where p is a positive integer.

a. Write the left Riemann sum for the integral with n subintervals.

b. It is a fact (proved by the 17th-century mathematicians Fermat and Pascal) that $\displaystyle\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1}\left(\frac{k}{n}\right)^p = \frac{1}{p+1}$. Use this fact to evaluate $I(p)$.

93. **An exact integration formula** Evaluate $\int_a^b \frac{dx}{x^2}$, where $0 < a < b$, using the definition of the definite integral and the following steps.

a. Assume $\{x_0, x_1, \ldots, x_n\}$ is a partition of $[a, b]$ with $\Delta x_k = x_k - x_{k-1}$, for $k = 1, 2, \ldots, n$. Show that $x_{k-1} \le \sqrt{x_{k-1}\, x_k} \le x_k$, for $k = 1, 2, \ldots, n$.

b. Show that $\frac{1}{x_{k-1}} - \frac{1}{x_k} = \frac{\Delta x_k}{x_{k-1}\, x_k}$, for $k = 1, 2, \ldots, n$.

c. Simplify the general Riemann sum for $\int_a^b \frac{dx}{x^2}$ using $x_k^* = \sqrt{x_{k-1}\, x_k}$.

d. Conclude that $\int_a^b \frac{dx}{x^2} = \frac{1}{a} - \frac{1}{b}$.

(*Source: The College Mathematics Journal*, 32, 4, Sep 2001)

94. Use Property 3 of Table 5.4 and Property 7 of Table 5.5 to prove Property 8 of Table 5.5.