5 Integration

Chapter Preview We are now at a critical point in the calculus story. Many would argue that this chapter is the cornerstone of calculus because it explains the relationship between the two processes of calculus: differentiation and integration. We begin by explaining why finding the area of regions bounded by the graphs of functions is such an important problem in calculus. Then you will see how antiderivatives lead to definite integrals, which are used to solve this problem. But there is more to the story. You will also see the remarkable connection between derivatives and integrals, which is expressed in the Fundamental Theorem of Calculus. In this chapter, we develop key properties of definite integrals, investigate a few of their many applications, and present the first of several powerful techniques for evaluating definite integrals.

5.1 Approximating Areas under Curves

The derivative of a function is associated with rates of change and slopes of tangent lines. We also know that antiderivatives (or indefinite integrals) reverse the derivative operation. **Figure 5.1** summarizes our current understanding and raises the question: What is the geometric meaning of the integral? The following example reveals a clue.

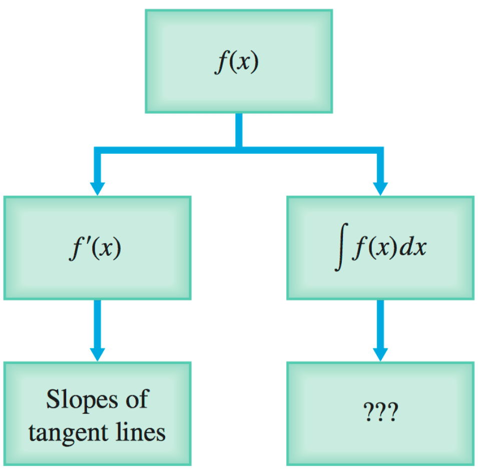


Figure 5.1

Area under a Velocity Curve »

Consider an object moving along a line with a known position function. You learned in previous chapters that the slope of the line tangent to the graph of the position function at a certain time gives the velocity v at that time. We now turn the situation around. If we know the velocity function of a moving object, what can we learn about its position function?

Imagine a car traveling at a constant velocity of 60 mi/hr along a straight highway over a two-hour period. The graph of the velocity function $v = 60$ on the interval $0 \le t \le 2$ is a horizontal line (**Figure 5.2**).

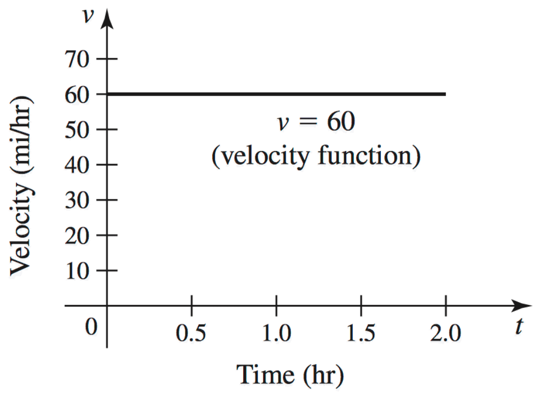


Figure 5.2

The displacement of the car between $t = 0$ and $t = 2$ hr is found by a familiar formula:

$$\begin{aligned}\text{displacement} &= \text{rate} \cdot \text{time}\\ &= 60 \text{ mi/hr} \cdot 2 \text{ hr} = 120 \text{ mi}.\end{aligned}$$

Note »

> Recall from Section 3.6 that the *displacement* of an object moving along a line is given by
>
> final position − initial position .
>
> If the velocity of an object is positive, its displacement equals the distance traveled.

This product is the area of the rectangle formed by the velocity curve and the t-axis between $t = 0$ and $t = 2$ (**Figure 5.3**). In this case (constant positive velocity), we see that the area between the velocity curve and the t-axis is the displacement of the moving object.

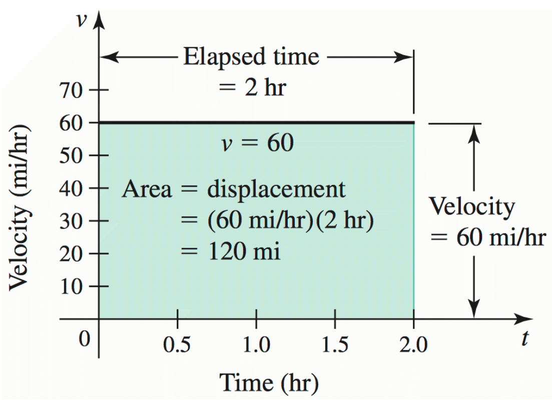


Figure 5.3

Note »

Quick Check 1 What is the displacement of an object that travels at a constant velocity of 10 mi/hr for a half hour, 20 mi/hr for the next half hour, and 30 mi/hr for the next hour? ◆

Answer »

Because objects do not necessarily move at a constant velocity, we first extend these ideas to positive velocities that *change* over an interval of time. One strategy is to divide the time interval into many subintervals and approximate the velocity on each subinterval by a constant velocity. Then the displacements on each subinterval are calculated and summed. This strategy produces only an approximation to the displacement; however, this approximation generally improves as the number of subintervals increases.

EXAMPLE 1 Approximating the displacement

Suppose the velocity in m/s of an object moving along a line is given by the function $v = t^2$, where $0 \le t \le 8$. Approximate the displacement of the object by dividing the time interval [0, 8] into n subintervals of equal length. On each subinterval, approximate the velocity by a constant equal to the value of v evaluated at the midpoint of the subinterval.

a. Begin by dividing [0, 8] into $n = 2$ subintervals: [0, 4] and [4, 8].

b. Divide [0, 8] into $n = 4$ subintervals: [0, 2], [2, 4], [4, 6], and [6, 8].

c. Divide [0, 8] into $n = 8$ subintervals of equal length.

SOLUTION »

a. We divide the interval [0, 8] into $n = 2$ subintervals, [0, 4] and [4, 8], each with length 4. The velocity on each subinterval is approximated using the value of v evaluated at the midpoint of that subinterval (**Figure 5.4**).

- We approximate the velocity on [0, 4] by $v(2) = 2^2 = 4$ m/s. Traveling at 4 m/s for 4 s results in a displacement of 4 m/s·4 s = 16 m.

Therefore, an approximation to the displacement over the entire interval [0, 8] is

$$(v(2)\cdot 4\text{ s}) + (v(6)\cdot 4\text{ s}) = (4\text{ m/s}\cdot 4\text{ s}) + (36\text{ m/s}\cdot 4\text{ s}) = 160\text{ m}.$$

b. With $n = 4$ (Figure 5.4), each subinterval has length 2. The approximate displacement over the entire interval is

$$\Big(\underbrace{1\text{ m/s}}_{v(1)}\cdot 2\text{ s}\Big) + \Big(\underbrace{9\text{ m/s}}_{v(3)}\cdot 2\text{ s}\Big) + \Big(\underbrace{25\text{ m/s}}_{v(5)}\cdot 2\text{ s}\Big) + \Big(\underbrace{49\text{ m/s}}_{v(7)}\cdot 2\text{ s}\Big) = 168\text{ m}.$$

c. With $n = 8$ subintervals (Figure 5.4), the approximation to the displacement is 170 m. In each case, the approximate displacement is the sum of the areas of the rectangles under the velocity curve.

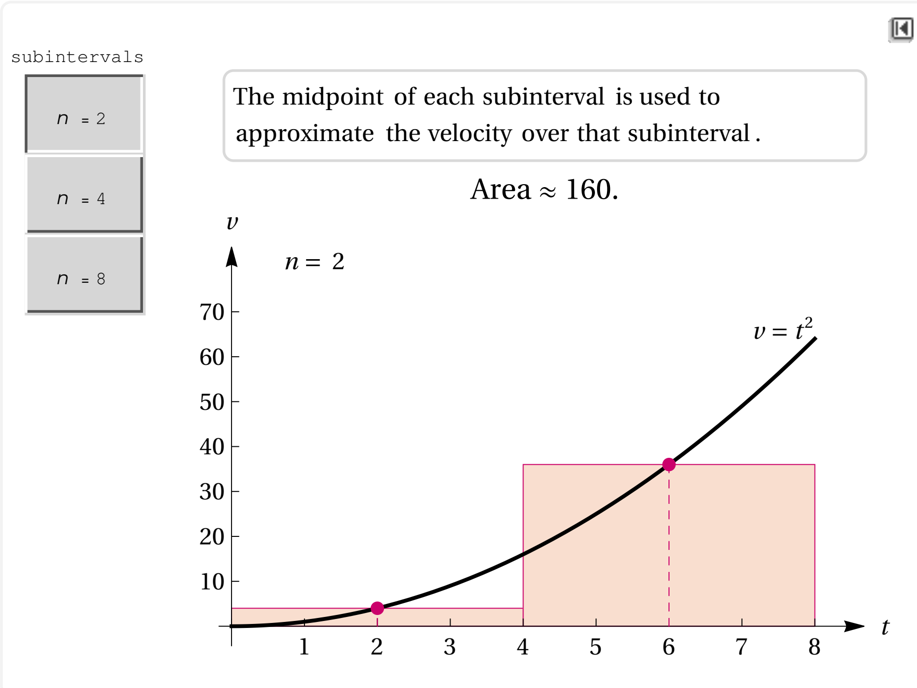


Figure 5.4

Related Exercises 3, 15–16 ◆

Quick Check 2 In Example 1, if we used $n = 32$ subintervals of equal length, what would be the length of each subinterval? Find the midpoint of the first and last subinterval. ◆

Answer »

The progression in Example 1 may be continued. Larger values of n mean more rectangles; in general, more rectangles give a better fit to the region under the curve (**Figure 5.5**).

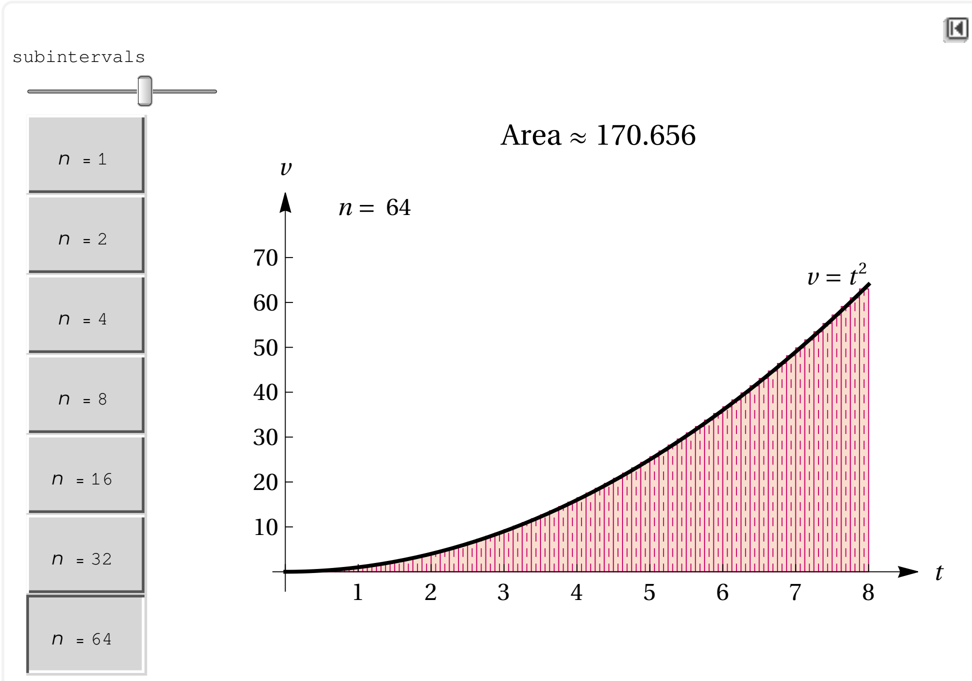


Figure 5.5

With the help of a calculator, we can generate the approximations in Table 5.1 using $n = 1, 2, 4, 8, 16, 32$, and 64 subintervals. Observe that as n increases, the approximations appear to approach a limit of approximately 170.7 m. The limit is the exact displacement, which is represented by the area of the region under the velocity curve. This strategy of taking limits of sums is developed fully in Section 5.2.

Table 5.1 Approximations to the area under the velocity curve $v = t^2$ on [0, 8]

Number of subintervals	Length of each subinterval	Approximate displacement (area under curve)
1	8 s	128.0 m
2	4 s	160.0 m
4	2 s	168.0 m
8	1 s	170.0 m
16	0.5 s	170.5 m
32	0.25 s	170.625 m
64	0.125 s	170.65625 m

Approximating Areas by Riemann Sums »

We wouldn't spend much time investigating areas under curves if the idea applied only to computing displacements from velocity curves. However, the problem of finding areas under curves arises frequently and turns out to be immensely important—as you will see in the next two chapters. For this reason, we now develop a systematic method for approximating areas under curves. Consider a function f that is continuous and nonnegative on an interval $[a, b]$. The goal is to approximate the area of the region R bounded by the graph of f and the x-axis from $x = a$ to $x = b$ (**Figure 5.6**).

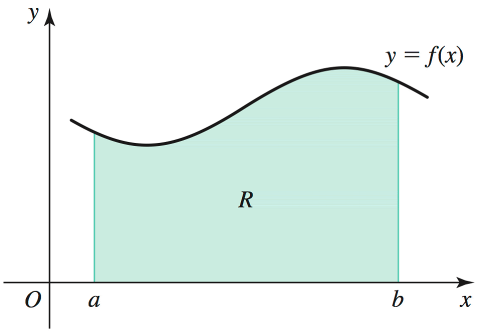


Figure 5.6

Note »

We begin by dividing the interval $[a, b]$ into n subintervals of equal length,

$$[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n],$$

where $a = x_0$ and $b = x_n$ (**Figure 5.7**). The length of each subinterval, denoted Δx, is found by dividing the length of the interval by n:

$$\Delta x = \frac{b-a}{n}.$$

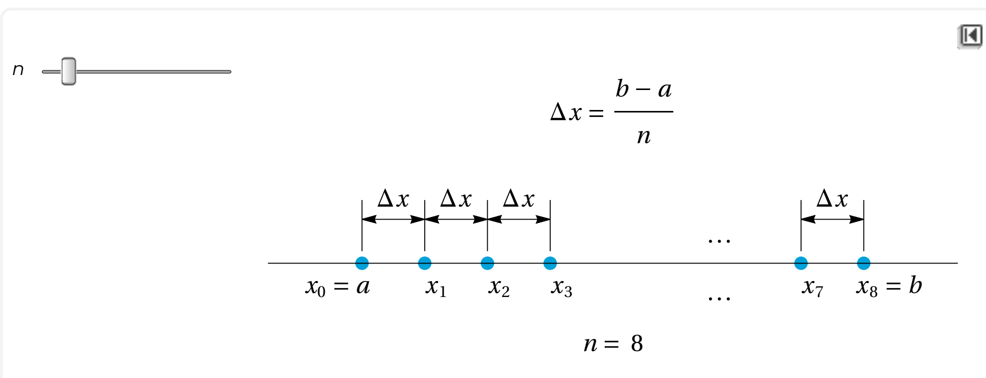


Figure 5.7

DEFINITION **Regular Partition**

Suppose $[a, b]$ is a closed interval containing n subintervals

$$[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$$

of equal length $\Delta x = \frac{b-a}{n}$ with $a = x_0$ and $b = x_n$. The endpoints $x_0, x_1, x_2, \ldots, x_{n-1}, x_n$ of the subintervals are called the **grid points** and they create a **regular partition** of the interval $[a, b]$. In general, the kth grid point is

$$x_k = a + k\,\Delta x, \text{ for } k = 0, 1, 2, \ldots, n.$$

Quick Check 3 If the interval $[1, 9]$ is partitioned into 4 subintervals of equal length, what is Δx? List the grid points x_0, x_1, x_2, x_3, and x_4. ◆

Answer »

In the kth subinterval $[x_{k-1}, x_k]$, we choose any point x_k^* and build a rectangle whose height is $f(x_k^*)$, the value of f at x_k^* (**Figure 5.8**).

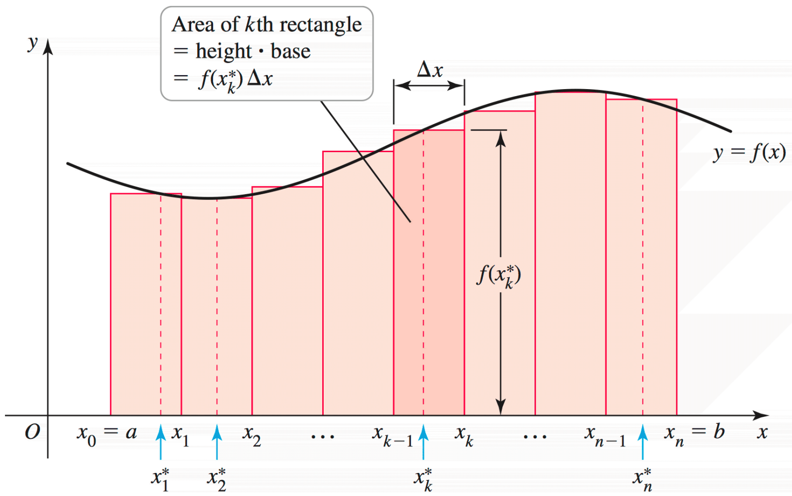


Figure 5.8

The area of the rectangle on the kth subinterval is

$$\text{height} \cdot \text{base} = f(x_k^*)\,\Delta x, \quad \text{where } k = 1, 2, \ldots, n.$$

Summing the areas of the rectangles in Figure 5.8, we obtain an approximation to the area of R, which is called a **Riemann sum**:

$$f(x_1^*)\,\Delta x + f(x_2^*)\,\Delta x + \cdots + f(x_n^*)\,\Delta x.$$

Three notable Riemann sums are the *left*, *right*, and *midpoint Riemann sums*.

Note »

DEFINITION **Riemann Sum**

Suppose f is defined on a closed interval $[a, b]$, which is divided into n subintervals of equal length Δx. If x_k^* is any point in the kth subinterval $[x_{k-1}, x_k]$, for $k = 1, 2, \ldots, n$, then

$$f(x_1^*)\,\Delta x + f(x_2^*)\,\Delta x + \cdots + f(x_n^*)\,\Delta x$$

is called a **Riemann sum** for f on $[a, b]$. (**Figure 5.9**) This sum is called

- a **left Riemann sum** if x_k^* is the left endpoint of $[x_{k-1}, x_k]$;
- a **right Riemann sum** if x_k^* is the right endpoint of $[x_{k-1}, x_k]$; and
- a **midpoint Riemann sum** if x_k^* is the midpoint of $[x_{k-1}, x_k]$, for $k = 1, 2, \ldots, n$.

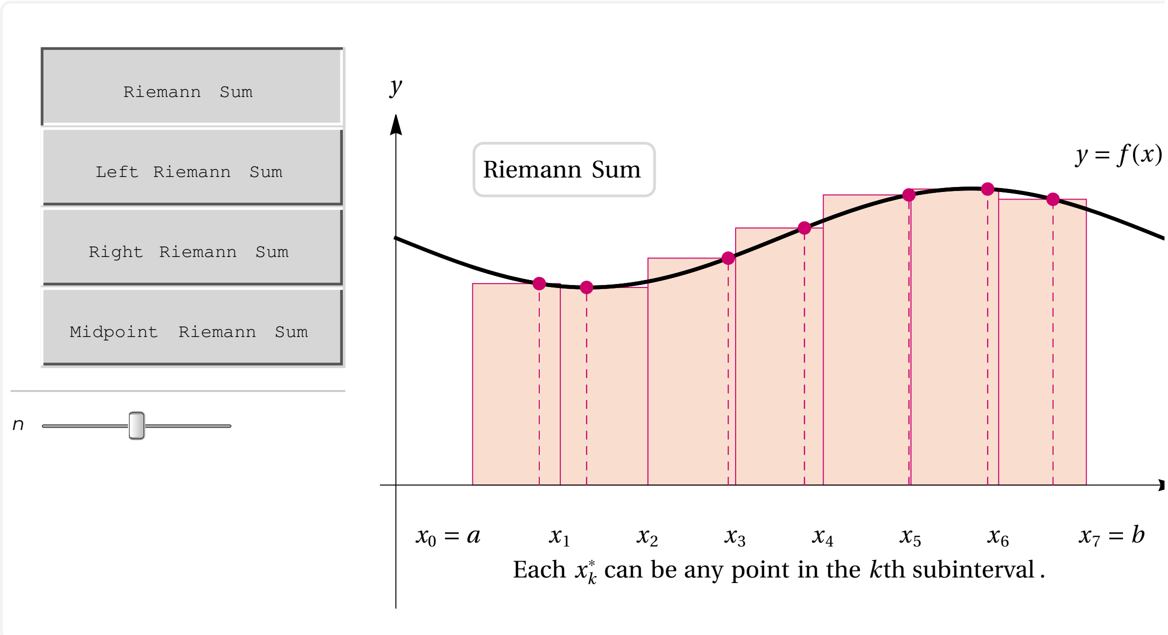


Figure 5.9

EXAMPLE 2 Left and right Riemann sums

Let R be the region bounded by the graph of $f(x) = 3\sqrt{x}$ and the x–axis between $x = 4$ and $x = 16$.

a. Approximate the area of R using a left Riemann sum with $n = 6$ subintervals. Illustrate the sum with the appropriate rectangles.

b. Approximate the area of R using a right Riemann sum with $n = 6$ subintervals. Illustrate the sum with the appropriate rectangles.

c. Do the area approximations in parts (a) and (b) underestimate or overestimate the actual area under the curve?

SOLUTION »

Dividing the interval $[a, b] = [4, 16]$ into $n = 6$ subintervals means the length of each subinterval is

$$\Delta x = \frac{b - a}{n} = \frac{16 - 4}{6} = 2;$$

therefore the grid points are 4, 6, 8, 10, 12, 14, and 16.

a. To find the left Riemann sum, we set $x_1^*, x_2^*, \ldots, x_6^*$ equal to the left endpoints of the six subintervals. The heights of the rectangles are $f(x_k^*)$, for $k = 1, 2, \ldots, 6$.

The resulting left Riemann sum (**Figure 5.12**) is

$$\begin{aligned} f(x_1^*)\,\Delta x + f(x_2^*)\,\Delta x + \cdots + f(x_6^*)\,\Delta x &= f(4)\cdot 2 + f(6)\cdot 2 + f(8)\cdot 2 + f(10)\cdot 2 + f(12)\cdot 2 + f(14)\cdot 2 \\ &= 3\sqrt{4}\cdot 2 + 3\sqrt{6}\cdot 2 + 3\sqrt{8}\cdot 2 + 3\sqrt{10}\cdot 2 + 3\sqrt{12}\cdot 2 + 3\sqrt{14}\cdot 2 \\ &\approx 105.876. \end{aligned}$$

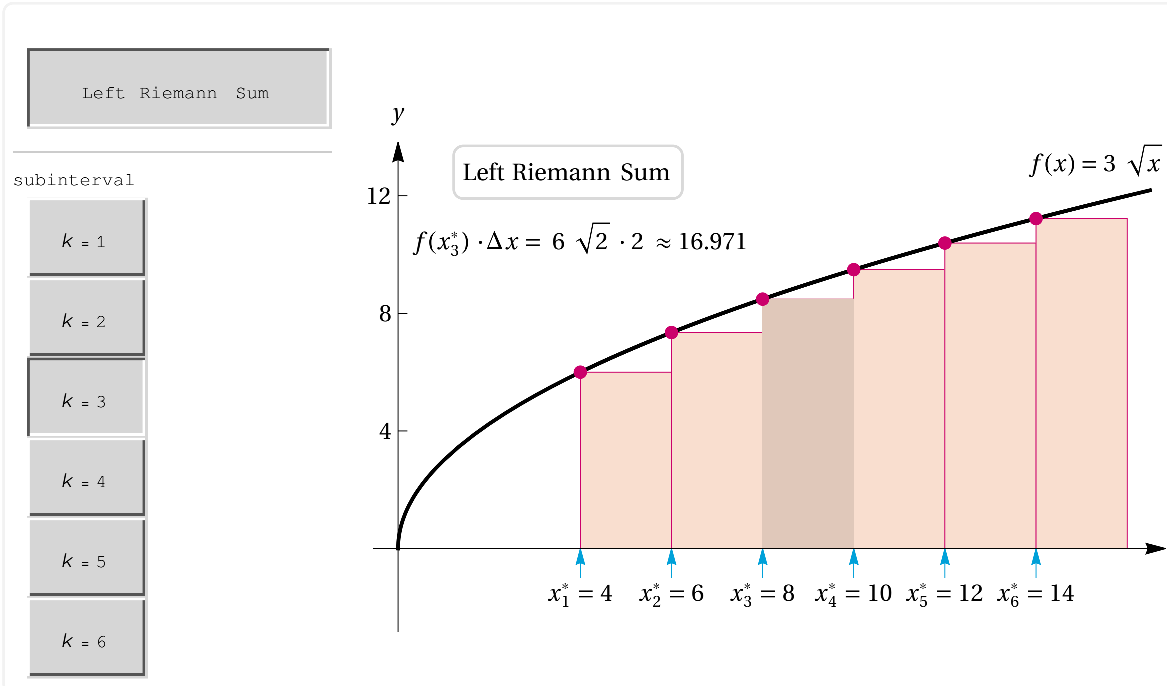


Figure 5.12

b. In a right Riemann sum the right endpoints are used for $x_1^*, x_2^*, \ldots, x_6^*$, and the heights of the rectangles are $f(x_k^*)$, for $k = 1, \ldots, 6$.

The resulting right Riemann sum (**Figure 5.13**) is

$$\begin{aligned} f(x_1^*)\,\Delta x + f(x_2^*)\,\Delta x + \cdots + f(x_6^*)\,\Delta x &= f(6)\cdot 2 + f(8)\cdot 2 + f(10)\cdot 2 + f(12)\cdot 2 + f(14)\cdot 2 + f(16)\cdot 2 \\ &= 3\sqrt{6}\cdot 2 + 3\sqrt{8}\cdot 2 + 3\sqrt{10}\cdot 2 + 3\sqrt{12}\cdot 2 + 3\sqrt{14}\cdot 2 + 3\sqrt{16}\cdot 2 \\ &\approx 117.876. \end{aligned}$$

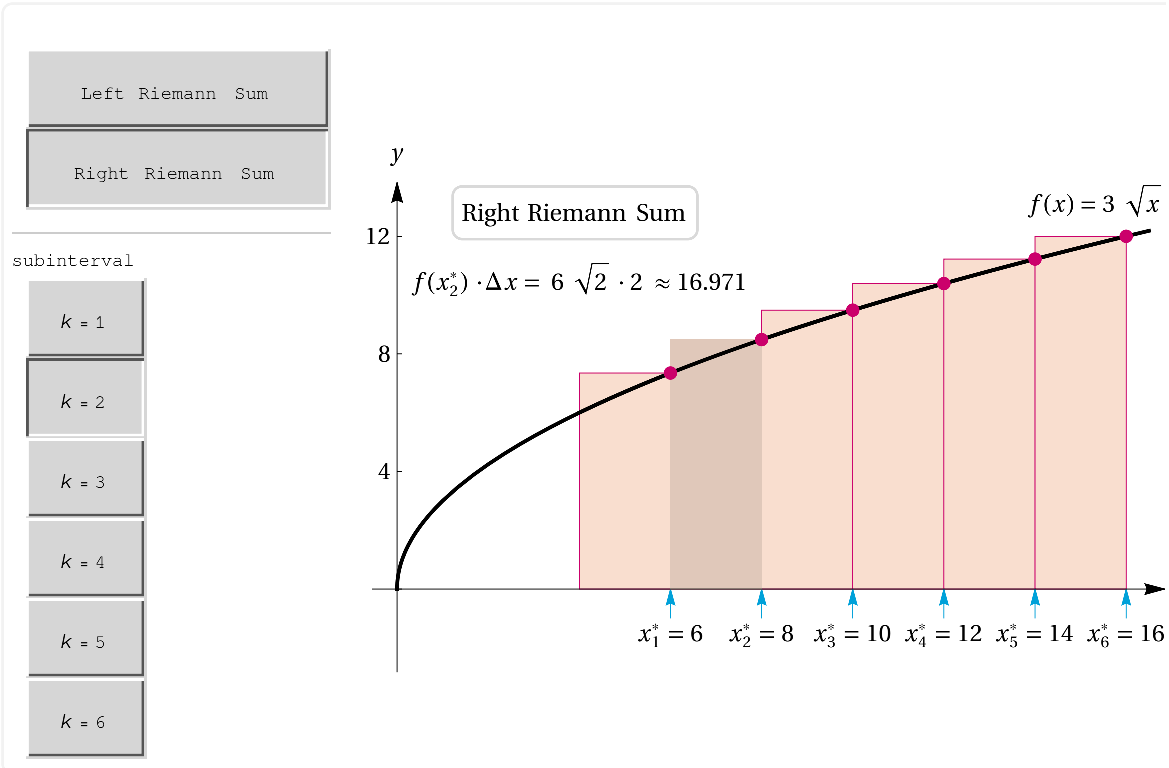


Figure 5.13

c. Looking at the graphs, we see that the left Riemann sum in part (a) underestimates the actual area of R, whereas the right Riemann sum in part (b) overestimates the area of R. Therefore, the area of R is between 105.876 and 117.876. These approximations improve as the number of rectangles increases.

Related Exercises 23–24, 29 ◆

Quick Check 4 If the function in Example 2 is instead $f(x) = 1/x$, does the left Riemann sum or the right Riemann sum overestimate the area under the curve? ◆

Answer »

EXAMPLE 3 A midpoint Riemann sum

Let R be the region bounded by the graph of $f(x) = 3\sqrt{x}$ and the x-axis between $x = 4$ and $x = 16$. Approximate the area of R using a midpoint Riemann sum with $n = 6$ subintervals. Illustrate the sum with the appropriate rectangles.

SOLUTION »

The grid points and the length of the subintervals are the same as in Example 2. To find the midpoint Riemann sum, we set $x_1^*, x_2^*, \ldots, x_6^*$ equal to the midpoints of the subintervals. The midpoint of the first subinterval is the average of x_0 and x_1, which is

$$x_1^* = \frac{x_0 + x_1}{2} = \frac{4 + 6}{2} = 5.$$

The remaining midpoints are also computed by averaging the two nearest grid points. The resulting midpoint Riemann sum (**Figure 5.14**) is

$$f(x_1^*)\,\Delta x + f(x_2^*)\,\Delta x + \cdots + f(x_6^*)\,\Delta x = f(5)\cdot 2 + f(7)\cdot 2 + f(9)\cdot 2 + f(11)\cdot 2 + f(13)\cdot 2 + f(15)\cdot 2$$
$$= 3\sqrt{5}\cdot 2 + 3\sqrt{7}\cdot 2 + 3\sqrt{9}\cdot 2 + 3\sqrt{11}\cdot 2 + 3\sqrt{13}\cdot 2 + 3\sqrt{15}\cdot 2$$
$$\approx 112.062.$$

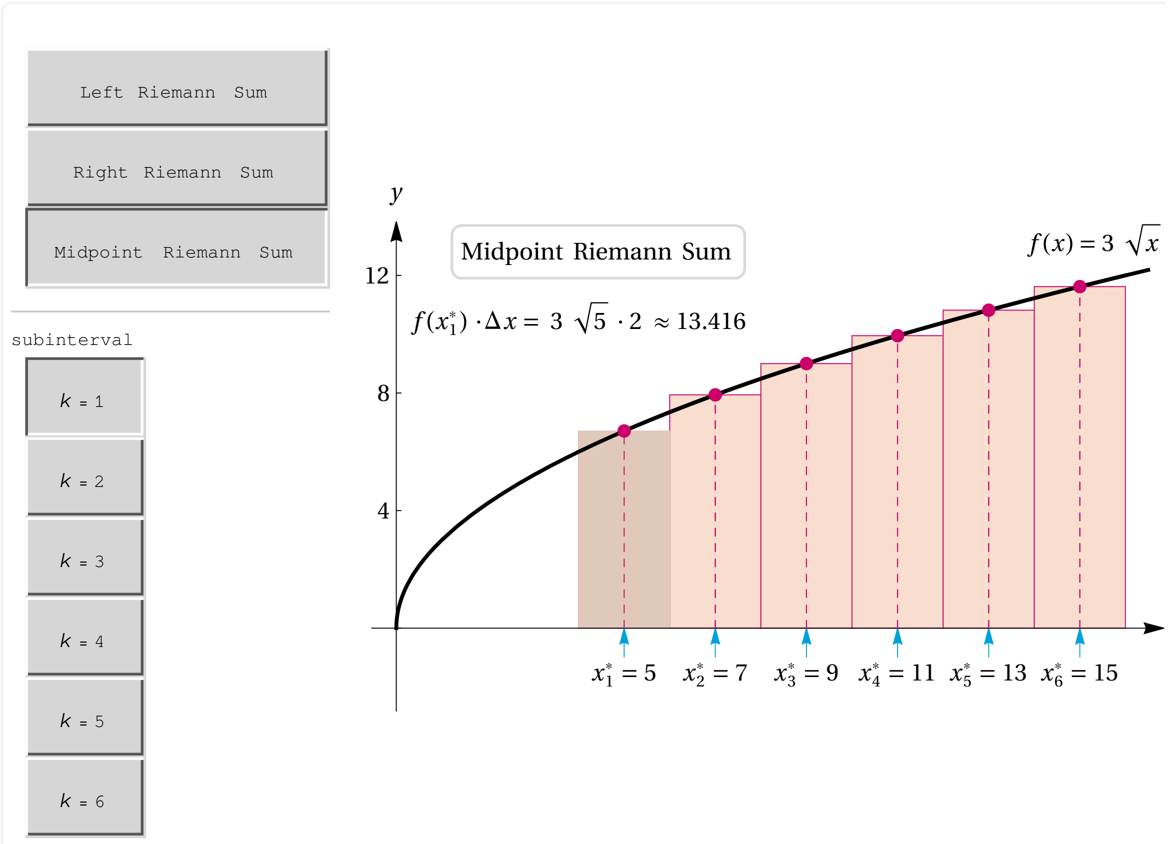


Figure 5.14

Comparing the midpoint Riemann sum (Figure 5.14) with the left and right Riemann sums (Figures 5.12 and 5.13) suggests that the midpoint sum is a more accurate estimate of the area under the curve. Indeed, in Section 5.3, we will learn that the exact area under the curve is 112.

Related Exercises 33–34, 39 ◆

EXAMPLE 4 Riemann sums from tables

Estimate the area A under the graph of f on the interval $[0, 2]$ using left and right Riemann sums with $n = 4$, when f is continuous but known only at the points in Table 5.2.

Table 5.2

x	$f(x)$
0	1
0.5	3
1.0	4.5
1.5	5.5
2.0	6.0

SOLUTION »

With $n = 4$ subintervals on the interval $[0, 2]$, $\Delta x = \frac{2}{4} = 0.5$. Using the left endpoint of each subinterval, the left Riemann sum is

$$\begin{aligned} A &\approx f(0)\,\Delta x + f(0.5)\,\Delta x + f(1.0)\,\Delta x + f(1.5)\,\Delta x \\ &= 1 \cdot 0.5 + 3 \cdot 0.5 + 4.5 \cdot 0.5 + 5.5 \cdot 0.5 = 7.0. \end{aligned}$$

Using the right endpoint of each subinterval, the right Riemann sum is

$$\begin{aligned} A &\approx f(0.5)\,\Delta x + f(1.0)\,\Delta x + f(1.5)\,\Delta x + f(2.0)\,\Delta x \\ &= 3 \cdot 0.5 + 4.5 \cdot 0.5 + 5.5 \cdot 0.5 + 6.0 \cdot 0.5 = 9.5. \end{aligned}$$

With only five function values, these estimates of the area are necessarily crude. Better estimates are obtained by using more subintervals and more function values.

Related Exercises 43–44 ◆

Sigma (Summation) Notation »

Working with Riemann sums is cumbersome with large numbers of subintervals. Therefore, we pause for a moment to introduce some notation that simplifies our work.

Sigma (or **summation**), **notation** is used to express sums in a compact way. For example, the sum $1 + 2 + 3 + \cdots + 10$ is represented in sigma notation as $\sum_{k=1}^{10} k$. Here is how the notation works. The symbol $\sum$ (*sigma*, the Greek capital S) stands for *sum*. The **index** k takes on all integer values from the lower limit $(k = 1)$ to the upper limit $(k = 10)$. The expression that immediately follows $\sum$ (the **summand**) is evaluated for each value of k, and the resulting values are summed. Here are some examples.

$$\sum_{k=1}^{99} k = 1 + 2 + 3 + \cdots + 99 = 4950$$

$$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n$$

$$\sum_{k=0}^{3} k^2 = 0^2 + 1^2 + 2^2 + 3^2 = 14$$

$$\sum_{k=1}^{4} (2\,k + 1) = 3 + 5 + 7 + 9 = 24$$

$$\sum_{k=-1}^{2} \left(k^2 + k\right) = \left((-1)^2 + (-1)\right) + \left(0^2 + 0\right) + \left(1^2 + 1\right) + \left(2^2 + 2\right) = 8$$

The index in a sum is a *dummy variable*. It is internal to the sum, so it does not matter what symbol you choose as an index. For example,

$$\sum_{k=1}^{99} k = \sum_{n=1}^{99} n = \sum_{p=1}^{99} p.$$

Two properties of sums are useful in upcoming work. Suppose $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$ are two sets of real numbers, and suppose c is a real number. Then we can factor constants out of a sum:

Constant Multiple Rule $$\sum_{k=1}^{n} c\,a_k = c \sum_{k=1}^{n} a_k.$$

We can also split a sum into two sums:

Addition Rule $$\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k.$$

In the coming examples and exercises, the following formulas for sums of powers of integers are essential.

THEOREM 5.1 Sums of Powers of Integers

Let n be a positive integer and c a real number.

$$\sum_{k=1}^{n} c = c\,n \qquad \sum_{k=1}^{n} k = \frac{n\,(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n\,(n+1)\,(2\,n+1)}{6} \qquad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

Note »

Formulas for $\sum_{k=1}^{n} k^p$, where p is a positive integer, have been known for centuries. The formulas for $p = 0, 1, 2,$ and 3 are relatively simple. The formulas become complicated as p increases.

Riemann Sums Using Sigma Notation »

With sigma notation, a Riemann sum has the convenient compact form

$$f(x_1^*)\,\Delta x + f(x_2^*)\,\Delta x + \cdots + f(x_n^*)\,\Delta x = \sum_{k=1}^{n} f(x_k^*)\,\Delta x.$$

To express left, right, and midpoint Riemann sums in sigma notation, we must identify the points x_k^*.

- For left Riemann sums, the left endpoints of the subintervals are $x_k^* = a + (k-1)\,\Delta x$, for $k = 1, \ldots, n$.
- For right Riemann sums, the right endpoints of the subintervals are $x_k^* = a + k\,\Delta x$, for $k = 1, \ldots, n$.
- For midpoint Riemann sums, the midpoints of the subintervals are $x_k^* = a + \left(k - \frac{1}{2}\right)\Delta x$, for $k = 1, \ldots, n$.

Note »

The three Riemann sums are written compactly as follows.

DEFINITION **Left, Right, and Midpoint Riemann Sums in Sigma Notation**

Suppose f is defined on a closed interval $[a, b]$, which is divided into n subintervals of equal length Δx. If x_k^* is a point in the kth subinterval $[x_{k-1}, x_k]$, for $k = 1, 2, \ldots, n$, then the **Riemann sum** of f on $[a, b]$ is $\sum_{k=1}^{n} f(x_k^*)\,\Delta x$. Three cases arise in practice.

- $\sum_{k=1}^{n} f(x_k^*)\,\Delta x$ is a **left Riemann sum** if $x_k^* = a + (k-1)\,\Delta x$.
- $\sum_{k=1}^{n} f(x_k^*)\,\Delta x$ is a **right Riemann sum** if $x_k^* = a + k\,\Delta x$.
- $\sum_{k=1}^{n} f(x_k^*)\,\Delta x$ is a **midpoint Riemann sum** if $x_k^* = a + \left(k - \frac{1}{2}\right)\Delta x$.

EXAMPLE 5 **Calculating Riemann sums**

Evaluate the left, right, and midpoint Riemann sums of $f(x) = x^3 + 1$ between $a = 0$ and $b = 2$ using $n = 50$ subintervals. Make a conjecture about the exact area under the curve (**Figure 5.15**).

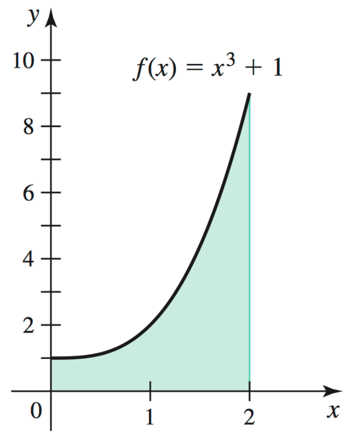


Figure 5.15

SOLUTION »

With $n = 50$, the length of each subinterval is

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{50} = \frac{1}{25} = 0.04.$$

The value of x_k^* for the left Riemann sum is

$$x_k^* = a + (k-1)\,\Delta x = 0 + 0.04\,(k-1) = 0.04\,k - 0.04,$$

for $k = 1, 2, \ldots, 50$. Therefore, the left Riemann sum, evaluated with a calculator, is

$$\sum_{k=1}^{n} f(x_k^*)\,\Delta x = \sum_{k=1}^{50} f(0.04\,k - 0.04)\,0.04 = 5.8416.$$

To evaluate the right Riemann sum, we let $x_k^* = a + k\,\Delta x = 0.04\,k$ and find that

$$\sum_{k=1}^{n} f(x_k^*)\,\Delta x = \sum_{k=1}^{50} f(0.04\,k)\,0.04 = 6.1616.$$

For the midpoint Riemann sum, we let

$$x_k^* = a + \left(k - \frac{1}{2}\right)\Delta x = 0 + 0.04\left(k - \frac{1}{2}\right) = 0.04\,k - 0.02.$$

The value of the sum is

$$\sum_{k=1}^{n} f(x_k^*)\,\Delta x = \sum_{k=1}^{50} f(0.04\,k - 0.02)\,0.04 = 5.9992.$$

Because f is increasing on $[0, 2]$, the left Riemann sum underestimates the area of the shaded region in Figure 5.15, while the right Riemann sum overestimates the area. Therefore, the exact area lies between 5.8416 and 6.1616. The midpoint Riemann sum usually gives the best estimate for increasing or decreasing functions.

Table 5.3 shows the left, right, and midpoint Riemann sum approximations for values of n up to 200. All three sets of approximations approach a value near 6, which is a reasonable estimate of the area under the curve. In Section 5.2, we show rigorously that the limit of all three Riemann sums as $n \to \infty$ is 6.

Table 5.3 Left, right, and midpoint Riemann sum approximations

n	L_n	R_n	M_n
20	5.61	6.41	5.995
40	5.8025	6.2025	5.99875
60	5.86778	6.13444	5.99944
80	5.90063	6.10063	5.99969
100	5.9204	6.0804	5.9998
120	5.93361	6.06694	5.99986
140	5.94306	6.05735	5.9999
160	5.95016	6.05016	5.99992
180	5.95568	6.04457	5.99994
200	5.9601	6.0401	5.99995

ALTERNATIVE SOLUTION

It is worth examining another approach to Example 5. Consider the right Riemann sum given previously:

$$\sum_{k=1}^{n} f(x_k^*)\,\Delta x = \sum_{k=1}^{50} f(0.04\,k)\,0.04.$$

Rather than evaluating this sum with a calculator, we note that $f(0.04\,k) = (0.04\,k)^3 + 1$ and then use the properties of sums:

$$\begin{aligned}\sum_{k=1}^{n} f(x_k^*)\,\Delta x &= \sum_{k=1}^{50} \underbrace{\left((0.04\,k)^3 + 1\right)}_{f(x_k^*)}\,\underbrace{0.04}_{\Delta x} \\ &= \sum_{k=1}^{50} (0.04\,k)^3\,0.04 + \sum_{k=1}^{50} 1\cdot 0.04 \qquad \sum (a_k + b_k) = \sum a_k + \sum b_k \\ &= (0.04)^4 \sum_{k=1}^{50} k^3 + 0.04 \times \sum_{k=1}^{50} 1. \qquad \sum c\,a_k = c \sum a_k\end{aligned}$$

Using the summation formulas for powers of integers in Theorem 5.1, we find that

$$\sum_{k=1}^{50} 1 = 50 \quad \text{and} \quad \sum_{k=1}^{50} k^3 = \frac{50^2 \cdot 51^2}{4}.$$

Substituting the values of these sums into the right Riemann sum yields

$$\sum_{k=1}^{50} f(x_k^*)\,\Delta x = \frac{3851}{625} = 6.1616,$$

confirming the result given by a calculator. The idea of evaluating Riemann sums for *arbitrary* values of n is used in Section 5.2, where we evaluate the limit of the Riemann sum as $n \to \infty$.

Related Exercises 51–52 ◆

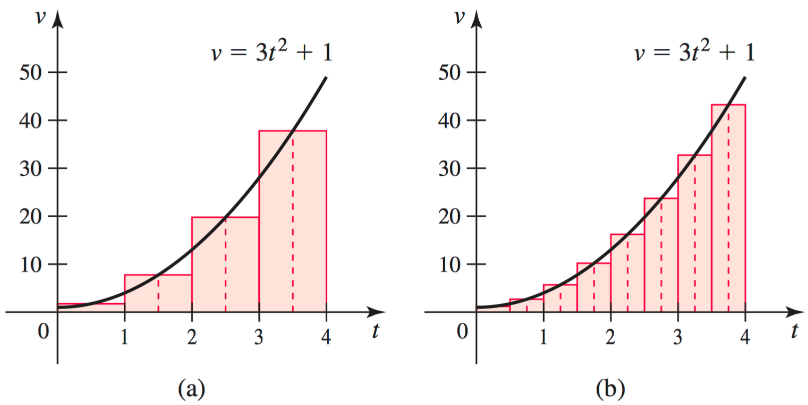
Exercises »

Getting Started »

Practice Exercises »

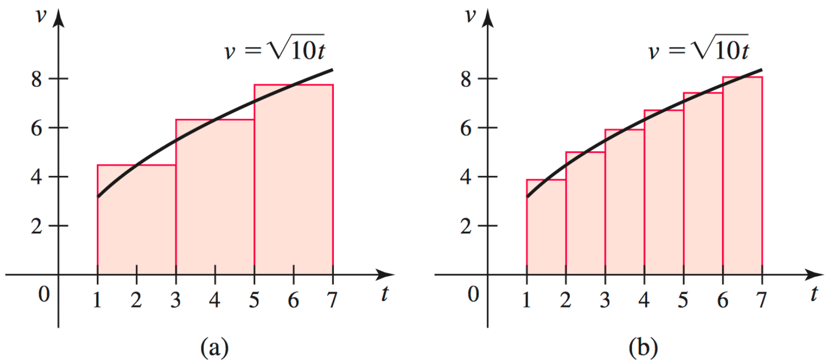
15. Approximating displacement The velocity in ft/s of an object moving along a line is given by $v = 3t^2 + 1$ on the interval $0 \le t \le 4$, where t is measured in seconds.

- **a.** Divide the interval [0, 4] into $n = 4$ subintervals, [0, 1], [1, 2], [2, 3], and [3, 4]. On each subinterval, assume the object moves at a constant velocity equal to v evaluated at the midpoint of the subinterval and use these approximations to estimate the displacement of the object on [0, 4] (see part (a) of the figure).
- **b.** Repeat part (a) for $n = 8$ subintervals (see part (b) of the figure).



T 16. Approximating displacement The velocity in ft/s of an object moving along a line is given by $v = \sqrt{10t}$ on the interval $1 \le t \le 7$, where t is measured in seconds.

- **a.** Divide the interval [1, 7] into $n = 3$ subintervals, [1, 3], [3, 5], and [5, 7]. On each subinterval, assume the object moves at a constant velocity equal to v evaluated at the midpoint of the subinterval and use these approximations to estimate the displacement of the object on [1, 7] (see part (a) of the figure).
- **b.** Repeat part (a) for $n = 6$ subintervals (see part (b) of the figure).



17–22. Approximating displacement *The velocity of an object is given by the following functions on a specified interval. Approximate the displacement of the object on this interval by subdividing the interval into n subintervals. Use the left endpoint of each subinterval to compute the height of the rectangles.*

17. $v = 2t + 1$ (m/s), for $0 \le t \le 8$; $n = 2$

18. $v = t^3 + 1$ (m/s), for $0 \le t \le 3$; $n = 3$

19. $v = \dfrac{1}{2t+1}$ (m/s), for $0 \le t \le 8$; $n = 4$

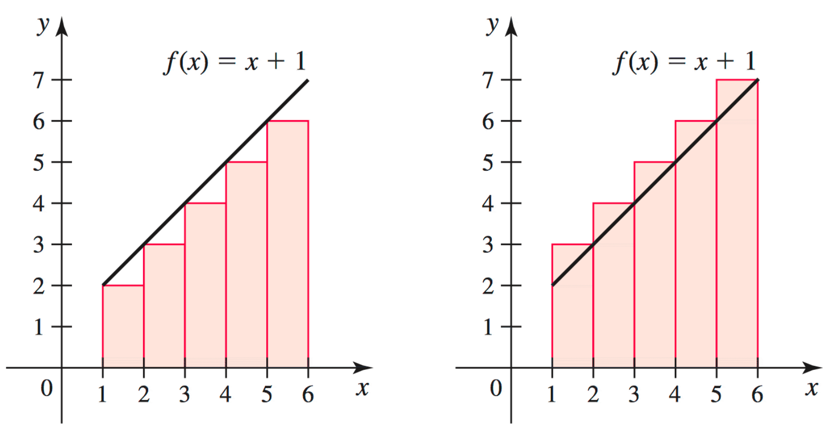
20. $v = \dfrac{t^2}{2} + 4$ (ft/s), for $0 \le t \le 12$; $n = 6$

T 21. $v = 4\sqrt{t+1}$ (mi/hr), for $0 \le t \le 15$; $n = 5$

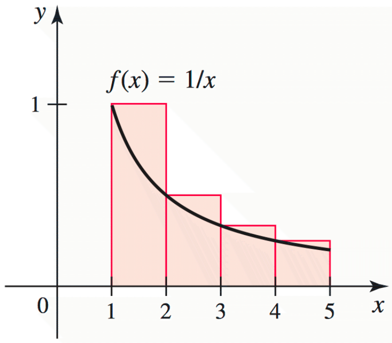
22. $v = \dfrac{t+3}{6}$ (m/s), for $0 \le t \le 4$; $n = 4$

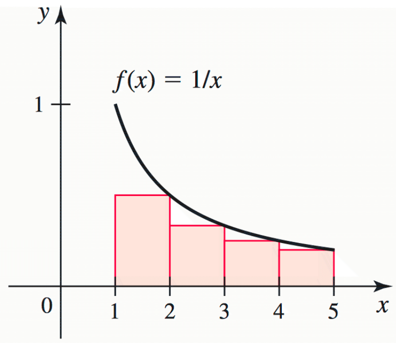
23–24. Left and right Riemann sums *Use the figures to calculate the left and right Riemann sums for f on the given interval and for the given value of n.*

23. $f(x) = x + 1$ on $[1, 6]$; $n = 5$



24. $f(x) = \dfrac{1}{x}$ on $[1, 5]$; $n = 4$





25–32. Left and right Riemann sums *Complete the following steps for the given function, interval, and value of n.*

- **a.** *Sketch the graph of the function on the given interval.*
- **b.** *Calculate Δx and the grid points $x_0, x_1, \ldots, x_n$.*
- **c.** *Illustrate the left and right Riemann sums. Then determine which Riemann sum underestimates and which sum overestimates the area under the curve.*
- **d.** *Calculate the left and right Riemann sums.*

25. $f(x) = x + 1$ on $[0, 4]$; $n = 4$

26. $f(x) = 9 - x$ on $[3, 8]$; $n = 5$

T **27.** $f(x) = \cos x$ on $\left[0, \frac{\pi}{2}\right]$; $n = 4$

28. $f(x) = \sin(\pi x/6)$ on $[0, 3]$; $n = 3$

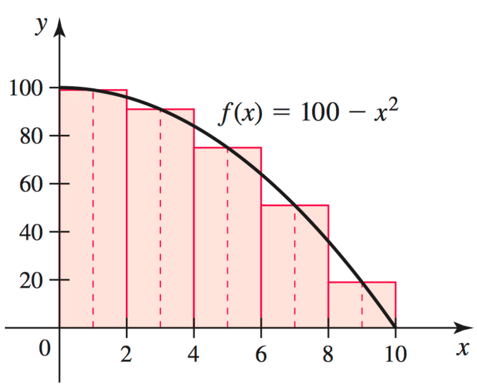
29. $f(x) = x^2 - 1$ on $[2, 4]$; $n = 4$

30. $f(x) = 2x^2$ on $[1, 6]$; $n = 5$

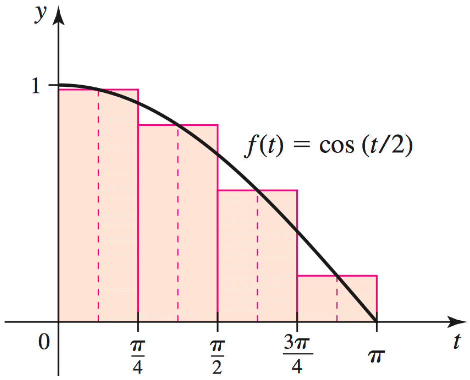
T **31.** $f(x) = \sqrt{x}$ on $[0, 3]$; $n = 6$

T **32.** $f(x) = 2^x$ on $[0, 1]$; $n = 4$

33. A midpoint Riemann sum Approximate the area of the region bounded by the graph of $f(x) = 100 - x^2$ and the x-axis on $[0, 10]$ with $n = 5$ subintervals. Use the midpoint of each subinterval to determine the height of each rectangle (see figure).



T **34. A midpoint Riemann sum** Approximate the area of the region bounded by the graph of $f(t) = \cos \frac{t}{2}$ and the t-axis on $[0, \pi]$ with $n = 4$ subintervals. Use the midpoint of each subinterval to determine the height of each rectangle (see figure).

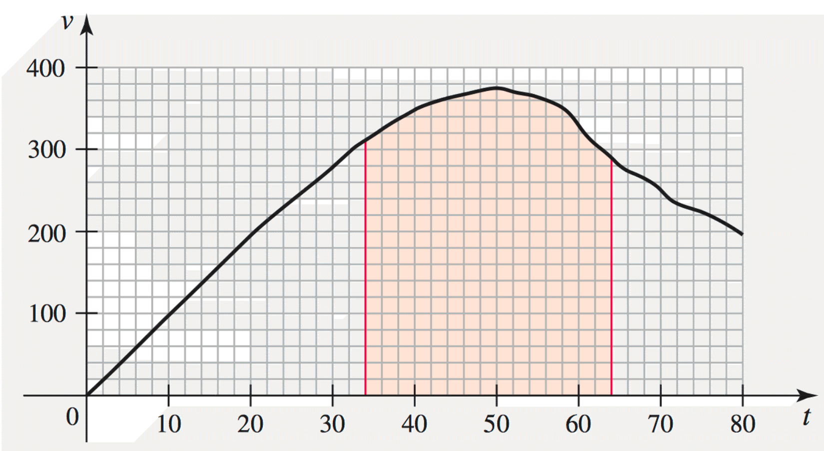


35. Free fall On October 14, 2012, Felix Baumgartner stepped off of a balloon capsule at an altitude of almost 39 km above Earth's surface and he began his free fall. His velocity in m/s during the fall is given in the figure. It is claimed that Felix reached the speed of sound 34 seconds into his fall and that he continued to fall at supersonic speed for 30 seconds. *(Source:* http://www.redbullstratos.com)

a. Divide the interval [34, 64] into $n = 5$ subintervals with the grid points $x_0 = 34$, $x_1 = 40$, $x_2 = 46$, $x_3 = 52$, $x_4 = 58$, and $x_5 = 64$. Use left and right Riemann sums to estimate how far Felix fell while traveling at supersonic speed.

b. It is claimed that the actual distance that Felix fell at supersonic speed was approximately 10,485 m. Which estimate in part (a) produced the more accurate estimate?

c. How could you obtain more accurate estimates of the total distance fallen than those found in part (a)?



36. Free fall Use geometry and the figure given in Exercise 35 to estimate how far Felix fell in the first 20 seconds of his free fall.

37–42. Midpoint Riemann sums *Complete the following steps for the given function, interval, and value of n.*

a. *Sketch the graph of the function on the given interval.*

b. *Calculate Δx and the grid points $x_0, x_1, \ldots, x_n$.*

c. *Illustrate the midpoint Riemann sum by sketching the appropriate rectangles.*

d. *Calculate the midpoint Riemann sum.*

37. $f(x) = 2x + 1$ on $[0, 4]$; $n = 4$

T **38.** $f(x) = 2\cos(\pi x/2)$ on $[0, 1]$; $n = 6$

T **39.** $f(x) = \sqrt{x}$ on $[1, 3]$; $n = 4$

40. $f(x) = x^2$ on $[0, 4]$; $n = 4$

41. $f(x) = \frac{1}{x}$ on $[1, 6]$; $n = 5$

42. $f(x) = 4 - x$ on $[-1, 4]$; $n = 5$

43–44. Riemann sums from tables *Evaluate the left and right Riemann sums for f over the given interval for the given value of n.*

43. $[0, 2]$; $n = 4$

x	0	0.5	1	1.5	2
$f(x)$	5	3	2	1	1

44. $[1, 5]$; $n = 8$

x	1	1.5	2	2.5	3	3.5	4	4.5	5
$f(x)$	0	2	3	2	2	1	0	2	3

45. Displacement from a table of velocities The velocities (in mi/hr) of an automobile moving along a straight highway over a two-hour period are given in the following table.

t (hr)	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2
v (mi/hr)	50	50	60	60	55	65	50	60	70

a. Sketch a smooth curve passing through the data points.

b. Find the midpoint Riemann sum approximation to the displacement on $[0, 2]$ with $n = 2$ and $n = 4$ subintervals.

46. Displacement from a table of velocities The velocities (in m/s) of an automobile moving along a straight freeway over a four-second period are given in the following table.

t (s)	0	0.5	1	1.5	2	2.5	3	3.5	4
v (m/s)	20	25	30	35	30	30	35	40	40

a. Sketch a smooth curve passing through the data points.

b. Find the midpoint Riemann sum approximation to the displacement on $[0, 4]$ with $n = 2$ and $n = 4$ subintervals.

47. Sigma notation Express the following sums using sigma notation. (Answers are not unique.)

a. $1+2+3+4+5$

b. $4+5+6+7+8+9$

c. $1^2+2^2+3^2+4^2$

d. $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$

48. Sigma notation Express the following sums using sigma notation. (Answers are not unique.)

a. $1+3+5+7+\cdots+99$

b. $4+9+14+\cdots+44$

c. $3+8+13+\cdots+63$

d. $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{49\cdot 50}$

49. Sigma notation Evaluate the following expressions.

a. $\sum_{k=1}^{10} k$

b. $\sum_{k=1}^{6} (2k+1)$

c. $\sum_{k=1}^{4} k^2$

d. $\sum_{n=1}^{5} \left(1+n^2\right)$

e. $\sum_{m=1}^{3} \frac{2m+2}{3}$

f. $\sum_{j=1}^{3} (3j-4)$

g. $\sum_{p=1}^{5} \left(2p+p^2\right)$

h. $\sum_{n=0}^{4} \sin \frac{n\pi}{2}$

T **50. Evaluating sums** Evaluate the following expressions by two methods. (i) Use Theorem 5.1. (ii) Use a calculator.

a. $\sum_{k=1}^{45} k$

b. $\sum_{k=1}^{45} (5k-1)$

c. $\sum_{k=1}^{75} 2k^2$

d. $\sum_{n=1}^{50} \left(1+n^2\right)$

e. $\sum_{m=1}^{75} \frac{2m+2}{3}$

f. $\sum_{j=1}^{20} (3j-4)$

g. $\sum_{p=1}^{35} \left(2p+p^2\right)$

h. $\sum_{n=0}^{40} \left(n^2+3n-1\right)$

T **51–54. Riemann sums for larger values of *n*** *Complete the following steps for the given function f and interval.*

> ***a.*** *For the given value of n, use sigma notation to write the left, right, and midpoint Riemann sums. Then evaluate each sum using a calculator.*
>
> ***b.*** *Based on the approximations found in part (a), estimate the area of the region bounded by the graph of f and the x-axis on the interval.*

51. $f(x) = 3\sqrt{x}$ on $[0, 4]$; $n = 40$

52. $f(x) = x^2 + 1$ on $[-1, 1]$; $n = 50$

53. $f(x) = x^2 - 1$ on $[2, 5]$; $n = 75$

54. $f(x) = \cos 2x$ on $\left[0, \frac{\pi}{4}\right]$; $n = 60$

T **55–58. Approximating areas with a calculator** *Use a calculator and right Riemann sums to approximate the area of the given region. Present your calculations in a table showing the approximations for $n = 10$, 30, 60, and 80 subintervals. Make a conjecture about the limit of Riemann sums as $n \to \infty$.*

55. The region bounded by the graph of $f(x) = 12 - 3x^2$ and the x-axis on the interval $[-1, 1]$.

56. The region bounded by the graph of $f(x) = 3x^2 + 1$ and the x-axis on the interval $[-1, 1]$.

57. The region bounded by the graph of $f(x) = \frac{1-\cos x}{2}$ and the x-axis on the interval $[-\pi, \pi]$.

58. The region bounded by the graph of $f(x) = (\sin x + \cos x)$ and the x-axis on the interval $[0, \pi/2]$.

59. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.

a. Consider the linear function $f(x) = 2x + 5$ and the region bounded by its graph and the x-axis on the interval [3, 6]. Suppose the area of this region is approximated using midpoint Riemann sums. Then the approximations give the exact area of the region for any number of subintervals.

b. A left Riemann sum always overestimates the area of a region bounded by a positive increasing function and the x-axis on an interval $[a, b]$.

c. For an increasing or decreasing nonconstant function on an interval $[a, b]$ and a given value of n, the value of the midpoint Riemann sum always lies between the values of the left and right Riemann sums.

T **60. Riemann sums for a semicircle** Let $f(x) = \sqrt{1 - x^2}$.

a. Show that the graph of f is the upper half of a circle of radius 1 centered at the origin.

b. Estimate the area between the graph of f and the x-axis on the interval $[-1, 1]$ using a midpoint Riemann sum with $n = 25$.

c. Repeat part (b) using $n = 75$ rectangles.

d. What happens to the midpoint Riemann sums on $[-1, 1]$ as $n \to \infty$?

T **61–64. Sigma notation for Riemann sums** *Use sigma notation to write the following Riemann sums. Then evaluate each Riemann sum using Theorem 5.1 or a calculator.*

61. The right Riemann sum for $f(x) = x + 1$ on [0, 4] with $n = 50$

62. The left Riemann sum for $f(x) = \frac{3}{x}$ on [1, 3] with $n = 30$

63. The midpoint Riemann sum for $f(x) = x^3$ on [3, 11] with $n = 32$

64. The midpoint Riemann sum for $f(x) = 1 + \cos \pi x$ on [0, 2] with $n = 50$

65–68. Identifying Riemann sums *Fill in the blanks with an interval and a value of n.*

65. $\sum_{k=1}^{4} f(1 + k) \cdot 1$ is a right Riemann sum for f on the interval [___, ___] with $n =$ ___.

66. $\sum_{k=1}^{4} f(2 + k) \cdot 1$ is a right Riemann sum for f on the interval [___, ___] with $n =$ ___.

67. $\sum_{k=1}^{4} f(1.5 + k) \cdot 1$ is a midpoint Riemann sum for f on the interval [___, ___] with $n =$ ___.

68. $\sum_{k=1}^{8} f\left(1.5 + \frac{k}{2}\right) \cdot \frac{1}{2}$ is a left Riemann sum for f on the interval [___, ___] with $n =$ ___.

69. Approximating areas Estimate the area of the region bounded by the graph of $f(x) = x^2 + 2$ and the x-axis on [0, 2] in the following ways.

a. Divide [0, 2] into $n = 4$ subintervals and approximate the area of the region using a left Riemann sum. Illustrate the solution geometrically.

a. Divide [0, 2] into $n = 4$ subintervals and approximate the area of the region using a midpoint Riemann sum. Illustrate the solution geometrically.

a. Divide [0, 2] into $n = 4$ subintervals and approximate the area of the region using a right Riemann sum. Illustrate the solution geometrically.

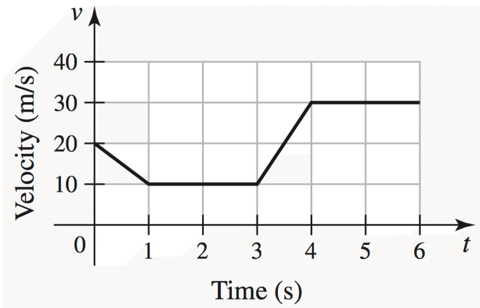
70. Displacement from a velocity graph Consider the velocity function for an object moving along a line (see figure).

a. Describe the motion of the object over the interval [0, 6].

b. Use geometry to find the displacement of the object between $t = 0$ and $t = 3$.

c. Use geometry to find the displacement of the object between $t = 3$ and $t = 5$.

d. Assuming the velocity remains 30 m/s, for $t \geq 4$, find the function that gives the displacement between $t = 0$ and any time $t \geq 4$.



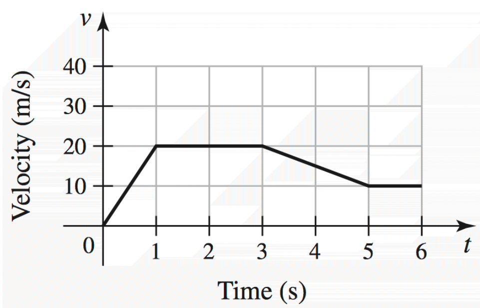
71. Displacement from a velocity graph Consider the velocity function for an object moving along a line (see figure).

a. Describe the motion of the object over the interval [0, 6].

b. Use geometry to find the displacement of the object between $t = 0$ and $t = 2$.

c. Use geometry to find the displacement of the object between $t = 2$ and $t = 5$.

d. Assuming that the velocity remains 10 m/s, for $t \geq 5$, find the function that gives the displacement between $t = 0$ and any time $t \geq 5$.



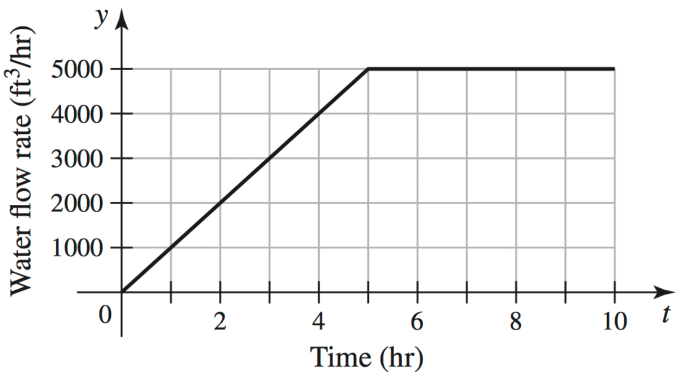
72. Flow rates Suppose a gauge at the outflow of a reservoir measures the flow rate of water in units of ft^3/hr. In Chapter 6, we show that the total amount of water that flows out of the reservoir is the area under the flow rate curve. Consider the flow rate function shown in the figure.

a. Find the amount of water (in units of ft^3) that flows out of the reservoir over the interval [0, 4].

b. Find the amount of water that flows out of the reservoir over the interval [8, 10].

c. Does more water flow out of the reservoir over the interval [0, 4] or [4, 6]?

d. Show that the units of your answer are consistent with the units of the variables on the axes.



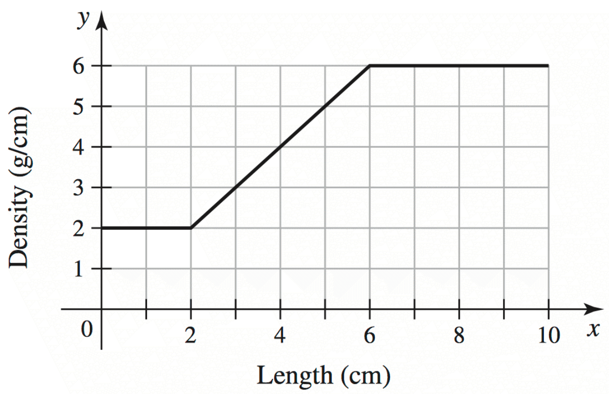
73. Mass from density A thin 10-cm rod is made of an alloy whose density varies along its length according to the function shown in the figure. Assume density is measured in units of g/cm. In Chapter 6, we show that the mass of the rod is the area under the density curve.

a. Find the mass of the left half of the rod $(0 \le x \le 5)$.

b. Find the mass of the right half of the rod $(5 \le x \le 10)$.

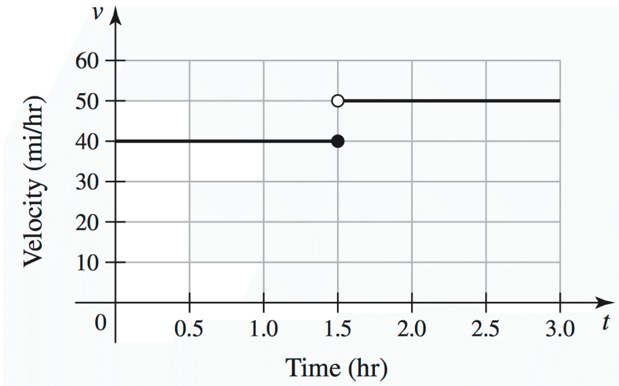
c. Find the mass of the entire rod $(0 \le x \le 10)$.

d. Find the point along the rod at which it will balance (called the center of mass).

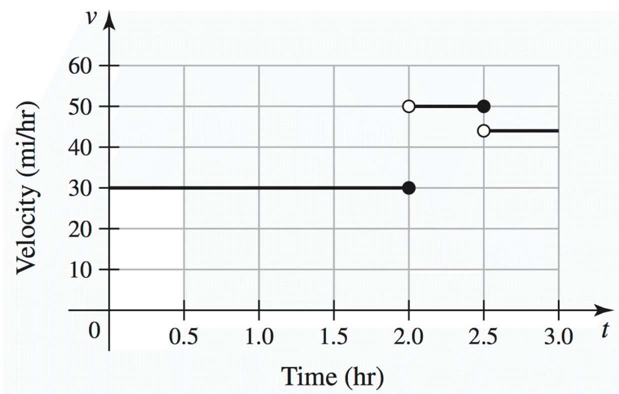


74–75. Displacement from velocity *The following functions describe the velocity of a car (in mi/hr) moving along a straight highway for a 3-hr interval. In each case, find the function that gives the displacement of the car over the interval* $[0, t]$, *where* $0 \le t \le 3$.

74. $v(t) = \begin{cases} 40 & \text{if } 0 \le t \le 1.5 \\ 50 & \text{if } 1.5 < t \le 3 \end{cases}$



75. $v(t) = \begin{cases} 30 & \text{if } 0 \le t \le 2 \\ 50 & \text{if } 2 < t \le 2.5 \\ 44 & \text{if } 2.5 < t \le 3 \end{cases}$



T **76–77. Functions with absolute value** *Use a calculator and the method of your choice to approximate the area of the following regions. Present your calculations in a table, showing approximations using* $n = 16$, 32, *and* 64 *subintervals. Make a conjecture about the limits of the approximations.*

76. The region bounded by the graph of $f(x) = |25 - x^2|$ and the x-axis on the interval $[0, 10]$

77. The region bounded by the graph of $f(x) = |1 - x^3|$ and the x-axis on the interval $[-1, 2]$

Explorations and Challenges »

78. Riemann sums for constant functions Let $f(x) = c$, where $c > 0$, be a constant function on $[a, b]$. Prove that any Riemann sum for any value of n gives the exact area of the region between the graph of f and the x-axis on $[a, b]$.

79. Riemann sums for linear functions Assume the linear function $f(x) = mx + c$ is positive on the interval $[a, b]$. Prove that the midpoint Riemann sum with any value of n gives the exact area of the region between the graph of f and the x-axis on $[a, b]$.

80. Shape of the graph for left Riemann sums Suppose a left Riemann sum is used to approximate the area of the region bounded by the graph of a positive function and the x-axis on the interval $[a, b]$. Fill in the following table to indicate whether the resulting approximation underestimates or overestimates the exact area in the four cases shown. Use a sketch to explain your reasoning in each case.

	Increasing on $[a, b]$	**Decreasing on $[a, b]$**
Concave up on $[a, b]$		
Concave down on $[a, b]$		

81. Shape of the graph for right Riemann sums Suppose a right Riemann sum is used to approximate the area of the region bounded by the graph of a positive function and the x-axis on the interval $[a, b]$. Fill in the following table to indicate whether the resulting approximation underestimates or overestimates the exact area in the four cases shown. Use a sketch to explain your reasoning in each case.

	Increasing on $[a, b]$	**Decreasing on $[a, b]$**
Concave up on $[a, b]$		
Concave down on $[a, b]$		