4.9 Antiderivatives

The goal of differentiation is to find the derivative f' of a given function f. The reverse process, called *antidifferentiation*, is equally important: Given a function f, we look for an *antiderivative* function F whose derivative is f; that is, a function F such that $F' = f$.

> **DEFINITION** **Antiderivative**
>
> A function F is an **antiderivative** of f on an interval I provided $F'(x) = f(x)$ for all x in I.

In this section, we revisit derivative formulas developed in previous chapters to discover corresponding antiderivative formulas.

Thinking Backward »

Consider the derivative formula $\frac{d}{dx}(x) = 1$. It implies that an antiderivative of $f(x) = 1$ is $F(x) = x$ because $F'(x) = f(x)$. Using the same logic, we can write

$$\frac{d}{dx}\left(x^2\right) = 2x \quad \Longrightarrow \quad \text{an antiderivative of } f(x) = 2x \text{ is } F(x) = x^2 \text{ and}$$
$$\frac{d}{dx}(\sin x) = \cos x \quad \Longrightarrow \quad \text{an antiderivative of } f(x) = \cos x \text{ is } F(x) = \sin x.$$

Each of these proposed antiderivative formulas is easily checked by showing that $F' = f$.

Quick Check 1 Verify by differentiation that x^4 is an antiderivative of $4x^3$. ◆

Answer »

$$\frac{d}{dx}\left(x^4\right) = 4x^3$$

An immediate question arises: Does a function have more than one antiderivative? To answer this question, let's focus on $f(x) = 1$ and the antiderivative, $F(x) = x$. Because the derivative of a constant C is zero, we see that $F(x) = x + C$ is also an antiderivative of $f(x) = 1$, which is easy to check:

$$F'(x) = \frac{d}{dx}(x + C) = 1 = f(x).$$

Therefore, $f(x) = 1$ actually has an infinite number of antiderivatives. For the same reason, any function of the form $F(x) = x^2 + C$ is an antiderivative of $f(x) = 2x$, and any function of the form $F(x) = \sin x + C$ is an antiderivative of $f(x) = \cos x$, where C is an arbitrary constant.

We might ask whether there are still *more* antiderivatives of a given function. The following theorem provides the answer.

THEOREM 4.14 **The Family of Antiderivatives**

Let F be any antiderivative of f on an interval I. Then *all* the antiderivatives of f on I have the form $F + C$, where C is an arbitrary constant.

Proof: Suppose F and G are antiderivatives of f on an interval I. Then $F' = f$ and $G' = f$, which implies that $F' = G'$ on I. Theorem 4.6 states that functions with equal derivatives differ by a constant. Therefore, $G = F + C$, and all antiderivatives of f have the form $F + C$, where C is an arbitrary constant. ◆

Theorem 4.14 says that while there are infinitely many antiderivatives of a function, they are all of one family, namely, those functions of the form $F + C$. Because the antiderivatives of a particular function differ by a constant, the graphs of the antiderivatives are vertical translations of one another (**Figure 4.82**).

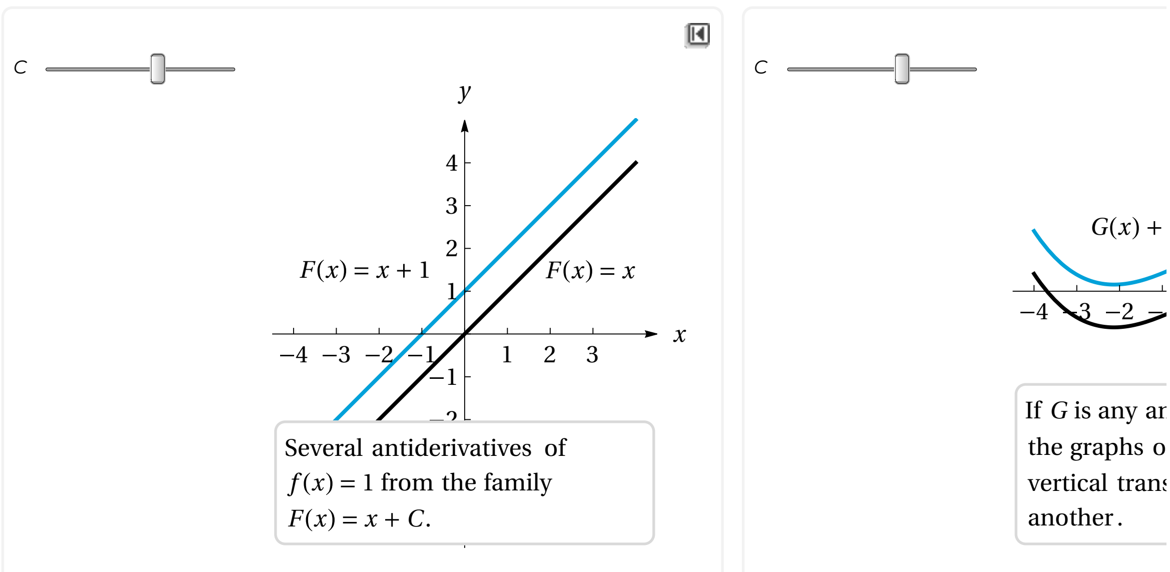


Figure 4.82

EXAMPLE 1 Finding antiderivatives

Use what you know about derivatives to find all antiderivatives of the following functions.

a. $f(x) = 3x^2$

b. $f(x) = -\dfrac{9}{x^{10}}$

c. $f(t) = \sin t$

SOLUTION »

a. Note that $\dfrac{d}{dx}\left(x^3\right) = 3x^2$. Therefore, an antiderivative of $f(x) = 3x^2$ is x^3. By Theorem 4.15, the complete family of antiderivatives is $F(x) = x^3 + C$, where C is an arbitrary constant.

b. Because $\dfrac{d}{dx}\left(x^{-9}\right) = -9x^{-10} = -9/x^{10}$, all antiderivatives of f are of the form $F(x) = x^{-9} + C$, where C is an arbitrary constant.

c. Recall that $\frac{d}{dt}(\cos t) = -\sin t$. We seek a function whose derivative is $\sin t$, not $-\sin t$. Observing that $\frac{d}{dt}(-\cos t) = \sin t$, it follows that the antiderivatives of $\sin t$ are $F(t) = -\cos t + C$, where C is an arbitrary constant.

Related Exercises 12–13, 17 ◆

Quick Check 2 Find the family of antiderivatives for each of $f(x) = \frac{1}{2\sqrt{x}}$, $g(x) = 4x^3$, and $h(x) = \sec^2 x$.

◆

Answer »

$\sqrt{x} + C$, $x^4 + C$, $\tan x + C$

Indefinite Integrals »

The notation $\frac{d}{dx}(f(x))$ means *take the derivative of* $f(x)$ with respect to x. We need analogous notation for antiderivatives. For historical reasons that become apparent in the next chapter, the notation that means *find the antiderivatives of* f is the **indefinite integral** $\int f(x)\,dx$. Every time an indefinite integral sign $\int$ appears, it is followed by a function called the **integrand**, which in turn is followed by the differential dx. For now dx simply means that x is the independent variable, or the **variable of integration**. The notation $\int f(x)\,dx$ represents *all* of the antiderivatives of f. When the integrand is a function of a variable different from x—say, $g(t)$—then we write $\int g(t)\,dt$ to represent the antiderivatives of g.

Using this new notation, the three results of Example 1 are written

$$\int 3x^2\,dx = x^3 + C, \quad \int\left(-\frac{9}{x^{10}}\right)dx = x^{-9} + C, \quad \text{and} \quad \int \sin x\,dx = -\cos x + C,$$

where C is an arbitrary constant called a **constant of integration**. The derivative formulas presented earlier in the text may be written in terms of indefinite integrals. We begin with the Power Rule.

THEOREM 4.15 **Power Rule for Indefinite Integrals**

$$\int x^p\,dx = \frac{x^{p+1}}{p+1} + C,$$

where $p \neq -1$ is a real number and C is an arbitrary constant.

Note »

Proof: The theorem says that the antiderivatives of $f(x) = x^p$ have the form $F(x) = \frac{x^{p+1}}{p+1} + C$. Differentiating F, we verify that $F'(x) = f(x)$, provided $p \neq -1$:

$$F'(x) = \frac{d}{dx}\left(\frac{x^{p+1}}{p+1} + C\right)$$
$$= \frac{d}{dx}\left(\frac{x^{p+1}}{p+1}\right) + \underbrace{\frac{d}{dx} C}_{0}$$
$$= \frac{(p+1)\, x^{(p+1)-1}}{p+1} + 0 = x^p. \blacklozenge$$

Note »

Theorems 3.4 and 3.5 (Section 3.3) state the Constant Multiple and Sum Rules for derivatives. Here are the corresponding antiderivative rules, which are proved by differentiation.

THEOREM 4.16 **Constant Multiple and Sum Rules**

Constant Multiple Rule:

$$\int c\, f(x)\, dx = c \int f(x)\, dx, \text{ for real numbers } c$$

Sum Rule:

$$\int (f(x) + g(x))\, dx = \int f(x)\, dx + \int g(x)\, dx$$

The following example shows how Theorems 4.15 and 4.16 are used.

EXAMPLE 2 Indefinite integrals

Determine the following indefinite integrals.

a. $\int \left(3x^5 + 2 - 5\sqrt{x}\right) dx$

b. $\int \left(\frac{4x^{19} - 5x^{-8}}{x^2}\right) dx$

c. $\int \left(z^2 + 1\right)(2z - 5)\, dz$

SOLUTION »

a.

Note »

$$\begin{aligned}\int\left(3x^5+2-5\sqrt{x}\right)dx &= \int 3x^5\,dx+\int 2\,dx-\int 5x^{1/2}\,dx && \text{Sum Rule}\\ &= 3\int x^5\,dx+2\int dx-5\int x^{1/2}\,dx && \text{Constant Multiple Rule}\\ &= 3\cdot\frac{x^6}{6}+2\cdot x-5\cdot\frac{x^{3/2}}{3/2}+C && \text{Power Rule}\\ &= \frac{x^6}{2}+2x+\frac{10}{3}x^{3/2}+C && \text{Simplify.}\end{aligned}$$

Note »

b.

$$\begin{aligned}\int\left(\frac{4x^{19}-5x^{-8}}{x^2}\right)dx &= \int\left(4x^{17}-5x^{-10}\right)dx && \text{Simplify the integrand.}\\ &= 4\int x^{17}\,dx-5\int x^{-10}\,dx && \text{Sum and Constant Multiple Rules}\\ &= 4\cdot\frac{x^{18}}{18}-5\cdot\frac{x^{-9}}{-9}+C && \text{Power Rule}\\ &= \frac{2x^{18}}{9}+\frac{5x^{-9}}{9}+C && \text{Simplify.}\end{aligned}$$

c.

$$\begin{aligned}\int\left(z^2+1\right)(2z-5)\,dz &= \int\left(2z^3-5z^2+2z-5\right)dz && \text{Expand integrand.}\\ &= \frac{1}{2}z^4-\frac{5}{3}z^3+z^2-5z+C && \text{Integrate each term.}\end{aligned}$$

Note »

All these results should be checked by differentiation.

Related Exercises 24, 25, 31, 35 ◆

Indefinite Integrals of Trigonometric Functions »

We used a familiar derivative formula in Example 1c to find the antiderivative of sin x. Our goal in this section is to write the other derivative results for trigonometric functions as indefinite integrals.

EXAMPLE 3 Indefinite integrals of trigonometric functions

Evaluate the following indefinite integrals.

a. $\int \sec^2 x\,dx$

b. $\int (2x+3\cos x)\,dx$

c. $\int \frac{\sin x}{\cos^2 x}\,dx$

SOLUTION »

a. The derivative result $\frac{d}{dx}(\tan x) = \sec^2 x$ is reversed to produce the indefinite integral $\int \sec^2 x \, dx = \tan x + C$.

Note »

b. We first split the integral into two integrals using Theorem 4.16:

$$\int (2x + 3\cos x)\, dx = 2\int x\, dx + 3\int \cos x\, dx. \quad \text{Sum and Constant Multiple Rules}$$

The first of these new integrals is handled by the Power Rule, and the second integral is evaluated by reversing the derivative result $\frac{d}{dx}(\sin x) = \cos x$:

$$\begin{aligned} \int (2x + 3\cos x)\, dx &= 2 \cdot \frac{x^2}{2} + 3\sin x + C. && \text{Power Rule; } \frac{d}{dx}(\sin x) = \cos x \Longrightarrow \int \cos x\, dx = \sin x + C \\ &= x^2 + 3\sin x + C. && \text{Simplify.} \end{aligned}$$

c. When we rewrite the integrand, a familiar derivative formula emerges:

$$\begin{aligned} \int \frac{\sin x}{\cos^2 x}\, dx &= \int \underbrace{\frac{1}{\cos x}}_{\sec x} \cdot \underbrace{\frac{\sin x}{\cos x}}_{\tan x}\, dx \\ &= \int \sec x \tan x\, dx && \text{Rewrite the integrand.} \\ &= \sec x + C. && \frac{d}{dx}(\sec x) = \sec x \tan x \Longrightarrow \int \sec x \tan x\, dx = \sec x + C \end{aligned}$$

Related Exercises 40, 41, 47 ◆

The ideas illustrated in Example 3 are used to obtain the integrals in Table 4.9, where we assume C is an arbitrary constant. Example 4 provides additional integrals involving trigonometric functions.

Table 4.9 Indefinite Integrals of Trigonometric Functions

1.	$\frac{d}{dx}(\sin x) = \cos x$	$\Longrightarrow$	$\int \cos x \, dx = \sin x + C$
2.	$\frac{d}{dx}(\cos x) = -\sin x$	$\Longrightarrow$	$\int \sin x \, dx = -\cos x + C$
3.	$\frac{d}{dx}(\tan x) = \sec^2 x$	$\Longrightarrow$	$\int \sec^2 x \, dx = \tan x + C$
4.	$\frac{d}{dx}(\cot x) = -\csc^2 x$	$\Longrightarrow$	$\int \csc^2 x \, dx = -\cot x + C$
5.	$\frac{d}{dx}(\sec x) = \sec x \tan x$	$\Longrightarrow$	$\int \sec x \tan x \, dx = \sec x + C$
6.	$\frac{d}{dx}(\csc x) = -\csc x \cot x$	$\Longrightarrow$	$\int \csc x \cot x \, dx = -\csc x + C$

Quick Check 3 Use differentiation to verify result 6 in Table 4.9: $\int \csc x \cot x \, dx = -\csc x + C$. ◆

Answer »

$$\frac{d}{dx}(-\csc x + C) = \csc x \cot x$$

EXAMPLE 4 Indefinite integrals involving trigonometric functions

Determine the following indefinite integrals.

a. $\int \left(\frac{2}{\pi} \sin x - 2 \csc^2 x \right) dx$

b. $\int \frac{4 \cos x + \sin^2 x}{\sin^2 x} \, dx$

SOLUTION »

a. Splitting up the integral (Theorem 4.16) and then using Table 4.9, we have

$$\begin{aligned} \int \left(\frac{2}{\pi} \sin x - 2 \csc^2 x \right) dx &= \frac{2}{\pi} \int \sin x \, dx - 2 \int \csc^2 x \, dx && \text{Sum and Constant Multiple Rules} \\ &= \frac{2}{\pi}(-\cos x) - 2(-\cot x) + C && \text{Results (2) and (4), Table 4.9} \\ &= 2 \cot x - \frac{2}{\pi} \cos x + C. && \text{Simplify.} \end{aligned}$$

b. Again, we split up the integral and then rewrite the first term in the integrand so that it matches result (6) in Table 4.9:

$$\int \frac{4\cos x + \sin^2 x}{\sin^2 x}\,dx = \int\left(\frac{4}{\sin x}\cdot\frac{\cos x}{\sin x} + \frac{\sin^2 x}{\sin^2 x}\right)dx$$

$$= 4\int \csc x \cot x\,dx + \int dx \qquad \text{Rewrite the integrand; Theorem 4.16.}$$

$$= -4\csc x + x + C. \qquad \text{Result (6), Table 4.9; } \int dx = x + C$$

Related Exercises 45, 50 ◆

Introduction to Differential Equations »

Motion Problems Revisited »

Antiderivatives allow us to revisit the topic of one-dimensional motion introduced in Section 3.6. Suppose the position of an object that moves along a line relative to an origin is $s(t)$, where $t \geq 0$ measures elapsed time. The velocity of the object is $v(t) = s'(t)$, which may now be read in terms of antiderivatives: *The position function is an antiderivative of the velocity*. If we are given the velocity function of an object and its position at a particular time, we can determine its position at all future times by solving an initial value problem.

Quick Check 5 Position is an antiderivative of velocity. But there are infinitely many antiderivatives that differ by a constant. Explain how two objects can have the same velocity function but two different position functions. ◆

Answer »

We also know that the acceleration $a(t)$ of an object moving in one dimension is the rate of change of the velocity, which means $a(t) = v'(t)$. In antiderivative terms, this says that the velocity is an antiderivative of the acceleration. Thus, if we are given the acceleration of an object and its velocity at a particular time, we can determine its velocity at all times. These ideas lie at the heart of modeling the motion of objects.

Note »

Initial Value Problems for Velocity and Position

Suppose an object moves along a line with a (known) velocity $v(t)$ for $t \geq 0$. Then its position is found by solving the initial value problem

$$s'(t) = v(t), \quad s(0) = s_0, \quad \text{where } s_0 \text{ is the (known) initial position.}$$

If the (known) acceleration of the object $a(t)$ is given, then its velocity is found by solving the initial value problem

$$v'(t) = a(t), \quad v(0) = v_0, \quad \text{where } v_0 \text{ is the (known) initial velocity.}$$

EXAMPLE 6 A race

Runner A begins at the point $s(0) = 0$ and runs on a straight and level road with velocity $v(t) = 2t$. Runner B begins with a head start at the point $S(0) = 8$ and runs with velocity $V(t) = 2$. Find the positions of the runners for $t \geq 0$ and determine who is ahead at $t = 6$ time units.

SOLUTION »

EXAMPLE 7 Motion with gravity

Neglecting air resistance, the motion of an object moving vertically near Earth's surface is determined by the acceleration due to gravity, which is approximately $9.8\ \text{m}/\text{s}^2$. Suppose a stone is thrown vertically upward at $t = 0$ with a velocity of 40 m/s from the edge of a cliff that is 100 m above a river.

a. Find the velocity $v(t)$ of the object, for $t \geq 0$.

b. Find the position $s(t)$ of the object, for $t \geq 0$.

c. Find the maximum height of the object above the river.

d. With what speed does the object strike the river?

Note »

The acceleration due to gravity at Earth's surface is approximately $g = 9.8\ \text{m}/\text{s}^2$, or $g = 32\ \text{ft}/\text{s}^2$. It varies even at sea level from about 9.8640 at the poles to 9.7982 at the equator. The equation $v'(t) = -g$ is an instance of Newton's Second Law of Motion, and assumes no other forces (such as air resistance) are present.

SOLUTION »

We establish a coordinate system in which the positive s-axis points vertically upward with $s = 0$ corresponding to the river (**Figure 4.86**). Let $s(t)$ be the position of the stone measured relative to the river for $t \geq 0$. The initial velocity of the stone is $v(0) = 40\ \text{m}/\text{s}$ and the initial position of the stone is $s(0) = 100\ \text{m}$.

a. The acceleration due to gravity points in the *negative* s-direction. Therefore, the initial value problem governing the motion of the object is

$$\text{acceleration} = v'(t) = -9.8, \quad v(0) = 40.$$

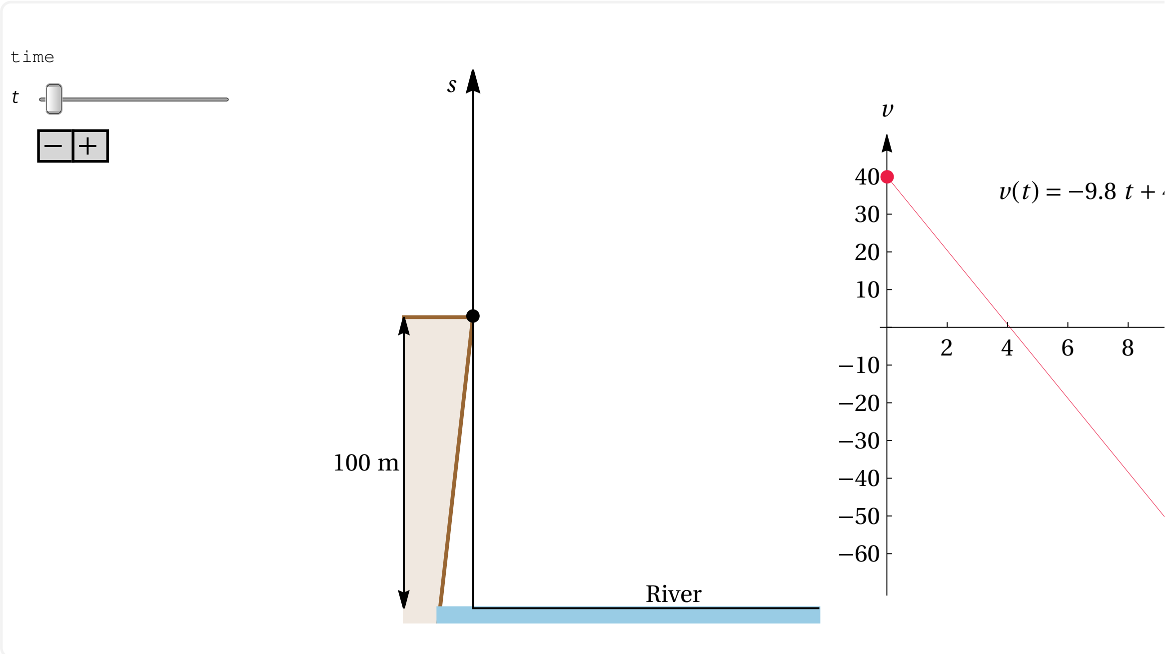
The antiderivatives of -9.8 are $v(t) = -9.8\,t + C$. The initial condition $v(0) = 40$ gives $C = 40$. Therefore, the velocity of the stone is

$$v(t) = -9.8\,t + 40.$$

As shown in **Figure 4.87**, the velocity decreases from its initial value $v(0) = 40$ until it reaches zero at the high point of the trajectory. This point is reached when

$$v(t) = -9.8\,t + 40 = 0,$$

or when $t \approx 4.1$ s. For $t > 4.1$, the velocity becomes increasingly negative as the stone falls to Earth.



Figures 4.86 and 4.87

b. Knowing the velocity of the stone, we can determine its position. The position function satisfies the initial value problem

$$v(t) = s'(t) = -9.8\,t + 40, \quad s(0) = 100.$$

The antiderivatives of $-9.8\,t + 40$ are

$$s(t) = -4.9\,t^2 + 40\,t + C.$$

The initial condition $s(0) = 100$ implies $C = 100$, so the position function of the stone is

$$s(t) = -4.9\,t^2 + 40\,t + 100,$$

as shown in **Figure 4.88**. The parabolic graph of the position function is not the actual trajectory of the stone; the stone moves vertically along the s-axis.

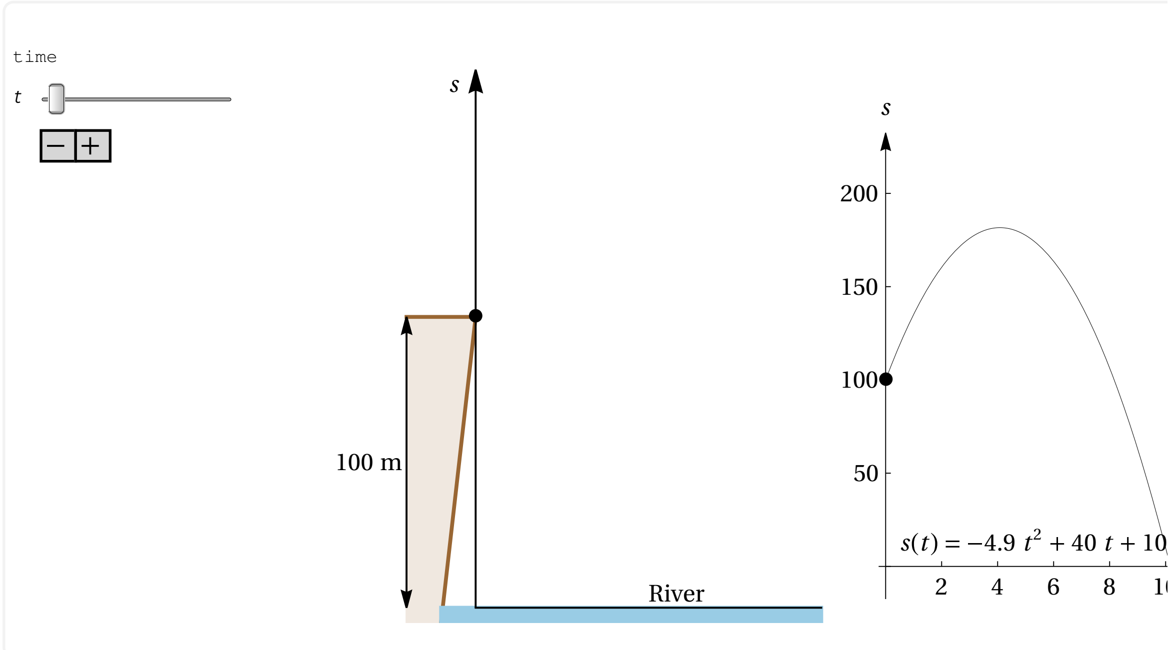


Figure 4.88

c. The position function of the stone increases for $0 < t < 4.1$. At $t \approx 4.1$ the stone reaches a high point of $s(4.1) \approx 182$ m.

d. For $t > 4.1$, the position function decreases, and the stone strikes the river when $s(t) = 0$. The roots of this equation are $t \approx 10.2$ and $t \approx -2.0$. Only the first root is relevant because the motion takes place for $t \geq 0$. Therefore, the stone strikes the ground at $t \approx 10.2$ s. Its speed (in m/s) at this instant is $|v(10.2)| \approx |-60| = 60$ m/s.

Related Exercises 89–90 ◆

Exercises »

Getting Started »

Practice Exercises »

11–22. Finding antiderivatives *Find all the antiderivatives of the following functions. Check your work by taking derivatives.*

11. $f(x) = 5\,x^4$

12. $g(x) = 11\,x^{10}$

13. $f(x) = 2\sin x + 1$

14. $g(x) = -4\cos x - x$

15. $p(x) = 3\sec^2 x$

16. $q(s) = \csc^2 s$

17. $f(y) = -\dfrac{2}{y^3}$

18. $h(z) = -6\,z^{-7}$

19. $f(x) = \dfrac{1}{\sqrt{x}}$

20. $h(y) = \sqrt[3]{y}$

21. $f(x) = \dfrac{7}{2}\,x^{5/2}$

22. $f(t) = \pi$

23–68. Indefinite integrals *Determine the following indefinite integrals. Check your work by differentiation.*

23. $\int (3\,x^5 - 5\,x^9)\,dx$

24. $\int (3\,u^{-2} - 4\,u^2 + 1)\,du$

25. $\int \left(4\,\sqrt{x} - \dfrac{4}{\sqrt{x}}\right) dx$

26. $\int \left(\dfrac{5}{t^2} + 4\,t^2\right) dt$

27. $\int (5\,s + 3)^2\,ds$

28. $\int 5\,m\,(12\,m^3 - 10\,m)\,dm$

29. $\int (3\,x^{1/3} + 4\,x^{-1/3} + 6)\,dx$

30. $\int 5\,\sqrt[4]{x}\,\,dx$

31. $\int (3\,x + 1)\,(4 - x)\,dx$

32. $\int (4\,z^{1/3} - z^{-1/3})\,dz$

33. $\int \left(\dfrac{3}{x^4} + 2 - \dfrac{3}{x^2}\right) dx$

34. $\int \sqrt[5]{r^2}\,\,dr$

35. $\displaystyle\int \frac{4x^4 - 6x^2}{x}\,dx$

36. $\displaystyle\int \frac{12t^8 - t}{t^{3/2}}\,dt$

37. $\displaystyle\int \frac{x^2 - 36}{x - 6}\,dx$

38. $\displaystyle\int \frac{y^3 - 9y^2 + 20y}{y - 4}\,dy$

39. $\displaystyle\int \left(\csc^2\theta + 2\theta^2 - 3\theta\right)d\theta$

40. $\displaystyle\int \left(\csc^2\theta + 1\right)d\theta$

41. $\displaystyle\int \frac{2 + 3\cos y}{\sin^2 y}\,dy$

42. $\displaystyle\int \sin t\,(4\csc t - \cot t)\,dt$

43. $\displaystyle\int \left(\sec^2 x - 1\right)dx$

44. $\displaystyle\int \frac{\sec^3 v - \sec^2 v}{\sec v - 1}\,dv$

45. $\displaystyle\int \left(\sec^2\theta + \sec\theta\tan\theta\right)d\theta$

46. $\displaystyle\int \frac{\sin\theta - 1}{\cos^2\theta}\,d\theta$

47. $\displaystyle\int \left(3t^2 + 2\csc^2 t\right)dt$

48. $\displaystyle\int \csc x\,(\cot x - \csc x)\,dx$

49. $\displaystyle\int \sec\theta\,(\tan\theta + \sec\theta + \cos\theta)\,d\theta$

50. $\displaystyle\int \frac{\csc^3 x + 1}{\csc x}\,dx$

51. $\displaystyle\int \frac{\tan x + \sec x}{\sec x}\,dx$

52. $\int \left(\sqrt[3]{x^2} + \sqrt{x^3} \right) dx$

53. $\int \sqrt{x} \left(2x^6 - 4\sqrt[3]{x} \right) dx$

54. $\int \frac{16\cos^2 w - 81\sin^2 w}{4\cos w - 9\sin w} dw$

55–60. Particular antiderivatives *For the following functions f, find the antiderivative F that satisfies the given condition.*

55. $f(x) = x^5 - 2x^2 + 1$; $F(0) = 1$

56. $f(x) = 4\sqrt{x} + 6$; $F(1) = 8$

57. $f(x) = 8x^3 + \sin x$; $F(0) = 2$

58. $f(t) = \sec^2 t$; $F\left(\frac{\pi}{4}\right) = 1$, $-\pi/2 < t < \pi/2$

59. $f(v) = \sec v \tan v$; $F(0) = 2$, $-\pi/2 < t < \pi/2$

60. $f(\theta) = 2\sin\theta - 4\cos\theta$; $F\left(\frac{\pi}{4}\right) = 2$

61–68. Solving initial value problems *Find the solution of the following initial value problems.*

61. $f'(x) = 2x - 3$; $f(0) = 4$

62. $g'(x) = 7x^6 - 4x^3 + 12$; $g(1) = 24$

63. $g'(x) = 7x\left(x^6 - \frac{1}{7}\right)$; $g(1) = 2$

64. $h'(t) = 1 + 6\sin t$; $h\left(\frac{\pi}{3}\right) = -3$

65. $f'(u) = 4(\cos u - \sin u)$; $f\left(\frac{\pi}{2}\right) = 0$

66. $f'(x) = \sin x + \cos x + 1$; $f(\pi) = 3$

67. $y'(\theta) = \frac{\sqrt{2}\cos^3\theta + 1}{\cos^2\theta}$; $y\left(\frac{\pi}{4}\right) = 3$, $-\pi/2 < \theta < \pi/2$

68. $v'(x) = 4x^{1/3} + 2x^{-1/3}$; $v(8) = 40$, $x > 0$

T **69–72. Graphing general solutions** *Graph several functions that satisfy each of the following differential equations. Then find and graph the particular function that satisfies the given initial condition.*

69. $f'(x) = 2x - 5$; $f(0) = 4$

70. $f'(x) = 3x^2 - 1;\ f(1) = 2$

71. $f'(x) = 3x + \sin x;\ f(0) = 3$

72. $f'(x) = \cos x - \sin x + 2;\ f(0) = 1$

73–78. Velocity to position *Given the following velocity functions of an object moving along a line, find the position function with the given initial position.*

73. $v(t) = 2t + 4;\ s(0) = 0$

74. $v(t) = 5;\ s(0) = 4$

75. $v(t) = 2\sqrt{t};\ s(0) = 1$

76. $v(t) = 2\cos t;\ s(0) = 0$

77. $v(t) = 6t^2 + 4t - 10;\ s(0) = 0$

78. $v(t) = 4t + \sin t;\ s(0) = 0$

79–84. Acceleration to position *Given the following acceleration functions of an object moving along a line, find the position function with the given initial velocity and position.*

79. $a(t) = -32;\ v(0) = 20,\ s(0) = 0$

80. $a(t) = 4;\ v(0) = -3,\ s(0) = 2$

81. $a(t) = 0.2t;\ v(0) = 0,\ s(0) = 1$

82. $a(t) = 2\cos t;\ v(0) = 1,\ s(0) = 0$

83. $a(t) = 2 + 3\sin t;\ v(0) = 1,\ s(0) = 10$

84. $a(t) = -9.8;\ v(0) = 5,\ s(0) = 0$

85. A car starting at rest accelerates at 16 ft/s^2 for 5 seconds on a straight road. How far does it travel during this time?

86. A car is moving at 60 mi/hr (88 ft/s) on a straight road when the driver steps on the brake pedal and begins decelerating at a constant rate of 10 ft/s^2 for 3 seconds. How far did the car go during this 3-second interval?

87–88. Races *The velocity function and initial position of Runners A and B are given. Analyze the race that results by graphing the position functions of the runners and finding the time and positions (if any) at which they first pass each other.*

87. A: $v(t) = \sin t;\ s(0) = 0$ B: $V(t) = \cos t;\ S(0) = 0$

88. A: $v(t) = 2t;\ s(0) = 0$ B: $V(t) = 3\sqrt{t};\ S(0) = 3$

T **89–92. Motion with gravity** *Consider the following descriptions of the vertical motion of an object subject only to the acceleration due to gravity. Begin with the acceleration equation* $a(t) = v'(t) = -g$, *where* $g = 9.8\ \text{m}/\text{s}^2$.

a. *Find the velocity of the object for all relevant times.*
b. *Find the position of the object for all relevant times.*
c. *Find the time when the object reaches its highest point. What is the height?*
d. *Find the time when the object strikes the ground.*

89. A softball is popped up vertically (from the ground) with a velocity of 30 m/s.

90. A stone is thrown vertically upward with a velocity of 30 m/s from the edge of a cliff 200 m above a river.

91. A payload is released at an elevation of 400 m from a hot-air balloon that is rising at a rate of 10 m/s.

92. A payload is dropped at an elevation of 400 m from a hot-air balloon that is descending at a rate of 10 m/s.

93. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.

a. $F(x) = x^3 - 4x + 100$ and $G(x) = x^3 - 4x - 100$ are antiderivatives of the same function.
b. If $F'(x) = f(x)$, then f is an antiderivative of F.
c. If $F'(x) = f(x)$, then $\int f(x)\,dx = F(x) + C$.
d. $f(x) = x^3 + 3$ and $g(x) = x^3 - 4$ are derivatives of the same function.
e. If $F'(x) = G'(x)$, then $F(x) = G(x)$.

Explorations and Challenges »

94–97. Functions from higher derivatives *Find the function F that satisfies the following differential equations and initial conditions.*

94. $F''(x) = 1;\ F'(0) = 3,\ F(0) = 4$

95. $F''(x) = \cos x;\ F'(0) = 3,\ F(\pi) = 4$

96. $F'''(x) = 4x;\ F''(0) = 0,\ F'(0) = 1,\ F(0) = 3$

97. $F'''(x) = 672x^5 + 24x;\ F''(0) = 0,\ F'(0) = 2,\ F(0) = 1$

98. **Mass on a spring** A mass oscillates up and down on the end of a spring. Find its position s relative to the equilibrium position if its acceleration is $a(t) = 2\sin t$ and its initial velocity and position are $v(0) = 3$ and $s(0) = 0$, respectively.

99. **Flow rate** A large tank is filled with water when an outflow valve is opened at $t = 0$. Water flows out at a rate, in gal/min, given by $Q'(t) = 0.1\left(100 - t^2\right)$, for $0 \le t \le 10$.

a. Find the amount of water $Q(t)$ that has flowed out of the tank after t minutes, given the initial condition $Q(0) = 0$.
b. Graph the flow function Q, for $0 \le t \le 10$.
c. How much water flows out of the tank in 10 min?

100. General head start problem Suppose object A is located at $s = 0$ at time $t = 0$ and starts moving along the s-axis with a velocity given by $v(t) = 2\,a\,t$, where $a > 0$. Object B is located at $s = c > 0$ at $t = 0$ and starts moving along the s-axis with a constant velocity given by $V(t) = b > 0$. Show that A always overtakes B at time

$$t = \frac{b + \sqrt{b^2 + 4\,a\,c}}{2\,a}.$$

101–104. Verifying indefinite integrals *Verify the following indefinite integrals by differentiation. These integrals are derived in later chapters.*

101. $\displaystyle\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx = 2\sin\sqrt{x} + C$

102. $\displaystyle\int \frac{x}{\sqrt{x^2+1}}\,dx = \sqrt{x^2+1} + C$

103. $\displaystyle\int x^2\cos x^3\,dx = \frac{1}{3}\sin x^3 + C$

104. $\displaystyle\int \frac{x}{\left(x^2-1\right)^2}\,dx = -\frac{1}{2\left(x^2-1\right)} + C$