4.6 Linear Approximation and Differentials

Imagine plotting a smooth curve with a graphing utility. Now pick a point P on the curve, draw the line tangent to the curve at P, and zoom in on it several times. As you successively enlarge the curve near P, it looks more and more like the tangent line (**Figure 4.63a**). This fundamental observation–that smooth curves appear straighter on smaller scales–is the basis of many important mathematical ideas, one of which is *linear approximation*.

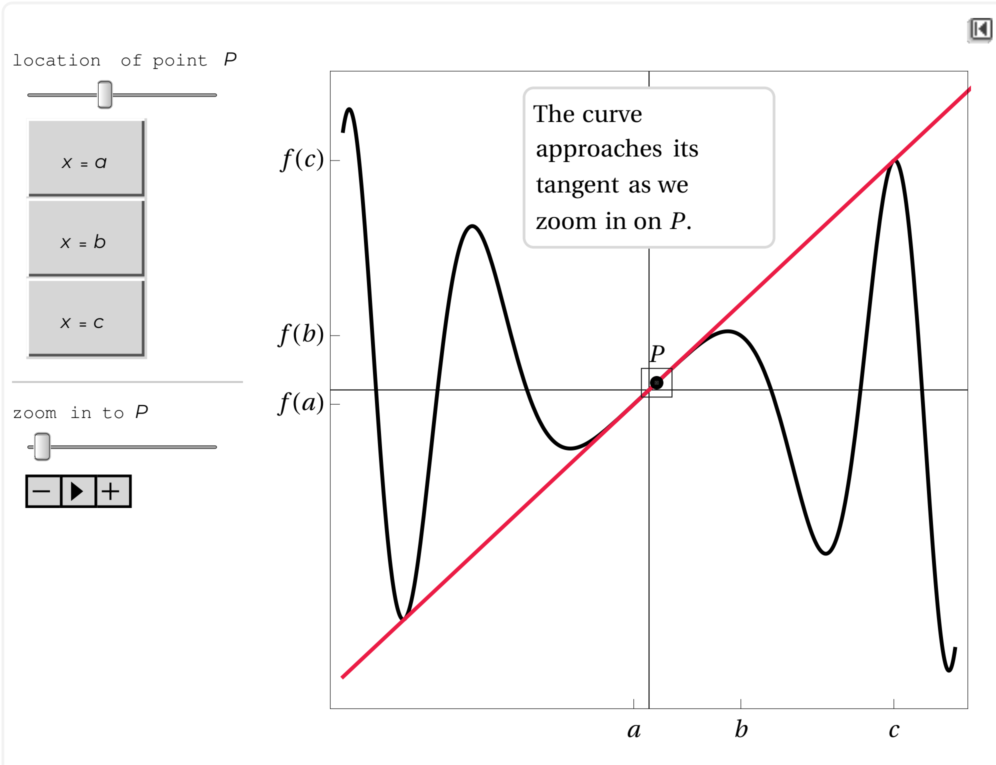

Figure 4.63 a

Now, consider a curve with a corner or cusp at a point Q (**Figure 4.63b**). No amount of magnification "straightens out" the curve or removes the corner at Q. The different behavior at P and Q is related to the idea of differentiability: The function in Figure 4.63a is differentiable at P, whereas the function in Figure 4.63b is not differentiable at Q. One of the requirements for the techniques presented in this section is that the function be differentiable at the point in question.

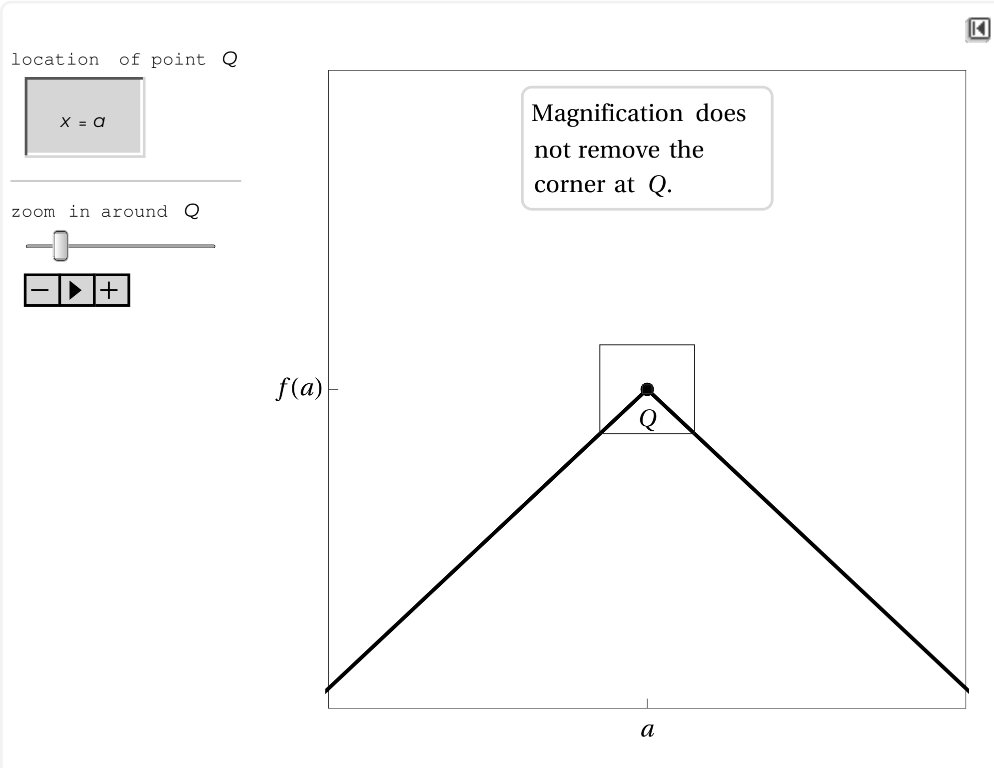

Figure 4.63 b

Linear Approximation »

Figure 4.63a suggests that when we zoom in on the graph of a smooth function at a point P, the curve approaches its tangent line at P. This fact is the key to understanding linear approximation. The idea is to use the line tangent to the curve at P to approximate the value of the function at points near P. Here's how it works.

Assume f is differentiable on an interval containing the point a. The slope of the line tangent to the curve at the point $(a, f(a))$ is $f'(a)$. Therefore, an equation of the tangent line is

$$y - f(a) = f'(a)(x-a) \quad \text{or} \quad y = \underbrace{f(a) + f'(a)(x-a)}_{L(x)}.$$

This tangent line is a new function L that we call the *linear approximation* to f at a (**Figure 4.64**). If f and f' are easy to evaluate at a, then the value of f at points near a is easily approximated using the linear approximation L. That is,

$$f(x) \approx L(x) = f(a) + f'(a)(x-a).$$

This approximation improves as x approaches a.

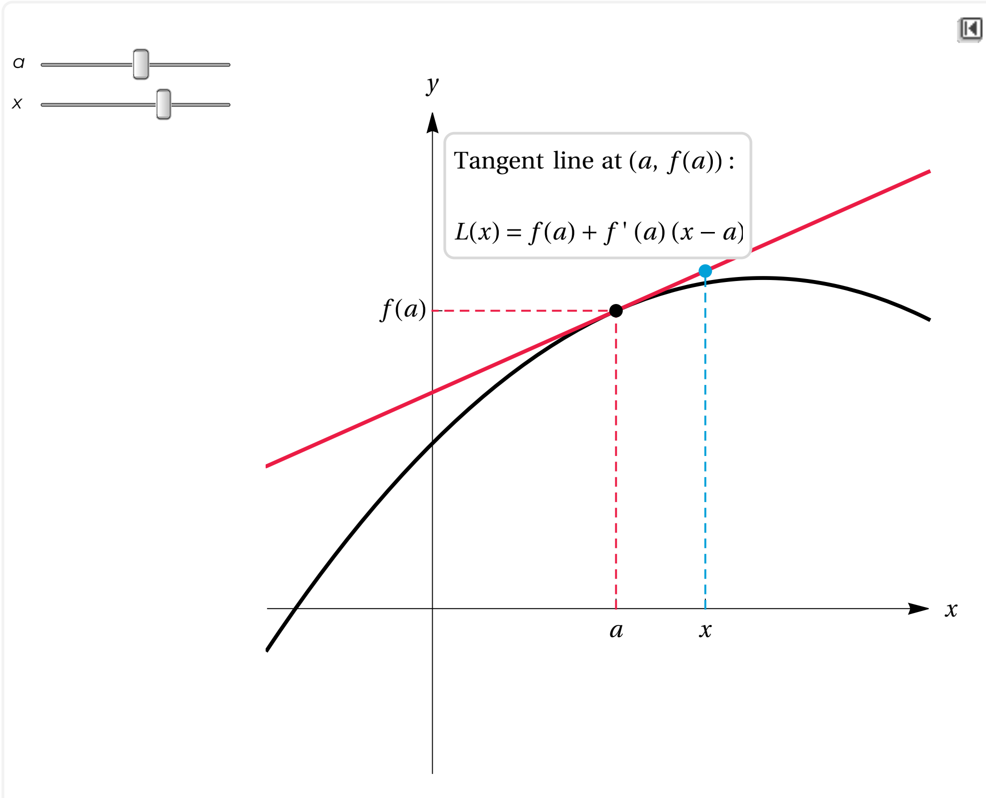

Figure 4.64

DEFINITION **Linear Approximation to f at a**

Suppose f is differentiable on an interval I containing the point a. The **linear approximation** to f at a is the linear function

$$L(x) = f(a) + f'(a)(x - a), \quad \text{for } x \text{ in } I.$$

Quick Check 1 Sketch the graph of a function f that is concave up at a point $(a, f(a))$. Sketch the linear approximation to f at a. Is the graph of the linear approximation above or below the graph of f? ◆

Answer »

The linear approximation lies below the graph of f for x near a.

EXAMPLE 1 Useful driving math

Suppose you are driving along a highway at a constant speed and you record the number of seconds it takes to travel between two consecutive mile markers. If it takes 60 seconds to travel one mile, then your average speed is $\dfrac{1\text{ mi}}{60\text{ s}}$ or 60 mi/hr. Now suppose you travel one mile in $60 + x$ seconds; for example, if it takes 62 seconds, then $x = 2$, and if it takes 57 seconds, then $x = -3$. The function

$$s(x) = \frac{3600}{60 + x} = 3600\,(60 + x)^{-1}$$

gives your average speed in mi/hr if you travel one mile in x seconds more or less than 60 seconds. For example, if you travel one mile in 62 seconds, then $x = 2$ and your average speed is $s(2) \approx 58.06$ mi/hr. If you travel one mile in 57 seconds, then $x = -3$ and your average speed is $s(-3) \approx 63.16$ mi/hr. Because you don't want to use a calculator while driving, you need an easy approximation to this function. Use linear approximation to derive such a formula.

Note »

In Example 1, notice that when x is positive, you are driving slower than 60 mi/hr; when x is negative, you are driving faster than 60 mi/hr.

SOLUTION »

The idea is to find the linear approximation to s at the point 0. We first use the Chain Rule to compute

$$s'(x) = -3600\,(60 + x)^{-2},$$

and then note that $s(0) = 60$ and $s'(0) = -3600 \cdot 60^{-2} = -1$. Using the linear approximation formula, we find that

$$s(x) \approx L(x) = s(0) + s'(0)(x - 0) = 60 - x.$$

For example, if you travel one mile in 62 seconds, then $x = 2$ and your average speed is approximately $L(2) = 58$ mi/hr, which is very close to the exact value given previously. If you travel one mile in 57 seconds, then $x = -3$ and your average speed is approximately $L(-3) = 63$ mi/hr, which again is close to the exact value.

Related Exercises 13–14 ◆

Quick Check 2 In Example 1, suppose you travel one mile in 75 seconds. What is the average speed given by the linear approximation formula? What is the exact average speed? Explain the discrepancy between the two values. ◆

Answer »

$L(15) = 45$, $s(15) = 48$; $x = 15$ is not close to 0.

EXAMPLE 2 Linear approximations and errors

a. Find the linear approximation to $f(x) = \sqrt{x}$ at $x = 1$ and use it to approximate $\sqrt{1.1}$.

b. Use linear approximation to estimate the value of $\sqrt{0.1}$.

SOLUTION »

a. We construct the linear approximation

$$L(x) = f(a) + f'(a)(x - a),$$

where $f(x) = \sqrt{x}$, $f'(x) = \dfrac{1}{2\sqrt{x}}$, and $a = 1$. Noting that $f(a) = f(1) = 1$ and $f'(a) = f'(1) = \dfrac{1}{2}$, we have

$$L(x) = 1 + \frac{1}{2}(x - 1) = \frac{1}{2}(x + 1),$$

which describes the line tangent to the curve at the point $(1, 1)$ (**Figure 4.65**).

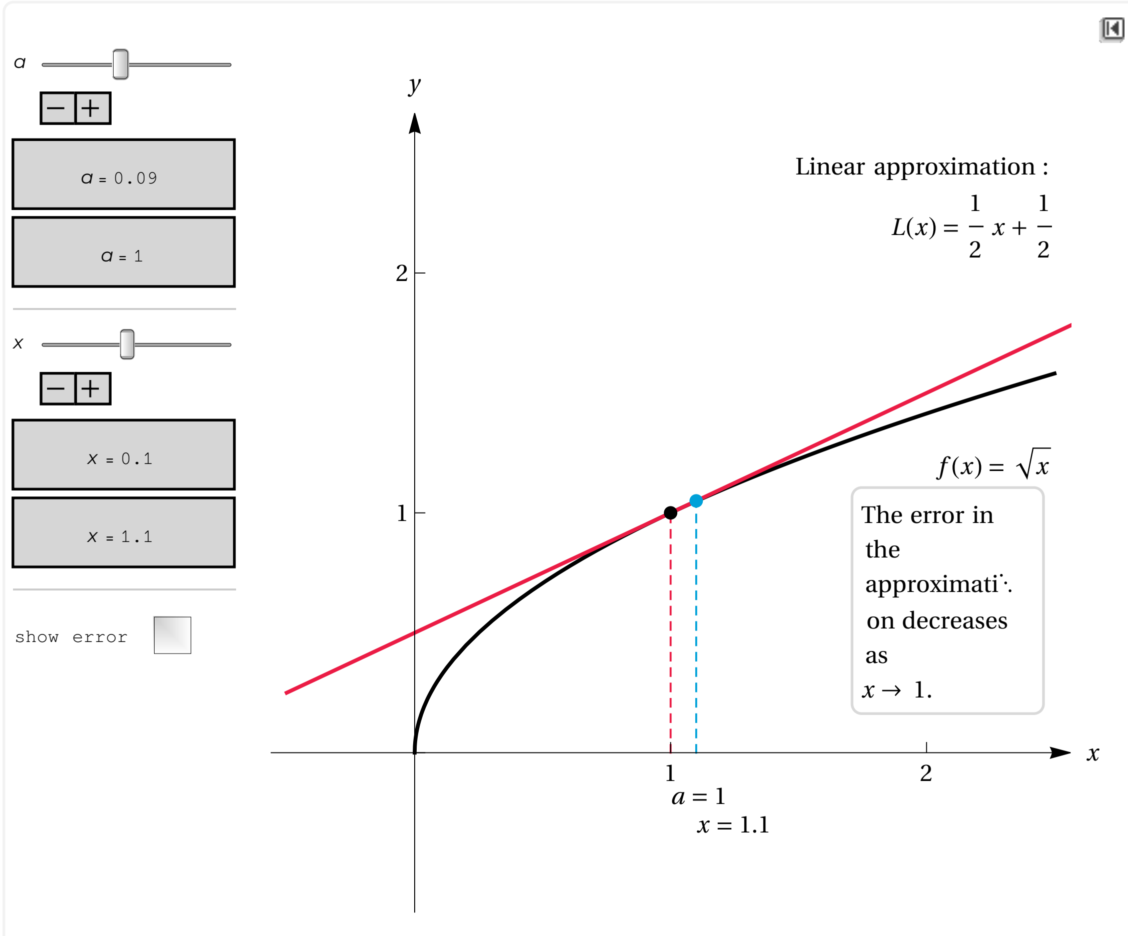

Figure 4.65

Because $x = 1.1$ is near $x = 1$, we approximate $\sqrt{1.1}$ by $L(1.1)$:

$$\sqrt{1.1} \approx L(1.1) = \frac{1}{2}(1.1 + 1) = 1.05.$$

The exact value is $f(1.1) = \sqrt{1.1} = 1.0488\ldots$; therefore, the linear approximation has an error of about 0.0012. Furthermore, our approximation is an *overestimate* because the tangent line lies above the graph of f. In Table 4.4 we see several approximations to $\sqrt{x}$ for x near 1 and the associated errors. Clearly, the errors decrease as x approaches 1.

Table 4.4

x	$L(x)$	Exact $\sqrt{x}$	Error
1.2	1.1	1.0954 …	4.6×10^{-3}
1.1	1.05	1.0488 …	1.2×10^{-3}
1.01	1.005	1.0049 …	1.2×10^{-5}
1.001	1.0005	1.0005 …	1.2×10^{-7}

b. If the linear approximation $L(x) = \frac{1}{2}(x+1)$ obtained in part (a) is used to approximate $\sqrt{0.1}$, we have

$$\sqrt{0.1} \approx L(0.1) = \frac{1}{2}(0.1+1) = 0.55.$$

A calculator gives $\sqrt{0.1} = 0.3162\ldots$, which shows that the approximation is well off the mark. The error arises because the tangent line through $(1, 1)$ is not close to the curve at $x = 0.1$ (Figure 4.65). For this reason, we seek a different value of a, with the requirement that it is near $x = 0.1$, and both $f(a)$ and $f'(a)$ are easily computed.

It is tempting to try $a = 0$, but $f'(0)$ is undefined. One choice that works well is $a = \frac{9}{100} = 0.09$. Using the linear approximation $L(x) = f(a) + f'(a)(x-a)$, we have

$$\begin{aligned}\sqrt{0.1} \approx L(0.1) &= \overbrace{\sqrt{\frac{9}{100}}}^{f(a)} + \overbrace{\frac{1}{2\sqrt{9/100}}}^{f'(a)}\Big(\overbrace{\frac{1}{10} - \frac{9}{100}}^{(x-a)}\Big)\\ &= \frac{3}{10} + \frac{10}{6}\left(\frac{1}{100}\right)\\ &= \frac{19}{60} \approx 0.3167.\end{aligned}$$

This approximation agrees with the exact value to three decimal places.

Note »

We choose $a = \frac{9}{100}$ because it is close to 0.1 and its square root is easy to evaluate.

Related Exercises 31–32 ◆

Quick Check 3 Suppose you want to use linear approximation to estimate $\sqrt{0.18}$. What is a good choice for a? ◆

Answer »

EXAMPLE 3 Linear approximation for the sine function

Find the linear approximation to $f(x) = \sin x$ at $x = 0$, and use it to approximate $\sin 2.5°$.

SOLUTION »

We begin by constructing a linear approximation $L(x) = f(a) + f'(a)(x-a)$, where $f(x) = \sin x$ and $a = 0$. Noting that $f(0) = 0$ and $f'(0) = \cos(0) = 1$, we have

$$L(x) = 0 + 1(x-0) = x.$$

Again, the linear approximation is the line tangent to the curve at the point $(0, 0)$ (**Figure 4.66**).

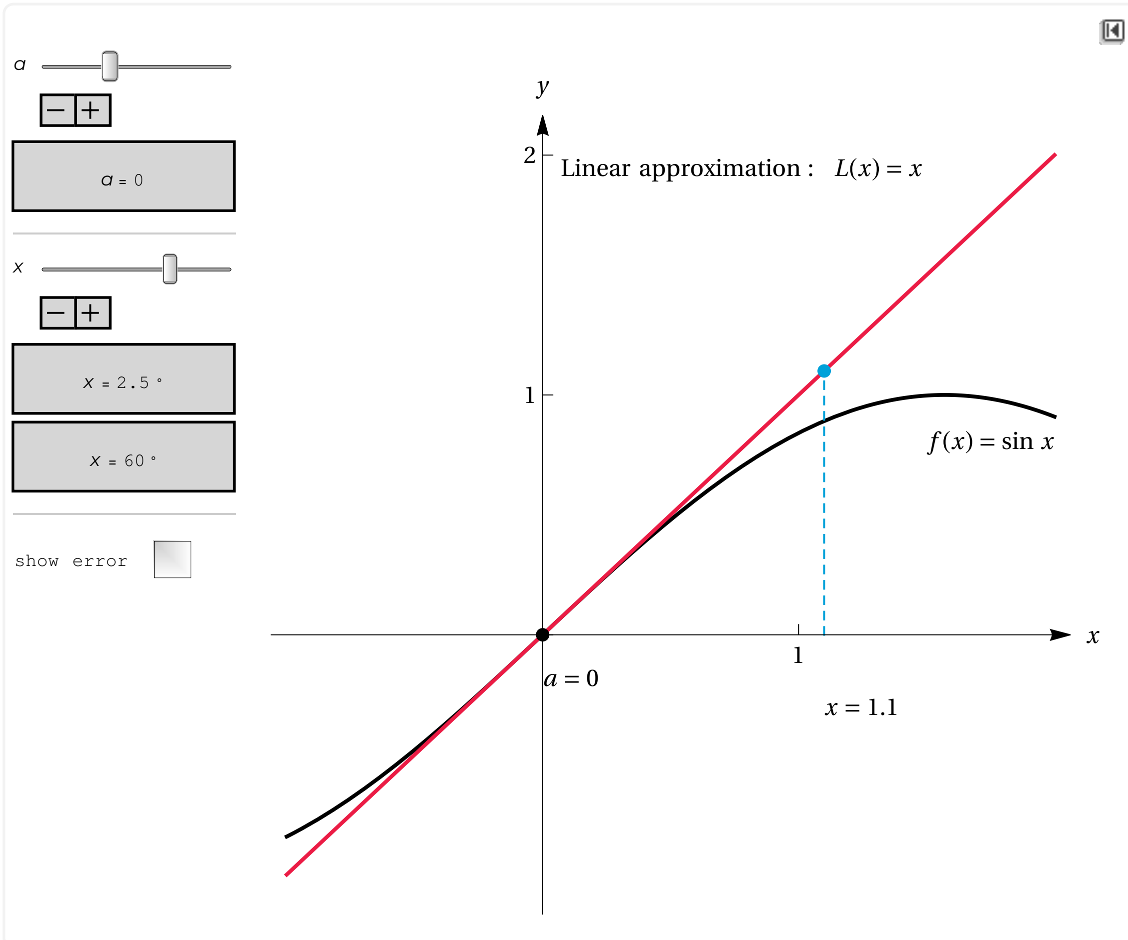

Figure 4.66

Before using $L(x)$ to approximate $\sin 2.5°$, we convert to radian measure (the derivative formulas for trigonometric functions require angles in radians):

$$2.5° = 2.5°\left(\frac{\pi}{180°}\right) = \frac{\pi}{72} \approx 0.04363 \text{ rad}.$$

Therefore, $\sin 2.5° \approx L(0.04363) = 0.04363$. A calculator gives $\sin 2.5° \approx 0.04362$, so the approximation is accurate to four decimal places.

Related Exercises 38, 46 ◆

In Examples 2 and 3, we used a calculator to check the accuracy of our approximations. This begs the question: Why bother with linear approximation when a calculator does a better job? There are some good answers to that question.

Linear approximation is actually just the first step in the process of *polynomial approximation*. While linear approximation does a decent job of estimating function values when x is near a, we can generally do better with higher-degree polynomials. These ideas are explored further in Chapter 11.

Linear approximation also allows us to discover simple approximations to complicated functions. In Example 3, we found the *small-angle approximation to the sine function*: $\sin x \approx x$ for x near 0.

Quick Check 4 Explain why the linear approximation to $f(x) = \cos x$ at $x = 0$ is $L(x) = 1$. ◆

Answer »

> Note that $f(0) = 1$ and $f'(0) = 0$, so $L(x) = 1$ (this is the line tangent to $y = \cos x$ at $(0, 1)$).

Linear Approximation and Concavity »

Additional insight into linear approximation is gained by bringing concavity into the picture. **Figure 4.67a** shows the graph of a function f and its linear approximation (tangent line) at the point $(a, f(a))$. In this particular case, f is concave up on an interval containing a, and the graph of L lies below the graph of f near a. As a result, the linear approximation evaluated at a point near a is less than the exact value of f at that point. In other words, the linear approximation *underestimates* values of f near a.

The contrasting case is shown in **Figure 4.67b**, where we see graphs of f and L when f is concave down on an interval containing a. Now the graph of L lies above the graph of f, which means the linear approximation *overestimates* values of f near a.

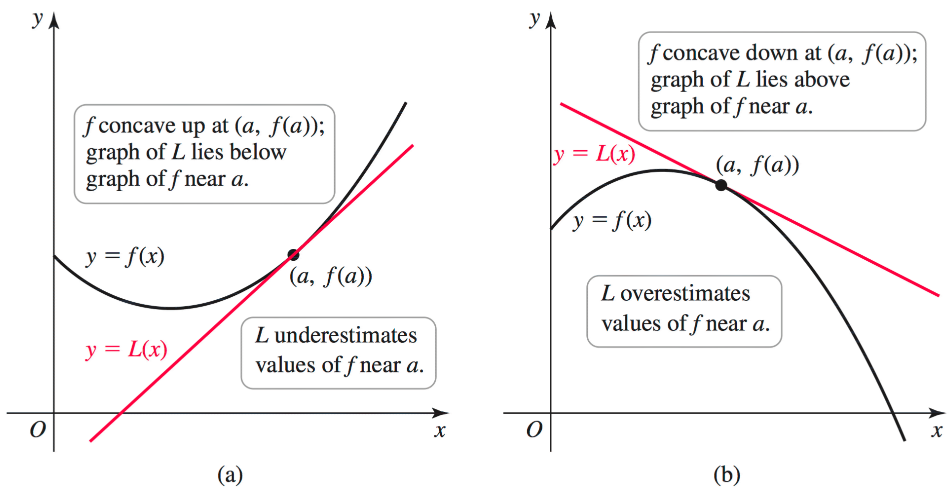

Figure 4.67

We can make another observation related to the degree of concavity (also called *curvature*). A large value of $|f''(a)|$ (large curvature) means that near $(a, f(a))$, the slope of the curve changes rapidly and the graph of f separates quickly from the tangent line. A small value of $|f''(a)|$ (small curvature) means that the slope of the curve changes slowly and the curve is relatively flat near $(a, f(a))$; therefore, the curve remains close to the tangent line. As a result, absolute errors in linear approximation are larger when $|f''(a)|$ is large.

EXAMPLE 4 Linear approximation and concavity

a. Find the linear approximation to $f(x) = x^{1/3}$ at $x = 1$ and $x = 27$.

b. Use the linear approximation of part (a) to approximate $\sqrt[3]{2}$ and $\sqrt[3]{26}$.

c. Are the approximations in part (b) overestimates or underestimates?

d. Compute the error in each approximation of part (b). Which error is greater? Explain.

SOLUTION »

a. Note that

$$f(1) = 1,\ f(27) = 3,\ f'(x) = \frac{1}{3x^{2/3}},\ f'(1) = \frac{1}{3},\ \text{and } f'(27) = \frac{1}{27}.$$

Therefore, the linear approximation at $x = 1$ is

$$L_1(x) = 1 + \frac{1}{3}(x - 1) = \frac{1}{3}x + \frac{2}{3},$$

and the linear approximation at $x = 27$ is

$$L_2(x) = 3 + \frac{1}{27}(x - 27) = \frac{1}{27}x + 2.$$

b. Using the results of part (a), we find that

$$\sqrt[3]{2} \approx L_1(2) = \frac{1}{3}\cdot 2 + \frac{2}{3} = \frac{4}{3} \approx 1.333$$

and

$$\sqrt[3]{26} \approx L_2(26) = \frac{1}{27}\cdot 26 + 2 \approx 2.963.$$

c. **Figure 4.68** show the graphs of f and the linear approximations L_1 and L_2 at $x = 1$ and $x = 27$, respectively (note the different scales on the two x-axis). We see that f is concave down at both points, which is confirmed by the fact that

$$f''(x) = -\frac{2}{9}x^{-5/3} < 0, \text{ for } x > 0.$$

Therefore, the linear approximations lie above the graph of f and both approximations are overestimates.

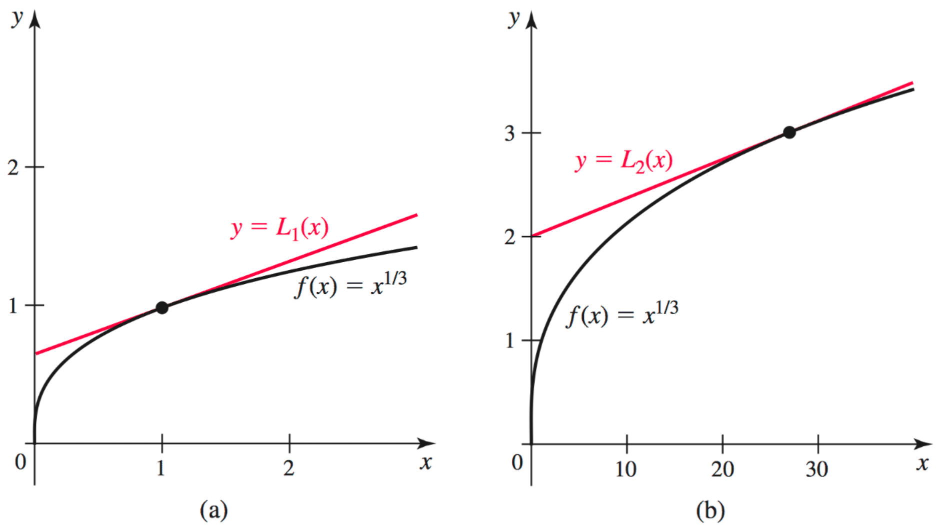

Figure 4.68

d. The errors in the two linear approximations are

$$\left|L_1(2) - 2^{1/3}\right| \approx 0.073 \quad \text{and} \quad \left|L_2(26) - 26^{1/3}\right| \approx 0.00047.$$

Because $|f''(1)| \approx 0.22$ and $|f''(27)| \approx 0.00091$, the curvature of f is much greater at $x = 1$ than at $x = 27$, explaining why the approximation of $\sqrt[3]{26}$ is more accurate than the approximation of $\sqrt[3]{2}$.

Related Exercises 47–50 ◆

A Variation on Linear Approximation »

Linear approximation says that a function f can be approximated as

$$f(x) \approx f(a) + f'(a)(x - a),$$

where a is fixed and x is a nearby point. We first rewrite this expression as

$$\underbrace{f(x) - f(a)}_{\Delta y} \approx f'(a)\underbrace{(x - a)}_{\Delta x}.$$

It is customary to use the notation Δ (capital Greek delta) to denote a change. The factor $x - a$ is the change in the x-coordinate between a and a nearby point x. Similarly, $f(x) - f(a)$ is the corresponding change in the y-coordinate (**Figure 4.69**). So we write this approximation as

$$\Delta y \approx f'(a)\,\Delta x.$$

In other words, a change in y (the function value) can be approximated by the corresponding change in x magnified or diminished by a factor of $f'(a)$. This interpretation states the familiar fact that $f'(a)$ is the rate of change of y with respect to x.

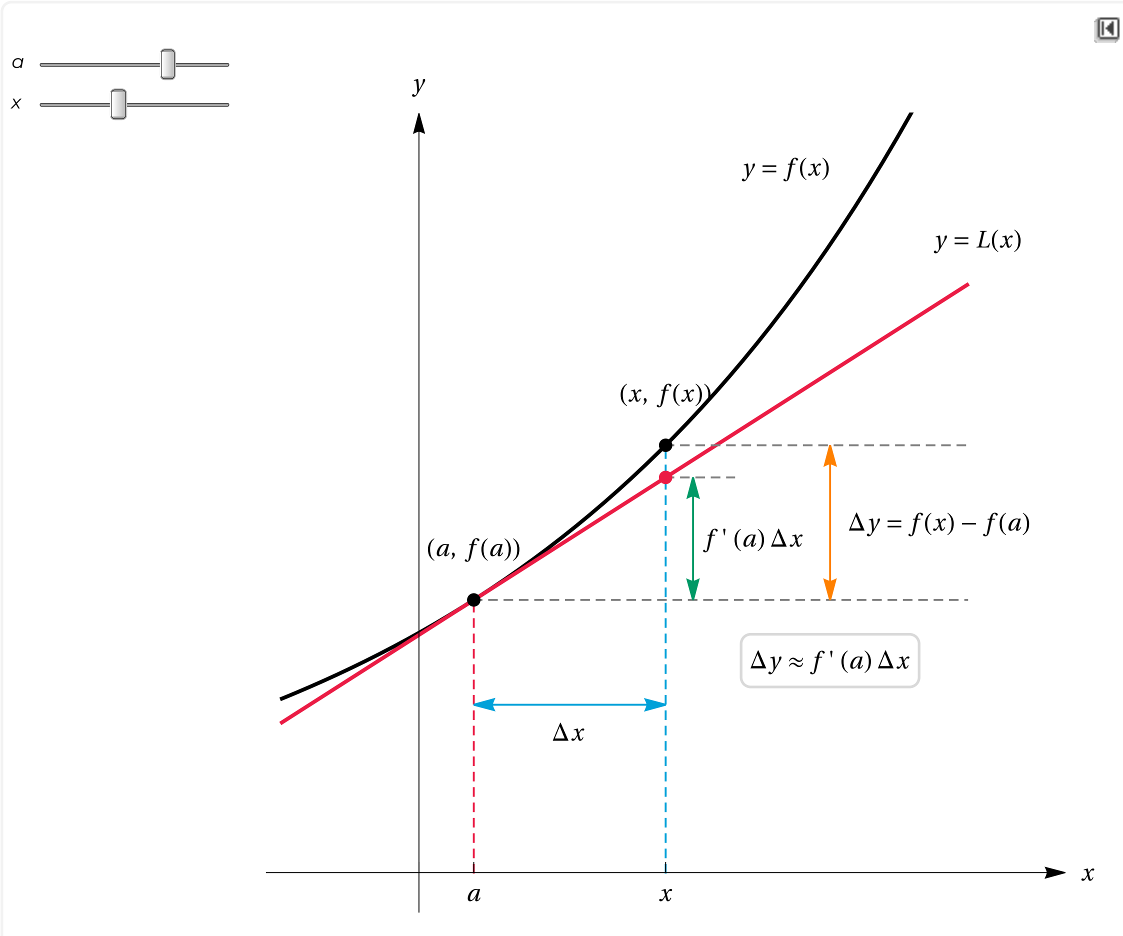

Figure 4.69

Relationship Between Δx and Δy

Suppose f is differentiable on an interval I containing the point a. The change in the value of f between two points a and $a + \Delta x$ is approximately

$$\Delta y \approx f'(a)\,\Delta x,$$

where $a + \Delta x$ is in I.

EXAMPLE 5 Estimating changes with linear approximations

a. Approximate the change in $y = f(x) = x^9 - 2x + 1$ when x changes from 1.00 to 1.05.

b. Approximate the change in the surface area of a spherical hot-air balloon when the radius decreases from 4 m to 3.9 m.

SOLUTION »

a. The change in y is $\Delta y \approx f'(a)\,\Delta x$, where $a = 1$, $\Delta x = 0.05$, and $f'(x) = 9x^8 - 2$. Substituting these values, we find that

$$\Delta y \approx f'(a)\,\Delta x = f'(1)\cdot 0.05 = 7\cdot 0.05 = 0.35.$$

If x increases from 1.00 to 1.05, then y increases by approximately 0.35.

b. The surface area of a sphere is $S = 4\pi r^2$, so the change in the surface area when the radius changes by Δr is $\Delta S \approx S'(a)\,\Delta r$. Substituting $S'(r) = 8\pi r$, $a = 4$, and $\Delta r = -0.1$, the approximate change in the surface area is

$$\Delta S \approx S'(a)\,\Delta r = S'(4)\cdot(-0.1) = 32\pi\cdot(-0.1) \approx -10.05.$$

The change in surface area is approximately $-10.05\ \mathrm{m}^2$; it is negative, reflecting a decrease.

Note »

Notice that the units in these calculations are consistent. If r has units of meters (m), S' has units of $\mathrm{m}^2/\mathrm{m} = \mathrm{m}$, so ΔS has units of m^2 as it should.

Related Exercises 55, 57 ◆

Quick Check 5 Given that the volume of a sphere is $V = \frac{4}{3}\pi r^3$, find an expression for the approximate change in the volume when the radius changes from a to $a + \Delta r$. ◆

Answer »

SUMMARY **Uses of Linear Approximation**

- To approximate f near $x = a$, use

$$f(x) \approx L(x) = f(a) + f'(a)(x - a).$$

- To approximate the change Δy in the dependent variable when the independent variable x changes from a to $a + \Delta x$, use

$$\Delta y \approx f'(a)\,\Delta x.$$

Differentials »

We now introduce an important concept that allows us to distinguish two related quantities:

- the change in the function $y = f(x)$ as x changes from a to $a + \Delta x$ (which we call Δy, as before), and
- the change in the linear approximation $y = L(x)$ as x changes from a to $a + \Delta x$ (which we will call the *differential* dy).

Consider a function $y = f(x)$ differentiable on an interval containing a. If the x-coordinate changes from a to $a + \Delta x$, the corresponding change in the function is *exactly*

$$\Delta y = f(a + \Delta x) - f(a).$$

Using the linear approximation $L(x) = f(a) + f'(a)(x - a)$, the change in L as x changes from a to $a + \Delta x$ is

$$\begin{aligned}\Delta L &= L(a + \Delta x) - L(a)\\ &= \underbrace{(f(a) + f'(a)(a + \Delta x - a))}_{L(a+\Delta x)} - \underbrace{(f(a) + f'(a)(a - a))}_{L(a)}\\ &= f'(a)\,\Delta x.\end{aligned}$$

To distinguish Δy and ΔL, we define two new variables called *differentials*. The differential dx is simply Δx; the differential dy is the change in the linear approximation, which is $\Delta L = f'(a)\,\Delta x$. Using this notation,

$$\Delta L = \underbrace{dy}_{\substack{\text{same} \\ \text{as } \Delta L}} = f'(a)\,\Delta x = f'(a)\underbrace{dx}_{\substack{\text{same} \\ \text{as } \Delta x}}.$$

Therefore, at the point a, we have $dy = f'(a)\,dx$. More generally, we replace the fixed point a with a variable point x and write

$$dy = f'(x)\,dx.$$

DEFINITION **Differentials**

Let f be differentiable on an interval containing x. A small change in x is denoted by the **differential** dx. The corresponding change in f is approximated by the **differential** $dy = f'(x)\,dx$; that is

$$\Delta y = f(x + dx) - f(x) \approx dy = f'(x)\,dx.$$

Note »

Figure 4.70 shows that if $\Delta x = dx$ is small, then the change in f, which is Δy, is well approximated by the change in the linear approximation, which is dy. Furthermore, the approximation $\Delta y \approx dy$ improves as dx approaches 0. The notation for differentials is consistent with the notation for the derivative: If we divide both sides of $dy = f'(x)\,dx$ by dx, we have (symbolically)

$$\frac{dy}{dx} = \frac{f'(x)\,dx}{dx} = f'(x).$$

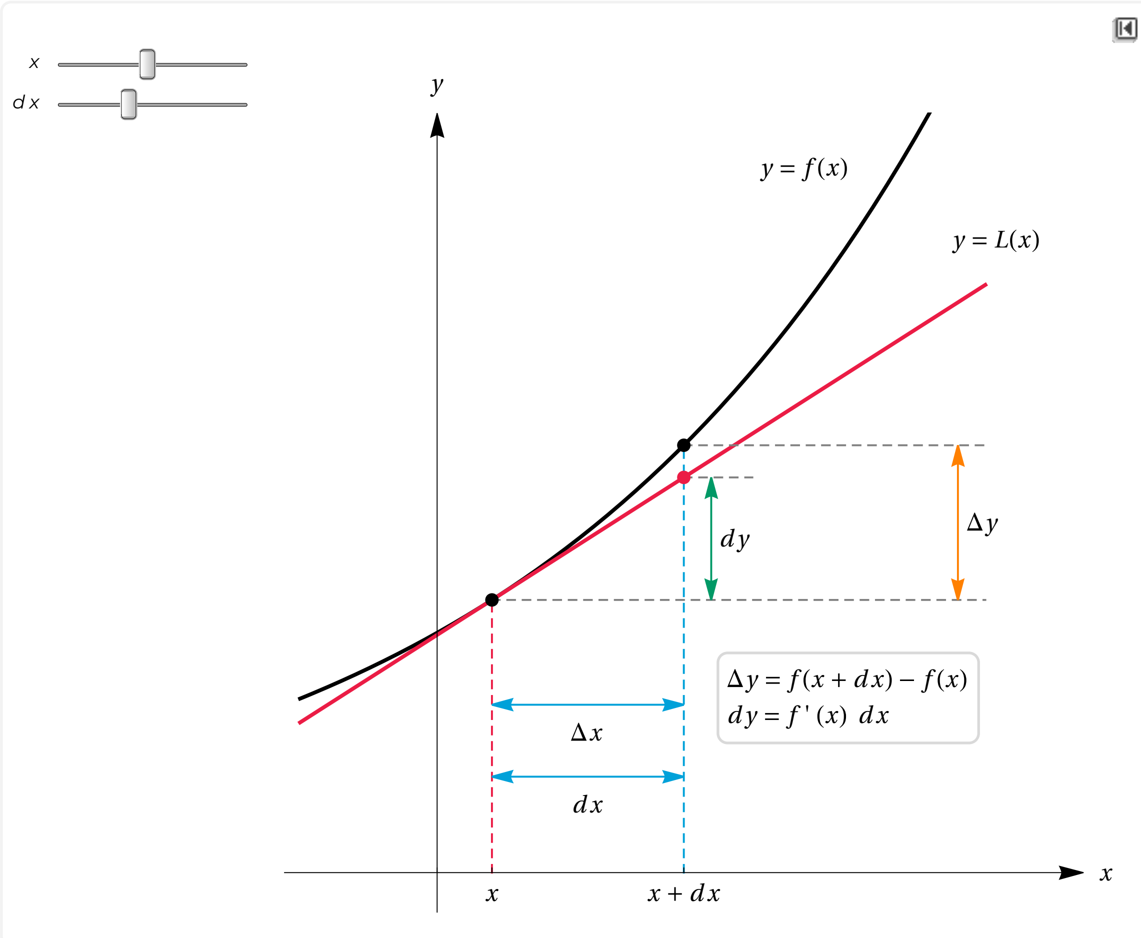

Figure 4.70

EXAMPLE 6 Differentials as change

Use the notation of differentials to write the approximate change in $f(x) = 3\cos^2 x$ given a small change dx.

SOLUTION »

With $f(x) = 3\cos^2 x$, we have $f'(x) = -6\cos x \sin x = -3\sin 2x$. Therefore,

$$dy = f'(x)\,dx = -3\sin 2x\,dx.$$

Note »

The interpretation is that a small change dx in the independent variable x produces an approximate change in the dependent variable of $dy = -3\sin 2x\,dx$. For example, if x increases from $x = \frac{\pi}{4}$ to $x = \frac{\pi}{4} + 0.1$, then $dx = 0.1$ and

$$dy = -3\sin\left(\frac{\pi}{2}\right)(0.1) = -0.3.$$

The approximate change in the function is -0.3, which means a decrease of approximately 0.3.

Related Exercises 60, 66 ◆

Exercises »

Getting Started »

Practice Exercises »

13–14. Estimating speed *Use the linear approximation given in Example 1 to answer the following questions.*

13. If you travel one mile in 59 seconds, what is your approximate average speed? What is your exact speed?

14. If you travel one mile in 63 seconds, what is your approximate average speed? What is your exact speed?

15–18. Estimating time *Suppose you want to travel D miles at a constant speed of* $(60 + x)$ *mi/hr, where x could be positive or negative. The time in minutes required to travel D miles is* $T(x) = 60\,D\,(60 + x)^{-1}$.

15. Show that the linear approximation to T at the point $x = 0$ is $T(x) \approx L(x) = D\left(1 - \dfrac{x}{60}\right)$.

16. Use the result of Exercise 15 to approximate the amount of time it takes to drive 45 miles at 62 mi/hr. What is the exact time required?

17. Use the result of Exercise 15 to approximate the amount of time it takes to drive 80 miles at 57 mi/hr. What is the exact time required?

18. Use the result of Exercise 15 to approximate the amount of time it takes to drive 93 miles at 63 mi/hr. What is the exact time required?

19–24. Linear approximation *Find the linear approximation to the following functions at the given point a.*

19. $f(x) = 4\,x^2 + x;\ a = 1$

20. $f(x) = x^3 - 5\,x + 3;\ a = 2$

21. $g(t) = \sqrt{2\,t + 9}\,;\ a = -4$

22. $h(w) = \sqrt{5\,w - 1}\,;\ a = 1$

23. $f(x) = \sqrt[3]{x}\,;\ a = 8$

24. $f(x) = 9\,(4\,x + 11)^{2/3};\ a = 4$

T **25–36. Linear approximation**

a. *Write the equation of the line that represents the linear approximation to the following functions at the given point a.*

b. *Use the linear approximation to estimate the given quantity.*

c. *Compute the percent error in your approximation,* $\dfrac{100\,|\text{approximation} - \text{exact}|}{|\text{exact}|}$*, where the exact value is given by a calculator.*

25. $f(x) = 12 - x^2$; $a = 2$; $f(2.1)$

26. $f(x) = \sin x$; $a = \pi/4$; $f(0.75)$

27. $f(x) = 1/(1 + x)$; $a = 0$; $f(-0.1)$

28. $f(x) = x/(x + 1)$; $a = 1$; $f(1.1)$

29. $f(x) = \cos x$; $a = 0$; $f(-0.01)$

30. $f(x) = x^{-3}$; $a = 1$; $f(1.05)$

31. $f(x) = (8 + x)^{-1/3}$; $a = 0$; $f(-0.1)$

32. $f(x) = \sqrt[4]{x}$; $a = 81$; $f(85)$

33. $f(x) = 1/(x + 1)$; $a = 0$; $1/1.1$

34. $f(x) = \cos x$; $a = \pi/4$; $\cos 0.8$

35. $f(x) = \sqrt[3]{64 + x}$; $a = 0$; $\sqrt[3]{62.5}$

36. $f(x) = \tan x$; $a = 0$; $\tan 3°$

37–46. Estimations with linear approximation *Use linear approximations to estimate the following quantities. Choose a value of a to produce a small error.*

37. $1/203$

38. $\tan(-2°)$

39. $\sqrt{146}$

40. $\sqrt[3]{65}$

41. $1/1.05$

42. $\sqrt{5/29}$

43. $\sin(\pi/4 + 0.1)$

44. $1/\sqrt{119}$

45. $1/\sqrt[3]{510}$

46. $\cos 31°$

47–50. Linear approximation and concavity *Carry out the following steps for the given functions f and points a.*

- ***a.*** *Find the linear approximation L to the function f at the point a.*
- ***b.*** *Graph f and L on the same set of axes.*
- ***c.*** *Based on the graphs in part (b), state whether linear approximations to f near $x = a$ are underestimates or overestimates.*

d. *Compute $f''(a)$ to confirm your conclusions in part (c).*

47. $f(x) = \dfrac{2}{x}$; $a = 1$

48. $f(x) = 5 - x^2$; $a = 2$

49. $f(x) = 1/\sqrt{x}$; $a = 1$

50. $f(x) = \sqrt{2}\cos x$; $a = \dfrac{\pi}{4}$

T **51.** **Error in driving speed** Consider again the average speed $s(x)$ and its linear approximation $L(x)$ discussed in Example 1. The error in using $L(x)$ to approximate $s(x)$ is given by $E(x) = |L(x) - s(x)|$. Use a graphing utility to determine the (approximate) values of x for which $E(x) \le 1$. What does your answer say about the accuracy of the average speeds estimated by $L(x)$ over this interval?

52. **Ideal Gas Law** The pressure P, temperature T, and volume V of an ideal gas are related by $PV = nRT$, where n is the number of moles of the gas and R is the universal gas constant. For the purposes of this exercise, let $nR = 1$; therefore, $P = T/V$.

a. Suppose that the volume is held constant and the temperature increases by $\Delta T = 0.05$. What is the approximate change in the pressure? Does the pressure increase or decrease?
b. Suppose that the temperature is held constant and the volume increases by $\Delta V = 0.1$. What is the approximate change in the pressure? Does the pressure increase or decrease?
c. Suppose that the pressure is held constant and the volume increases by $\Delta V = 0.1$. What is the approximate change in the temperature? Does the temperature increase or decrease?

53. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.

a. The linear approximation to $f(x) = x^2$ at $x = 0$ is $L(x) = 0$.
b. Linear approximation at $x = 0$ provides a good approximation to $f(x) = |x|$.
c. If $f(x) = mx + b$, then the linear approximation to f at any point a is $L(x) = f(x)$.

54. **Time function** Show that the function $T(x) = 60D(60 + x)^{-1}$ gives the time in minutes required to drive D miles at $60 + x$ miles per hour.

55–58. Approximating changes

55. Approximate the change in the volume of a sphere when its radius changes from $r = 5$ ft to $r = 5.1$ ft $\left(V(r) = \dfrac{4}{3}\pi r^3\right)$.

56. Approximate the change in the volume of a right circular cone of fixed height $h = 4$ m when its radius increases from $r = 3$ m to $r = 3.05$ m $\left(V(r) = \dfrac{1}{3}\pi r^2 h\right)$.

57. Approximate the change in the lateral surface area (excluding the area of the base) of a right circular cone with fixed height $h = 6$ m when its radius decreases from $r = 10$ m to $r = 9.9$ m $\left(S = \pi r\sqrt{r^2 + h^2}\right)$.

58. Approximate the change in the magnitude of the electrostatic force between two charges when the distance between them increases from $r = 20$ m to $r = 21$ m $(F(r) = 0.01/r^2)$.

59–66. Differentials *Consider the following functions and express the relationship between a small change in x and the corresponding change in y in the form* $dy = f'(x)\,dx$.

59. $f(x) = 2x + 1$

60. $f(x) = \sin^2 x$

61. $f(x) = \dfrac{1}{x^3}$

62. $f(x) = \sqrt{x^2 + 1}$

63. $f(x) = 2 - a\cos x$, a constant

64. $f(x) = \dfrac{x + 4}{4 - x}$

65. $f(x) = 3x^3 - 4x$

66. $f(x) = \tan x$

Explorations and Challenges »

T **67. Errors in approximations** Suppose $f(x) = \sqrt[3]{x}$ is to be approximated near $x = 8$. Find the linear approximation to f at 8. Then complete the following table, showing the errors in various approximations. Use a calculator to obtain the exact values. The percent error is $100\,|\text{approximation} - \text{exact}|/|\text{exact}|$. Comment on the behavior of the errors as x approaches 8.

x	Linear approx.	Exact value	Percent error
8.1			
8.01			
8.001			
8.0001			
7.9999			
7.999			
7.99			
7.9			

T **68. Errors in approximations** Suppose $f(x) = \dfrac{1}{x+1}$ is to be approximated near $x = 0$. Find the linear approximation to f at 0. Then complete the following table showing the errors in various approximations. Use a calculator to obtain the exact values. The percent error is $100\,|\text{approximation} - \text{exact}|/|\text{exact}|$. Comment on the behavior of the errors as x approaches 0.

x	**Linear approx.**	**Exact value**	**Percent error**
0.1			
0.01			
0.001			
0.0001			
−0.0001			
−0.001			
−0.01			
−0.1			

69. **Linear approximation and the second derivative** Draw the graph of a function f such that $f(1) = f'(1) = f''(1) = 1$. Draw the linear approximation to the function at the point $(1, 1)$. Now draw the graph of another function g such that $g(1) = g'(1) = 1$ and $g''(1) = 10$. (It is not possible to represent the second derivative exactly, but your graphs should reflect the fact that $f''(1)$ is relatively small compared to $g''(1)$.) Now suppose linear approximations are used to approximate $f(1.1)$ and $g(1.1)$.

a. Which function has the more accurate linear approximation near $x = 1$ and why?

b. Explain why the error in the linear approximation to f near a point a is proportional to the magnitude of $f''(a)$.