4.5 Optimization Problems

The theme of this section is *optimization*, a topic arising in many disciplines that rely on mathematics. A structural engineer may seek the dimensions of a beam that maximizes strength for a specified cost. A packaging designer may seek the dimensions of a container that maximizes the capacity of the container for a given surface area. Airline strategists need to find the best allocation of airliners among several hubs in order to minimize fuel costs and maximize passenger miles. In all these examples, the challenge is to find an *efficient* way to carry out a task, where “efficient” could mean least expensive, most profitable, least time consuming, or, as you will see, many other measures.

To introduce the ideas behind optimization problems, think about pairs of nonnegative real numbers x and y between 0 and 20 with the property that their sum is 20, that is $x + y = 20$. Of all possible pairs, which has the greatest product?

Table 4.3 displays a few cases showing how the product of two nonnegative numbers varies while their sum remains constant. The condition that $x + y = 20$ is called a **constraint**: It tells us to consider only (nonnegative) values of x and y satisfying this equation.

Table 4.3

x	y	$x + y$	$P = x\,y$
1	19	20	19
5.5	14.5	20	79.75
9	11	20	99
13	7	20	91
18	2	20	36

The quantity that we wish to maximize (or minimize in other cases) is called the **objective function**; in this case, the objective function is the product $P = x\,y$. From Table 4.3 it appears that the product is greatest if both x and y are near the middle of the interval $[0, 20]$.

This simple problem has all the essential features of optimization problems. At their heart, all optimization problems take the following form:

> *What is the maximum (minimum) value of an objective function subject to the given constraint(s)?*

For the problem at hand, this question would be stated as "What pair of nonnegative numbers maximizes $P = x\,y$ subject to the constraint $x + y = 20$?" The first step is to use the constraint to express the objective function $P = x\,y$ in terms of a single variable. In this case, the constraint is

$$x + y = 20, \quad \text{or} \quad y = 20 - x\,.$$

Substituting for y, the objective function becomes

$$P = x\,y = x\,(20 - x) = 20\,x - x^2\,,$$

which is a function of the single variable x. Notice that the values of x lie in the interval $0 \le x \le 20$ with $P(0) = P(20) = 0$.

Note »

> In this problem it is just as easy to eliminate x as y. In other problems, eliminating one variable may result in less work than eliminating other variables.

To maximize P, we first find the critical points by solving

$$P'(x) = 20 - 2x = 0$$

to obtain the solution $x = 10$. To find the absolute maximum value of P on the interval $[0, 20]$, we check the endpoints and the critical points. Because $P(0) = P(20) = 0$ and $P(10) = 100$, we conclude that P has its absolute maximum value at $x = 10$. By the constraint $x + y = 20$, the numbers with the greatest product are $x = y = 10$, and their product is $P = 100$.

Figure 4.53 summarizes this problem. We see the constraint line $x + y = 20$ in the xy-plane. Above the line is the objective function $P = xy$. As x and y vary along the constraint line, the objective function changes, reaching a maximum value of 100 when $x = y = 10$.

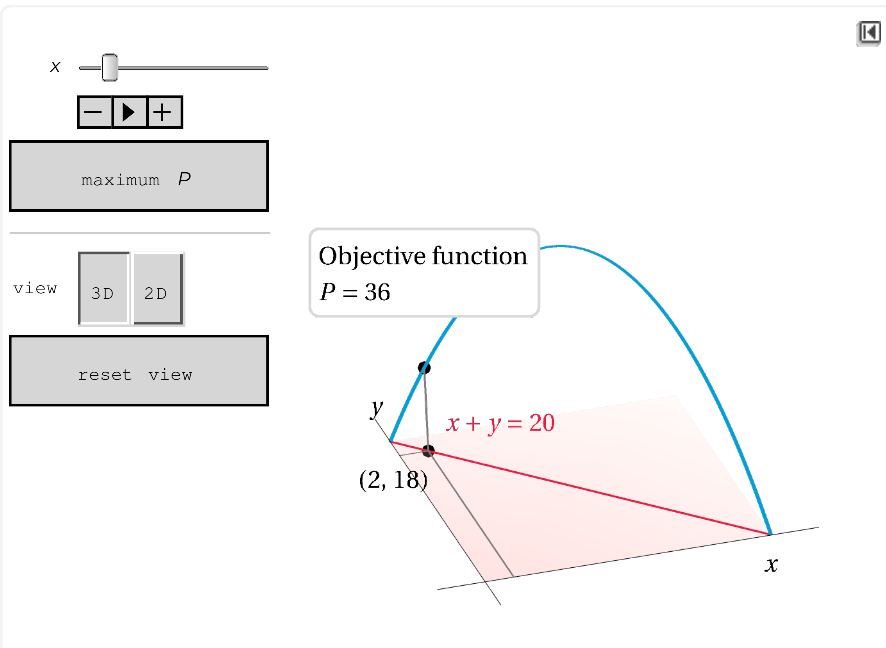


Figure 4.53

Quick Check 1 Verify that in the previous example the same result is obtained if the constraint $x + y = 20$ is used to eliminate x rather than y. ◆

Most optimization problems have the same basic structure as the preceding example: There is an objective function, which may involve several variables, and one or more constraints. The methods of calculus (Sections 4.1 and 4.3) are used to find the minimum or maximum values of the objective function.

EXAMPLE 1 Rancher's dilemma

A rancher has 400 ft of fence for constructing a rectangular corral. One side of the corral will be formed by a barn and requires no fence. Three exterior fences and two interior fences partition the corral into three rectangular regions as shown in **Figure 4.54**. What dimensions of the corral maximize the enclosed area? What is the area of that corral?

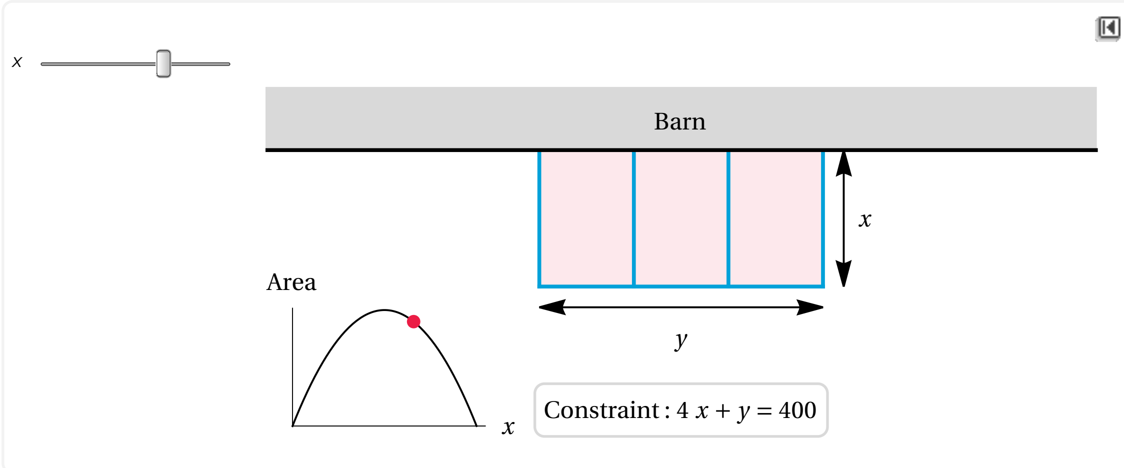


Figure 4.54

SOLUTION »

We first sketch the corral (Figure 4.54), where x is the width and y is the length of the corral. The amount of fence required is $4x + y$, so the constraint is $4x + y = 400$, or $y = 400 - 4x$.

The objective function to be maximized is the area of the corral, $A = xy$. Using $y = 400 - 4x$, we eliminate y and express A as a function of x:

$$A = xy = x(400 - 4x) = 400x - 4x^2.$$

Notice that the width of the corral must be at least $x = 0$, and it cannot exceed $x = 100$ (because 400 ft of fence are available). Therefore, we maximize $A(x) = 400x - 4x^2$, for $0 \le x \le 100$. The critical points of the objective function satisfy

$$A'(x) = 400 - 8x = 0,$$

which has the solution $x = 50$. To find the absolute maximum value of A, we check the endpoints of $[0, 100]$ and the critical point $x = 50$. Because $A(0) = A(100) = 0$ and $A(50) = 10{,}000$, the absolute maximum value of A occurs when $x = 50$. Using the constraint, the optimal length of the corral is $y = 400 - 4(50) = 200$ ft. Therefore, the maximum area of 10,000 ft^2 is achieved with dimensions $x = 50$ ft and $y = 200$ ft. The objective function A is shown in **Figure 4.55**.

Note »

> Recall from Section 4.1 that, when they exist, the absolute extreme values occur at critical points or endpoints.

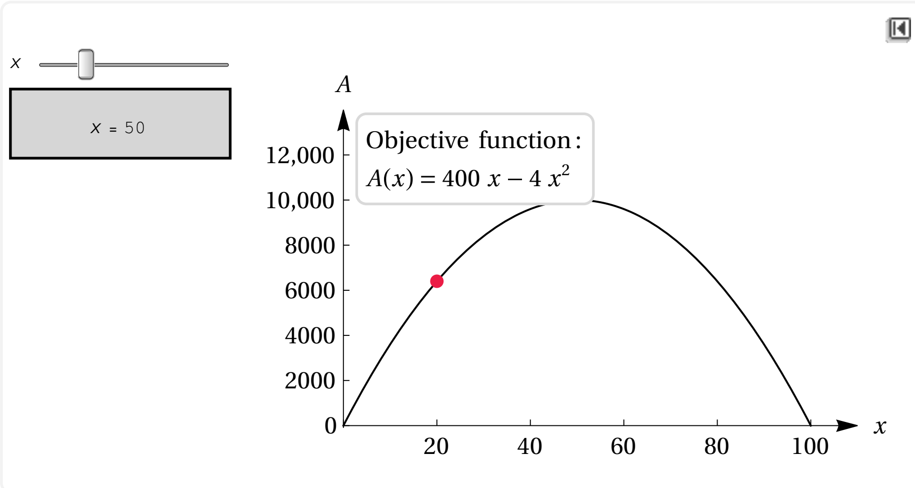


Figure 4.55

Related Exercises 12, 16 ◆

Quick Check 2 Find the objective function in Example 1 (in terms of x) (i) if there is no interior fence and (ii) if there is one interior fence that forms a right angle with the barn, as in Figure 4.54. ◆

Answer »

(i) $A = 400\,x - 2\,x^2$, (ii) $A = 400\,x - 3\,x^2$

EXAMPLE 2 Airline regulations

Suppose an airline policy states that all baggage must be box-shaped with a sum of length, width, and height not exceeding 64 in. What are the dimensions and volume of a square-based box with the greatest volume under these conditions?

SOLUTION »

We sketch a square-based box whose length and width are w and whose height is h (**Figure 4.56**).

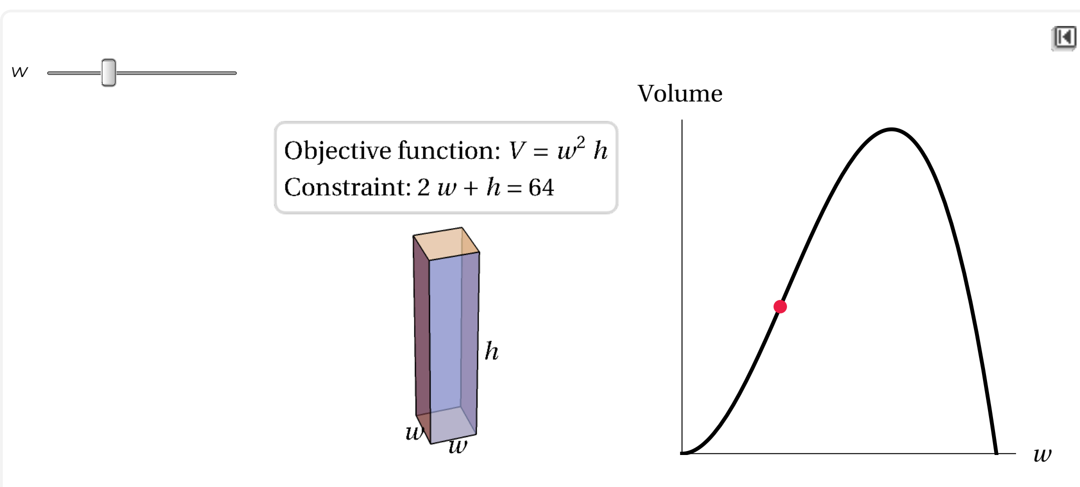


Figure 4.56

By the airline policy, the constraint is $2\,w + h = 64$. The objective function is the volume, $V = w^2\,h$. Either w or h

may be eliminated from the objective function; the constraint $h = 64 - 2w$ implies that the volume is

$$V = w^2 h = w^2(64 - 2w) = 64w^2 - 2w^3.$$

The objective function has now been expressed in terms of a single variable. Notice that w is nonnegative and cannot exceed 32, so the domain of V is $0 \le w \le 32$. The critical points satisfy

$$V'(w) = 128w - 6w^2 = 2w(64 - 3w) = 0,$$

which has roots $w = 0$ and $w = \frac{64}{3}$. By the First (or Second) Derivative Test, $w = \frac{64}{3}$ corresponds to a local maximum. At the endpoints, $V(0) = V(32) = 0$. Therefore, the volume function has an absolute maximum of $V\left(\frac{64}{3}\right) \approx 9709 \text{ in}^3$. The dimensions of the optimal box are $w = \frac{64}{3}$ in and $h = 64 - 2w = \frac{64}{3}$ in, so the optimal box is a cube. A graph of the volume function is shown in **Figure 4.57**.

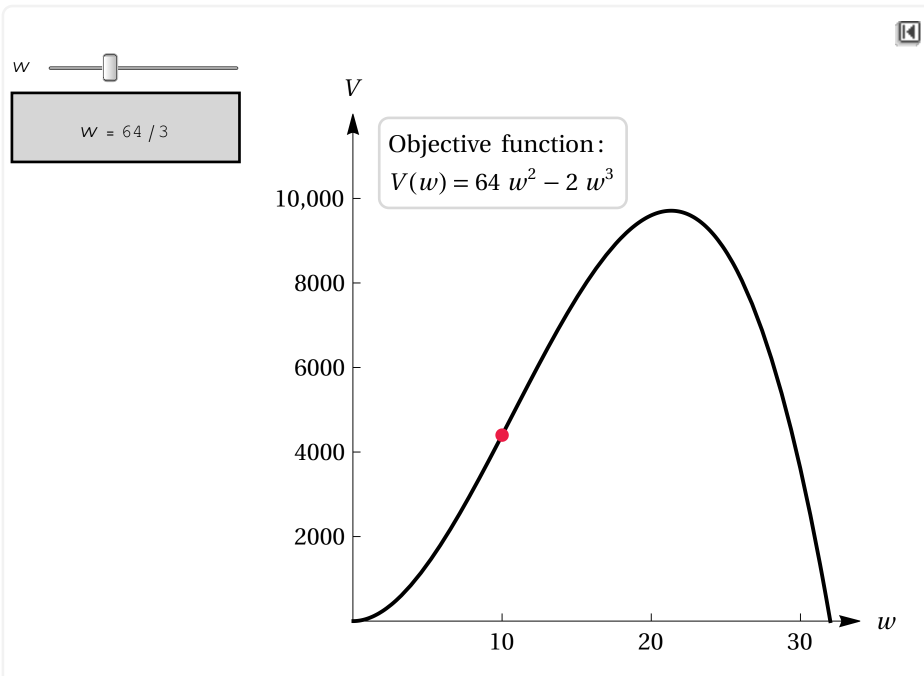


Figure 4.57

Related Exercises 19–20 ◆

Quick Check 3 Find the objective function in Example 2 (in terms of w) if the constraint is that the sum of length and width and height cannot exceed 108 in. ◆

Answer »

$V = 108w^2 - 2w^3$

Optimization Guidelines

With two examples providing some insight, we present a procedure for solving optimization problems. These guidelines provide a general framework, but the details may vary depending upon the problem.

Guidelines for Optimization Problems

1. Read the problem carefully, identify the variables, and organize the given information with a picture.

2. Identify the objective function (the function to be optimized). Write it in terms of the variables of the problem.

3. Identify the constraint(s). Write them in terms of the variables of the problem.

4. Use the constraint(s) to eliminate all but one independent variable of the objective function.

5. With the objective function expressed in terms of a single variable, find the interval of interest for that variable.

6. Use methods of calculus to find the absolute maximum or minimum value of the objective function on the interval of interest. If necessary, check the endpoints.

EXAMPLE 3 Walking and swimming

Suppose you are standing on the shore of a circular pond with radius 1 mile and you want to get to a point on the shore directly opposite your position (on the other end of the diameter). You plan to swim at 2 mi/hr from your current position to another point P on the shore and then walk at 3 mi/hr along the shore to the terminal point (**Figure 4.58**). How should you choose P to minimize the total time for the trip?

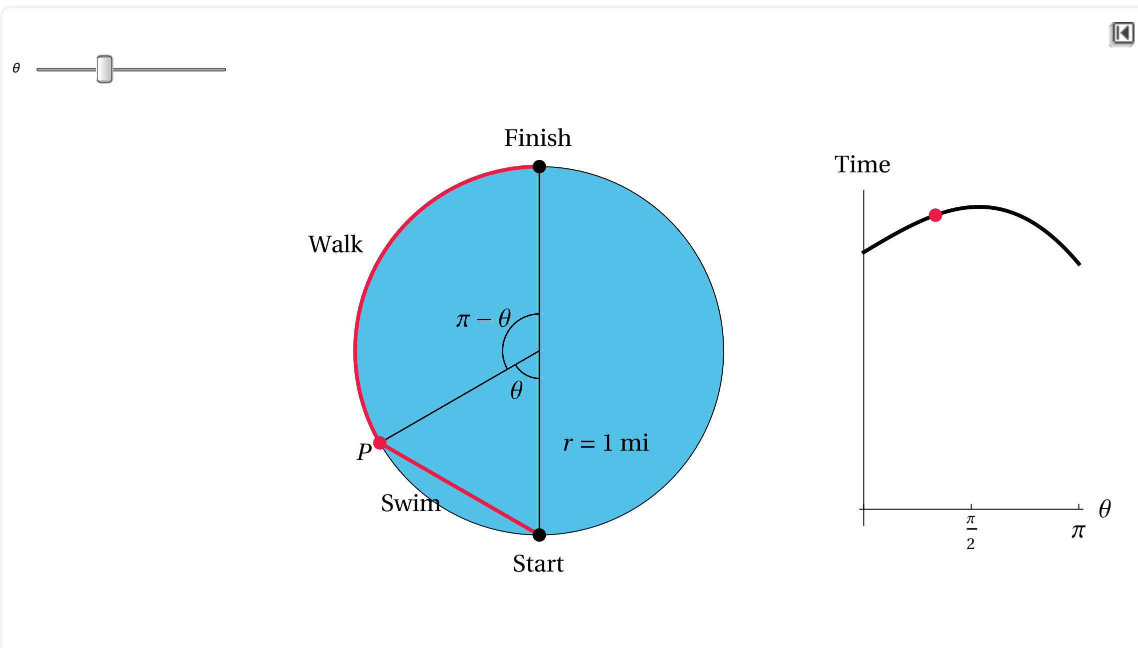


Figure 4.58

SOLUTION »

As shown in Figure 4.58, the initial point is chosen arbitrarily, and the terminal point is at the other end of a diameter. The easiest way to describe the transition point P is to refer to the central angle θ. If $\theta = 0$, then the entire trip is done by walking; if $\theta = \pi$, the entire trip is done by swimming. So the interval of interest is $0 \le \theta \le \pi$.

Note »

The objective function is the total travel time as it varies with θ. For each leg of the trip (swim and walk), the travel time is the distance traveled divided by the speed. We need a few facts from circular geometry. The length of the swimming leg is the length of the chord of the circle corresponding to the angle θ. For a circle of radius r, this chord length is given by $2r\sin\frac{\theta}{2}$ (**Figure 4.59**). So, the time for the swimming leg (with $r = 1$ and speed 2 mi/hr) is

$$\text{time} = \frac{\text{distance}}{\text{rate}} = \frac{2\sin(\theta/2)}{2} = \sin\frac{\theta}{2}.$$

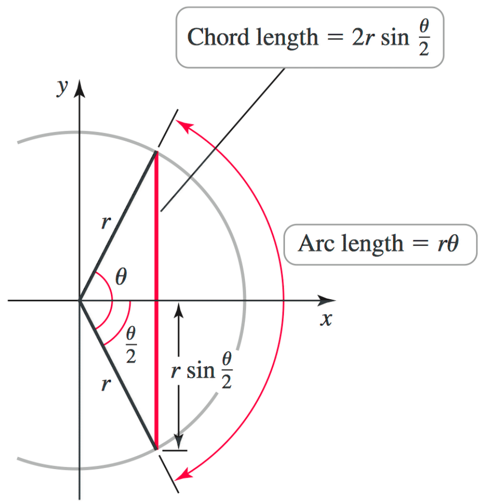


Figure 4.59

Note »

> To show that the chord length of a circle is $2r\sin\frac{\theta}{2}$, draw a line from the center of the circle to the midpoint of the chord. This line bisects the angle θ. Using a right triangle, half the length of the chord is $r\sin\frac{\theta}{2}$.

The length of the walking leg is the length of the arc of the circle corresponding to the angle $\pi - \theta$. For a circle of radius r, the arc length corresponding to an angle θ is $r\theta$ (Figure 4.59). Therefore, the time for the walking leg (with an angle $\pi - \theta$, $r = 1$, and speed 3 mi/hr) is

$$\text{time} = \frac{\text{distance}}{\text{rate}} = \frac{\pi - \theta}{3}.$$

The total travel time for the trip is the objective function

$$T(\theta) = \sin\frac{\theta}{2} + \frac{\pi - \theta}{3}, \quad \text{for } 0 \le \theta \le \pi.$$

We now analyze the objective function. The critical points of T satisfy

$$\frac{dT}{d\theta} = \frac{1}{2}\cos\frac{\theta}{2} - \frac{1}{3} = 0 \quad \text{or} \quad \cos\frac{\theta}{2} = \frac{2}{3}.$$

Using a calculator, the only solution in the interval $[0, \pi]$ is $\theta \approx 1.68$ rad $\approx 96°$, which is the critical point.

Evaluating the objective function at the critical point and at the endpoints, we find that $T(1.68) \approx 1.23$ hr, $T(0) = \frac{\pi}{3} = 1.05$ hr, and $T(\pi) = 1$ hr. We conclude that the minimum travel time is $T(\pi) = 1$ hr when the entire trip is done swimming. The *maximum* travel time, corresponding to $\theta \approx 96°$, is $T \approx 1.23$ hr.

The objective function is shown in **Figure 4.60**. In general, the maximum and minimum travel times depend on the walking and swimming speeds (Exercise 26).

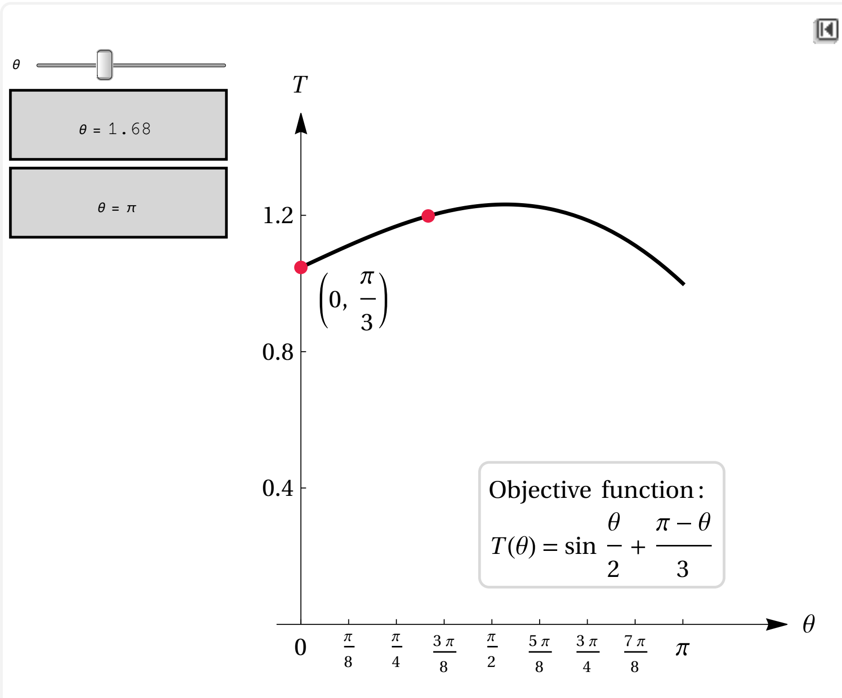


Figure 4.60

Related Exercises 26–27 ◆

EXAMPLE 4 Ladder over the fence

An 8-foot-tall fence runs parallel to the side of a house 3 feet away (**Figure 4.61**). What is the length of the shortest ladder that clears the fence and reaches the house? Assume that the vertical wall of the house and the horizontal ground have infinite extent (see Exercise 31 for more realistic assumptions).

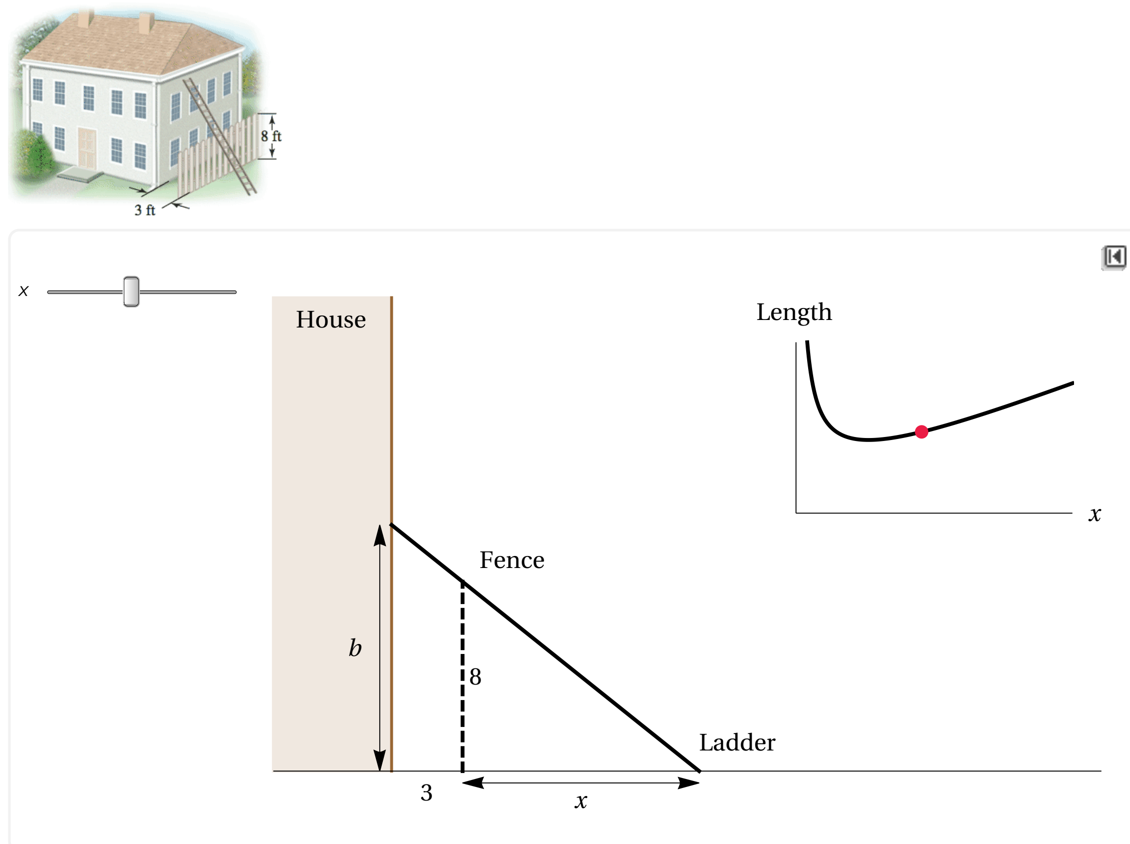


Figure 4.61

SOLUTION »

Let's first ask why we expect a minimum ladder length. You could put the foot of the ladder far from the fence so that it clears the fence at a shallow angle, but the ladder would be long. Or you could put the foot of the ladder close to the fence so that it clears the fence at a steep angle, but again, the ladder would be long. Somewhere between these extremes is a ladder position that minimizes the ladder length.

The objective function in this problem is the ladder length L. The position of the ladder is specified by x, the distance between the foot of the ladder and the fence (Figure 4.61). The goal is to express L as a function of x, where $x > 0$.

The Pythagorean theorem gives the relationship

$$L^2 = (x+3)^2 + b^2,$$

where b is the height of the top of the ladder above the ground. Similar triangles give the constraint $\dfrac{8}{x} = \dfrac{b}{x+3}$. We now solve the constraint equation for b and substitute to express L^2 in terms of x:

$$L^2 = (x+3)^2 + \underbrace{\left(\frac{8\,(x+3)}{x}\right)}_{b}{}^2 = (x+3)^2\left(1 + \frac{64}{x^2}\right).$$

At this juncture, we could find the critical points of L by first solving the preceding equation for L, and then solving $L' = 0$. However, the solution is simplified considerably if we note that L is a nonnegative function. Therefore, L and L^2 have local extrema at the same points, so we choose to minimize L^2. The derivative of L^2 is

$$\begin{aligned} \frac{d}{dx}\left((x+3)^2\left(1+\frac{64}{x^2}\right)\right) &= 2(x+3)\left(1+\frac{64}{x^2}\right)+(x+3)^2\left(-\frac{128}{x^2}\right) && \text{Chain Rule and Product Rule} \\ &= 2(x+3)\left(1+\frac{64}{x^2}-(x+3)\frac{64}{x^2}\right) && \text{Factor.} \\ &= \frac{2(x+3)\left(x^3-192\right)}{x^3}. && \text{Simpilify.} \end{aligned}$$

Because $x > 0$, we have $x + 3 \neq 0$; therefore, the condition $\frac{d}{dx}\left(L^2\right) = 0$ becomes $x^3 - 192 = 0$, or $x = 4\sqrt[3]{3} \approx 5.77$. By the First Derivative Test, this critical point corresponds to a local minimum. By Theorem 4.9, this solitary local minimum is also the absolute minimum on the interval $(0, \infty)$. Therefore, the minimum ladder length occurs when the foot of the ladder is approximately 5.77 ft from the fence. We find that $L^2(5.77) \approx 224.77$ and the minimum ladder length is $\sqrt{224.77} \approx 15$ ft.

Related Exercises 30–31 ◆

EXAMPLE 5 Water tower

A water storage tank in a small community is built in the shape of a right circular cylinder with a capacity of 32,000 ft^3 (about 240,000 gallons). The interior wall and floor of the tank must be cleaned and treated annually. Labor costs for cleaning the wall are twice as high per square foot as the cost to clean the floor. Find the radius and height of the tank that minimize the cleaning cost.

SOLUTION »

Figure 4.62 shows a water tank with radius r and height h.

Figure 4.62

The objective function describes the cost to clean the tank, which depends on the interior surface area of the tank. The area of the tank's circular floor is πr^2. Because the circumference of the tank is $2\pi r$, the lateral surface area of the wall is $2\pi r h$. Therefore, the cost to clean the tank is proportional to

$$C = 2 \cdot \underbrace{2\pi r h}_{\text{area of wall}} + \underbrace{\pi r^2}_{\text{area of floor}}.$$

It costs twice as much per square foot to clean the wall, so the area of the wall is multiplied by 2.

Note »

To find a constraint equation, we start with the volume of a cylinder $V = \pi r^2 h$ and replace V with 32,000 to obtain $\pi r^2 h = 32{,}000$, or $h = 32{,}000/(\pi r^2)$. Substituting this expression for h into the cost equation, we obtain a function of the radius:

$$C(r) = 4\pi r \frac{32{,}000}{\pi r^2} + \pi r^2 = \frac{128{,}000}{r} + \pi r^2.$$

To minimize C, we find its critical points on the interval $0 < r < \infty$:

$$C'(r) = -\frac{128{,}000}{r^2} + 2\pi r = 0 \quad \text{Set the derivative equal to 0.}$$

$$2\pi r = \frac{128{,}000}{r^2} \quad \text{Rearrange equation.}$$

$$r^3 = \frac{64{,}000}{\pi} \quad \text{Multiply by } r^2\ (r > 0) \text{ and divide by } 2\pi.$$

$$r = \frac{40}{\sqrt[3]{\pi}} \approx 27.3 \text{ ft.} \quad \text{Solve for } r.$$

This solitary critical point corresponds to a local minimum (confirm that $C''(r) > 0$ and use the Second Derivative Test), so by Theorem 4.9, it also corresponds to the absolute minimum. Because $h = 32{,}000/(\pi r^2)$, the dimensions of the tank that minimize the cleaning cost are

$$r \approx 27.3 \text{ ft} \quad \text{and} \quad h = \frac{32{,}000}{\pi r^2} \approx 13.7 \text{ ft.}$$

A bit of algebra shows that r is exactly twice as large as h, so the optimal tank is 4 times as wide as it is high.

Related Exercises 21, 35, 42 ◆

Exercises »

Getting Started »

Practice Exercises »

11. **Maximum-area rectangles** Of all rectangles with a perimeter of 10, which one has the maximum area? (Give the dimensions.)

12. **Maximum-area rectangles** Of all rectangles with a fixed perimeter of P, which one has the maximum area? (Give the dimensions in terms of P.)

13. Minimum-perimeter rectangles Of all rectangles of area 100, which one has the minimum perimeter?

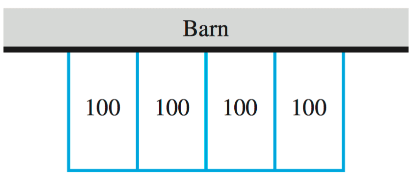
14. Minimum-perimeter rectangles Of all rectangles with a fixed area A, which one has the minimum perimeter? (Give the dimensions in terms of A.)

15. Minimum sum Find positive numbers x and y satisfying the equation $xy = 12$ such that the sum $2x + y$ is as small as possible.

16. Pen problems

a. A rectangular pen is built with one side against a barn. Two hundred meters of fencing are used for the other three sides of the pen. What dimensions maximize the area of the pen?

b. A rancher plans to make four identical and adjacent rectangular pens against a barn, each with an area of 100 m^2 (see figure). What are the dimensions of each pen that minimize the amount of fence that must be used?



17. Rectangles beneath a semicircle A rectangle is constructed with its base on the diameter of a semicircle with radius 5 and its other two vertices on the semicircle. What are the dimensions of the rectangle with maximum area?

18. Rectangles beneath a parabola A rectangle is constructed with its base on the x-axis and two of its vertices on the parabola $y = 48 - x^2$. What are the dimensions of the rectangle with the maximum area? What is the area?

19. Minimum-surface-area box Of all boxes with a square base and a volume of 8 m^3, which one has the minimum surface area? (Give its dimensions.)

20. Maximum-volume box Suppose an airline policy states that all baggage must be box-shaped with a sum of length, width, and height not exceeding 108 in. What are the dimensions and volume of a square-based box with the greatest volume under these conditions?

21. Shipping crates A square-based, box-shaped shipping crate is designed to have a volume of 16 ft^3. The material used to make the base costs twice as much (per square foot) as the material in the sides, and the material used to make the top costs half as much (per square foot) as the material in the sides. What are the dimensions of the crate that minimize the cost of materials?

22. Closest point on a line What point on the line $y = 3x + 4$ is closest to the origin?

T **23. Closest point on a curve** What point on the parabola $y = 1 - x^2$ is closest to the point $(1, 1)$?

24. Minimum distance Find the point P on the curve $y = x^2$ that is closest to the point $(18, 0)$. What is the least distance between P and $(18, 0)$? (*Hint:* Use synthetic division.)

25. Minimum distance Find the point P on the line $y = 3x$ that is closest to the point $(50, 0)$. What is the least distance between P and $(50, 0)$?

26. Walking and swimming A man wishes to get from an initial point on the shore of a circular pond with radius 1 mi to a point on the shore directly opposite (on the other end of the diameter). He plans to swim from the initial point to another point on the shore and then walk along the shore to the terminal point.

a. If he swims at 2 mi/hr and walks at 4 mi/hr, what are the maximum and minimum times for the trip?

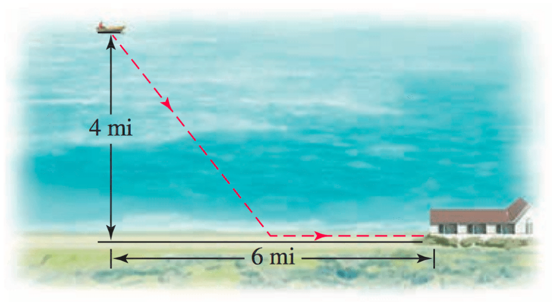
b. If he swims at 2 mi/hr and walks at 1.5 mi/hr, what are the maximum and minimum times for the trip?

c. If he swims at 2 mi/hr, what is the minimum walking speed for which it is quickest to walk the entire distance?

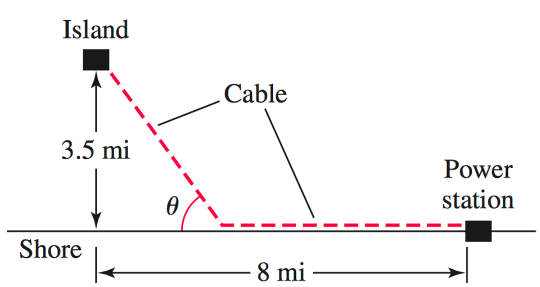
27. Walking and rowing A boat on the ocean is 4 mi from the nearest point on a straight shoreline; that point is 6 mi from a restaurant on the shore (see figure). A woman plans to row the boat straight to a point on the shore and then walk along the shore to the restaurant.

a. If she walks at 3 mi/hr and rows at 2 mi/hr, at which point on the shore should she land to minimize the total travel time?

b. If she walks at 3 mi/hr, what is the minimum speed at which she must row so that the quickest way to the restaurant is to row directly (with no walking)?



28. Laying cable An island is 3.5 mi from the nearest point on a straight shoreline; that point is 8 mi from a power station (see figure). A utility company plans to lay electrical cable underwater from the island to the shore and then underground along the shore to the power station. Assume it costs \$2400/mi to lay underwater cable and \$1200/mi to lay underground cable. At what point should the underwater cable meet the shore in order to minimize the cost of the project?



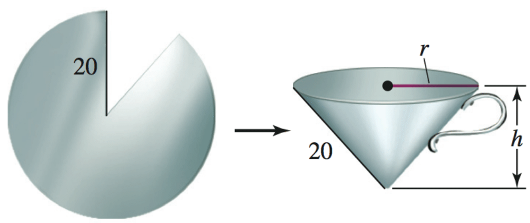
29. Laying cable again Solve the problem in Exercise 28, but this time minimize the cost with respect to the smaller angle θ between the underwater cable and the shore. (You should get the same answer.)

30. Shortest ladder A 10-ft-tall fence runs parallel to the wall of a house at a distance of 4 ft. Find the length of the shortest ladder that extends from the ground to the house without touching the fence. Assume the vertical wall of the house and the horizontal ground have infinite extent.

31. Shortest ladder—more realistic An 8-ft-tall fence runs parallel to the wall of a house at a distance of 5 ft. Find the length of the shortest ladder that extends from the ground to the house without touching the fence. Assume the vertical wall of the house is 20 ft high and the horizontal ground extends 20 ft from the fence.

32. Circle and square A piece of wire of length 60 is cut, and the resulting two pieces are formed to make a circle and a square. Where should the wire be cut to (a) maximize and (b) minimize the combined area of the circle and the square?

33. Maximum-volume cone A cone is constructed by cutting a sector from a circular sheet of metal with radius 20. The cut sheet is then folded up and welded (see figure). Find the radius and height of the cone with maximum volume that can be formed in this way.



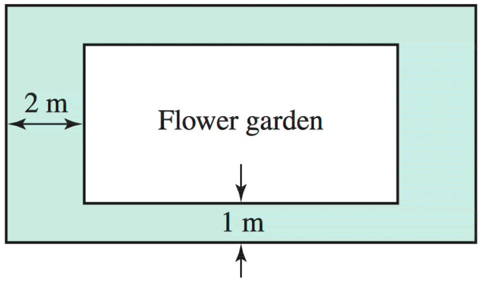
34. Slant height and cones Among all right circular cones with a slant height of 3, what are the dimensions (radius and height) that maximize the volume of the cone? The slant height of a cone is the distance from the outer edge of the base to the vertex.

35. Optimal soda can

- **a. Classical problem** Find the radius and height of a cylindrical soda can with a volume of 354 cm^3 that minimize the surface area.
- **b. Real problem** Compare your answer in part (a) to a real soda can, which has a volume of 354 cm^3, a radius of 3.1 cm, and a height of 12.0 cm, to conclude that real soda cans do not seem to have an optimal design. Then use the fact that real soda cans have a double thickness in their top and bottom surfaces to find the radius and height that minimizes the surface area of a real can (the surface areas of the top and bottom are now twice their values in part (a)). Are these dimensions closer to the dimensions of a real soda can?

36. Covering a marble Imagine a flat-bottomed cylindrical pot with a circular cross section of radius 4. A marble with radius $0 < r < 4$ is placed in the bottom of the pot. What is the radius of the marble that requires the most water to cover it completely?

37. Optimal garden A rectangular flower garden with an area of 30 m^2 is surrounded by a grass border 1 m wide on two sides and 2 m wide on the other two sides (see figure). What dimensions of the garden minimize the combined area of the garden and borders?

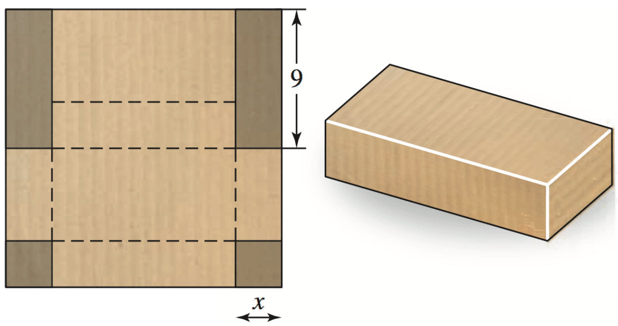


38. Rectangles beneath a line

a. A rectangle is constructed with one side on the positive x-axis, one side on the positive y-axis, and the vertex opposite the origin on the line $y = 10 - 2x$. What dimensions maximize the area of the rectangle? What is the maximum area?

b. Is it possible to construct a rectangle with a greater area than that found in part (a) by placing one side of the rectangle on the line $y = 10 - 2x$ and the two vertices not on that line on the positive x- and y-axes? Find the dimensions of the rectangle of maximum area that can be constructed in this way.

39. Designing a box Two squares of length x are cut out of adjacent corners of an $18'' \times 18''$ piece of cardboard and two rectangles of length $9''$ and width x are cut out of the other two corners of the cardboard (see figure). The resulting piece of cardboard is then folded along the dashed lines to form an enclosed box. Find the dimensions and volume of the largest box that can be formed in this way.



40. Folded boxes

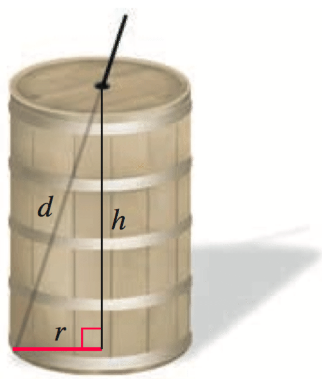
a. Squares with sides of length x are cut out of each corner of a rectangular piece of cardboard measuring 5 ft by 8 ft. The resulting piece of cardboard is then folded into a box without a lid. Find the volume of the largest box that can be formed in this way.

b. Squares with sides of length x are cut out of each corner of a square piece of cardboard with sides of length ℓ. Find the volume of the largest open box that can be formed in this way.

41. Norman window A window consists of rectangular pane of glass surmounted by a semicircular pane of glass (see figure). If the perimeter of the window is 20 feet, determine the dimensions of the window that maximize the area of the window.

42. Light transmission A window consists of a rectangular pane of clear glass surmounted by a semicircular pane of tinted glass. The clear glass transmits twice as much light per unit of surface area as the tinted glass. Of all such windows with a fixed perimeter P, what are the dimensions of the window that transmits the most light?

43. Kepler's wine barrel Several mathematical stories originated with the second wedding of the mathematician and astronomer Johannes Kepler. Here is one: While shopping for wine for his wedding, Kepler noticed that the price of a barrel of wine (here assumed to be a cylinder) was determined solely by the length d of a dipstick that was inserted diagonally through a centered hole in the top of the barrel to the edge of the base of the barrel (see figure). Kepler realized that this measurement does not determine the volume of the barrel and that for a fixed value of d, the volume varies with the radius r and height h of the barrel. For a fixed value of d, what is the ratio r/h that maximizes the volume of the barrel?



44. Maximizing profit Suppose you own a tour bus and you book groups of 20 to 70 people for a day tour. The cost per person is \$30 minus \$0.25 for every ticket sold. If gas and other miscellaneous costs are \$200, how many tickets should you sell to maximize your profit? Treat the number of tickets as a nonnegative real number.

45. Maximum-volume cylinder in a sphere Find the dimensions of the right circular cylinder of maximum volume that can be placed inside of a sphere of radius R.

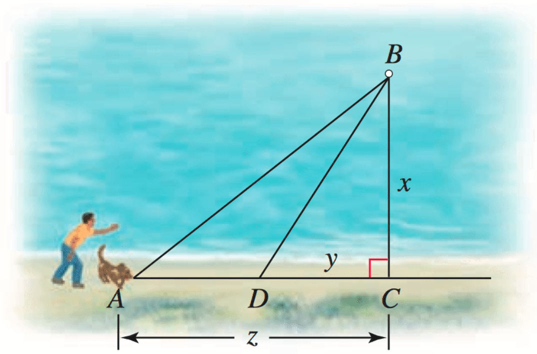
46. Cone in a sphere Find the height h, radius r, and volume of a right circular cone with maximum volume that is inscribed in a sphere of radius R.

47. Cone in a cone A right circular cone is inscribed inside a larger right circular cone with a volume of 150 cm^3. The axes of the cones coincide and the vertex of the inner cone touches the center of the base of the outer cone. Find the ratio of the heights of the cones that maximizes the volume of the inner cone.

48. Do dogs know calculus? A mathematician stands on a beach with his dog at point A. He throws a tennis ball so that it hits the water at point B. The dog, wanting to get to the tennis ball as quickly as possible, runs along the straight beach line to point D and then swims from point D to point B to retrieve his ball. Assume C is the point on the edge of the beach closest to the tennis ball (see figure).

- **a.** Assume the dog runs at speed r and swims at speed s, where $r > s$ and both are measured in meters/second. Also assume the lengths of BC, CD, and AC are x, y, and z, respectively. Find a function $T(y)$ representing the total time it takes for the dog to get to the ball.
- **b.** Verify that the value of y that minimizes the time it takes to retrieve the ball is
$$y = \frac{x}{\sqrt{r/s+1}\ \sqrt{r/s-1}}.$$
- **c.** If the dog runs at 8 m/s and swims at 1 m/s, what ratio y/x produces the fastest retrieving time?
- **d.** A dog named Elvis who runs at 6.4 m/s and swims at 0.910 m/s was found to use an average ratio of $\frac{y}{x}$ of 0.144 to retrieve his ball. Does Elvis appear to know calculus?

(*Source:* T. Pennings, *Do Dogs Know Calculus? The College Mathematics Journal*, 34, 3, May 2003)

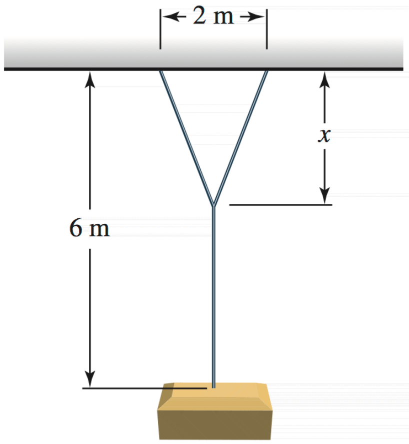


49. Travel costs A simple model for travel costs involves the cost of gasoline and the cost of a driver. Specifically, assume gasoline costs $\$p$/gallon and the vehicle gets g miles per gallon. Also, assume the driver earns $\$w$/hour.

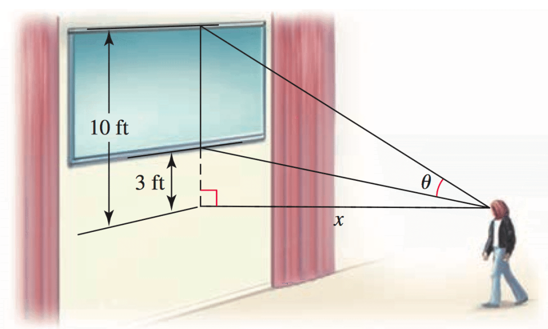
- **a.** A plausible function to describe how gas mileage (in mi/gal) varies with speed is $g(v) = v(85 - v)/60$. Evaluate $g(0)$, $g(40)$, and $g(60)$ and explain why these values are reasonable.
- **b.** At what speed does the gas mileage function have its maximum?
- **c.** Explain why $C(v) = Lp/g(v) + Lw/v$ gives the cost of the trip in dollars, where L is the length of the trip and v is the constant speed. Show that the dimensions are consistent.
- **d.** Let $L = 400$ mi, $p = \$4$/gal, and $w = \$20$/hr. At what (constant) speed should the vehicle be driven to minimize the cost of the trip?
- **e.** Should the optimal speed be increased or decreased (compared with part (d)) if L is increased from 400 mi to 500 mi? Explain.
- **f.** Should the optimal speed be increased or decreased (compared with part (d)) if p is increased from \$4/gal to \$4.20/gal? Explain.

g. Should the optimal speed be increased or decreased (compared with part (d)) if w is decreased from \$20/hr to \$15/hr? Explain.

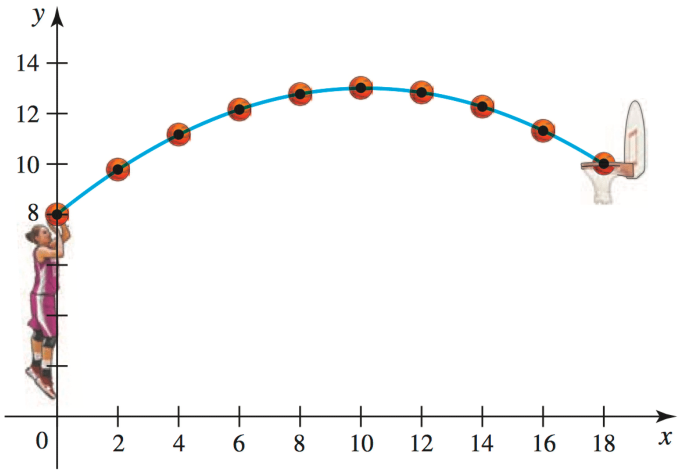
50. **Suspension system** A load must be suspended 6 m below a high ceiling using cables attached to two supports that are 2 m apart (see figure). How far below the ceiling (x in the figure) should the cables be joined to minimize the total length of cable used?



51. **Viewing angles** An auditorium with a flat floor has a large screen on one wall. The lower edge of the screen is 3 ft above eye level and the upper edge of the screen is 10 ft above eye level (see figure). How far from the screen should you stand to maximize your viewing angle? (*Hint:* Use the identity $\tan(\alpha - \beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$.)



T **52. Basketball shot** A basketball is shot with an initial velocity of v ft/s at an angle of 45° to the floor. The center of the basketball is 8 ft above the floor at a horizontal distance of 18 feet from the center of the basketball hoop when it is released. The height h (in feet) of the center of the basketball after it has traveled a horizontal distance of x feet is modeled by the function $h(x) = -\dfrac{32x^2}{v^2} + x + 8$ (see figure).



a. Find the initial velocity v if the center of the basketball passes the center of the hoop that is located 10 ft above the floor. Assume the ball does not hit the front of the hoop (otherwise it might not pass though the center of the hoop). The validity of this assumption is explored in the remainder of this exercise.

b. During the flight of the basketball, show that the distance s from the center of the basketball to the front of the hoop is

$$s = \sqrt{(x - 17.25)^2 + \left(-\frac{4\,x^2}{81} + x - 2\right)^2}\,.$$

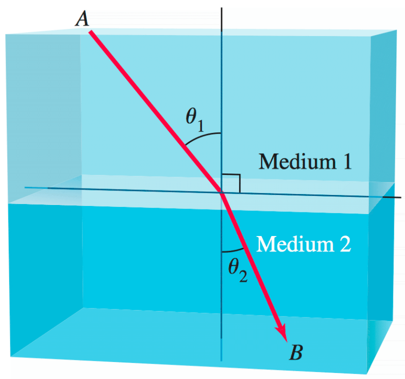
(*Hint:* The diameter of basketball hoop is 18 inches.)

c. Determine whether the assumption that the basketball does not hit the front of the hoop in part (a) is valid. Use the fact that the diameter of a women's basketball is about 9.23 inches. (*Hint*: The ball will hit the front of the hoop if, during its flight, the distance from the center of the ball and the front of the hoop is less than the radius of the basketball.)

d. A men's basketball has a diameter of about 9.5 inches. Would this larger ball lead to a different conclusion than in part (c)?

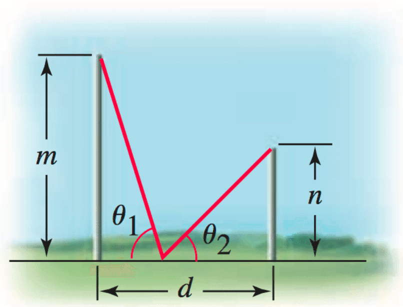
53. Light sources The intensity of a light source at a distance is directly proportional to the strength of the source and inversely proportional to the square of the distance from the source. Two light sources, one twice as strong as the other, are 12 m apart. At what point on the line segment joining the sources is the intensity the weakest?

54. Snell's Law Suppose a light source at A is in a medium in which light travels at a speed v_1 and the point B is in a medium in which light travels at a speed v_2 (see figure). Using Fermat's Principle, which states that light travels along the path that requires the minimum travel time (Exercise 55), show that the path taken between points A and B satisfies $(\sin\theta_1)/v_1 = (\sin\theta_2)/v_2$.



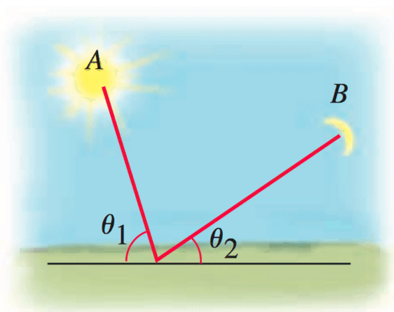
55. Fermat's Principle

a. Two poles of heights m and n are separated by a horizontal distance d. A rope is stretched from the top of one pole to the ground and then to the top of the other pole. Show that the configuration that requires the least amount of rope occurs when $\theta_1 = \theta_2$ (see figure).



(a)

b. Fermat's Principle states that when light travels between two points in the same medium (at a constant speed), it travels on the path that minimizes the travel time. Show that when light from a source A reflects off of a surface and is received at point B, the angle of incidence equals the angle of reflection, or $\theta_1 = \theta_2$ (see figure).

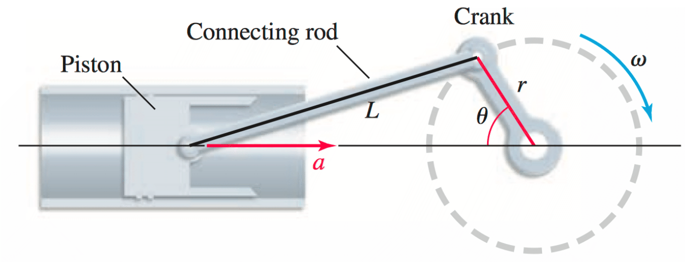


(b)

56. Crankshaft A crank of radius r rotates with an angular frequency ω. It is connected to a piston by a connecting rod of length L (see figure). The acceleration of the piston varies with the position of the crank according to the function

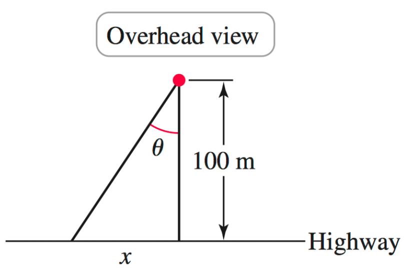
$$a(\theta) = \omega^2 r\left(\cos\theta + \frac{r\cos 2\theta}{L}\right).$$

For $\omega = 1$, $L = 2$, and $r = 1$, find the values of θ, with $0 \le \theta \le 2\pi$, for which the acceleration of the piston is a maximum and minimum.



57. Making silos A grain silo consists of a cylindrical concrete tower surmounted by a metal hemispherical dome. The metal in the dome costs 1.5 times as much as the concrete (per unit of surface area). If the volume of the silo is 750 m^3, what are the dimensions of the silo (radius and height of the cylindrical tower) that minimize the cost of the materials? Assume the silo has no floor and no flat ceiling under the dome.

58. Searchlight problem—narrow beam A searchlight is 100 m from the nearest point on a straight highway (see figure). As it rotates, the searchlight casts a horizontal beam that intersects the highway in a point. If the light revolves at a rate of $\pi/6$ rad/s, find the rate at which the beam sweeps along the highway as a function of θ. For what value of θ is this rate maximized?



59. Minimizing related functions Complete each of the following parts.

a. What value of x minimizes $f(x) = (x - a_1)^2 + (x - a_2)^2$, for constants a_1 and a_2?

b. What value of x minimizes $f(x) = (x - a_1)^2 + (x - a_2)^2 + (x - a_3)^2$, for constants a_1, a_2, and a_3?

c. Based upon the answers to parts (a) and (b), make a conjecture about the values of x that minimize $f(x) = (x - a_1)^2 + (x - a_2)^2 + \cdots + (x - a_n)^2$ for a positive integer n and constants $a_1, a_2, \ldots, a_n$. Use calculus to verify your conjecture.

(*Source:* T. Apostol, *Calculus,* Vol. 1, John Wiley and Sons, 1967)

Explorations and Challenges »

T **60. Gliding mammals** Many species of small mammals (such as flying squirrels and marsupial gliders) have the ability to walk and glide. Recent research suggests that these animals choose the most energy-efficient means of travel. According to one empirical model, the energy required for a glider with body mass m to walk a horizontal distance D is $8.46\,D\,m^{2/3}$ (where m is measured in grams, D is measured in meters, and energy is measured in microliters of oxygen consumed in respiration). The energy cost of climbing to a height $D\tan\theta$ and gliding a horizontal distance D at an angle of θ is modeled by $1.36\,m\,D\tan\theta$ (where $\theta = 0$ represents horizontal flight and $\theta > 45°$ represents controlled falling). Therefore, the function

$$S(m, \theta) = 8.46\,m^{2/3} - 1.36\,m\tan\theta$$

gives the energy difference per horizontal meter traveled between walking and gliding: If $S > 0$ for given values of m and θ, then it is more costly to walk than glide.

a. For what glide angles is it more efficient for a 200-gram animal to glide rather than walk?

b. Find an equation that relates θ to m in the case that walking and gliding are equally efficient. Does the angle θ increase or decrease as the mass m increases?

c. To make gliding more efficient than walking, do larger gliders have a larger or smaller selection of glide angles than smaller gliders?

d. Let $\theta = 25°$ (a typical glide angle). Graph S as a function of m, for $0 \le m \le 3000$. For what values of m is gliding more efficient?

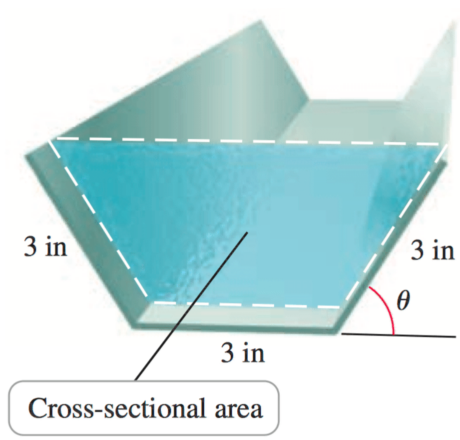
e. For $\theta = 25°$, what value of m (call it m^*) maximizes S?

f. Does m^*, as defined in part (e), increase or decrease with increasing θ? That is, as a glider reduces its glide angle, does its optimal size become larger or smaller?

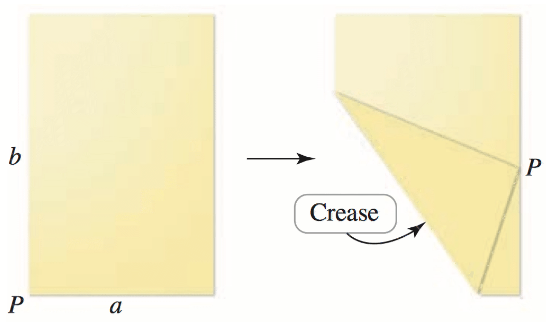
g. Assuming Dumbo is a gliding elephant whose weight is 1 metric ton $(10^6$ g), what glide angle would Dumbo use to be more efficient at gliding than walking?

(*Source:* R. Dial, *Energetic savings and the body size distribution of gliding mammals*, *Evolutionary Ecology Research*, 5, 2003)

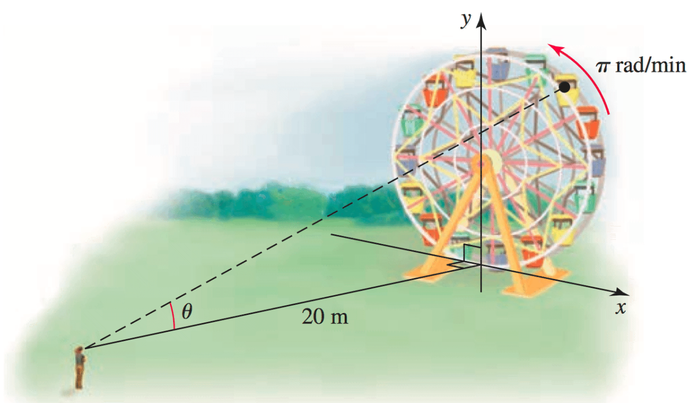
61. Metal rain gutters A rain gutter is made from sheets of metal 9 in wide. The gutters have a 3-in base and two 3-in sides, folded up at an angle θ (see figure). What angle θ maximizes the cross-sectional area of the gutter?



62. Crease-length problem A rectangular sheet of paper of width a and length b, where $0 < a < b$, is folded by taking one corner of the sheet and placing it at a point P on the opposite long side of the sheet (see figure). The fold is then flattened to form a crease across the sheet. Assuming that the fold is made so that there is no flap extending beyond the original sheet, find the point P that produces the crease of minimum length. What is the length of that crease?



T 63. Watching a Ferris wheel An observer stands 20 m from the bottom of a Ferris wheel on a line that is perpendicular to the face of the wheel, with her eyes at the level of the bottom of the wheel. The wheel revolves at a rate of π rad/min and the observer's line of sight with a specific seat on the Ferris wheel makes an angle θ with the horizontal (see figure). At what time during a full revolution is θ changing most rapidly?

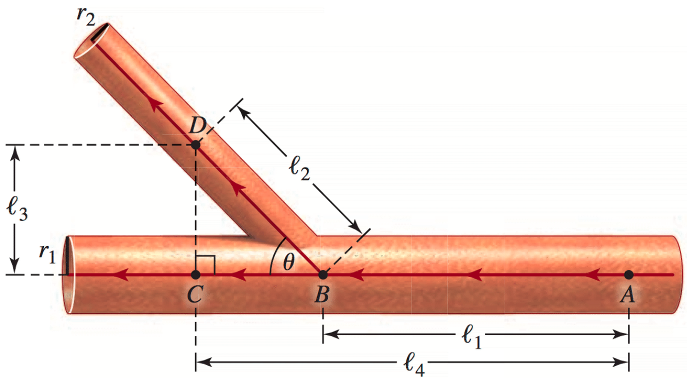


64. Blood flow The resistance to blood flow in the circulatory system is one measure of how hard the heart works to pump blood through blood vessels. Lower resistance may correspond to a healthier, higher efficiency circulatory system. Consider a smaller straight blood vessel of radius r_2 that branches off of a larger straight blood vessel of radius r_1 at an angle θ (see figure with given lengths ℓ_1, ℓ_2, ℓ_3, and ℓ_4). Using Poiseuille's Law, it can be shown that the total resistance T to blood flowing along the path from points A to B to D is $T = k\left(\dfrac{\ell_1}{r_1^{\,4}} + \dfrac{\ell_2}{r_2^{\,4}}\right)$, where $k > 0$ is a constant.

a. Show that $T = k\left(\dfrac{\ell_4 - \ell_3 \cot\theta}{r_1^{\,4}} + \dfrac{\ell_3 \csc\theta}{r_2^{\,4}}\right)$, assuming that the line segment from A to C to is perpendicular to the line segment from C to D and $0 < \theta < \dfrac{\pi}{2}$.

b. Show that the total resistance T in minimized when $\cos\theta = \left(\dfrac{r_2}{r_1}\right)^4$.

c. If the radius of the smaller vessel is 85% of the radius of the larger vessel, then find the value of θ that minimizes T. State your answer in degrees. (*Source*: *Blood Vessel Branching: Beyond the Standard Calculus Problem*, *Mathematics Magazine*, 84, 2011)



65. Sum of isosceles distances

a. An isosceles triangle has a base of length 4 and two sides of length $2\sqrt{2}$. Let P be a point on the perpendicular bisector of the base. Find the location P that minimizes the sum of the distances between P and the three vertices.

b. Assume in part (a) that the height of the isosceles triangle is $h > 0$ and its base has length 4. Show that the location of P that gives a minimum solution is independent of h for $h \geq \dfrac{2}{\sqrt{3}}$.

66. Cylinder and cones (Putnam Exam 1938) Right circular cones of height h and radius r are attached to each end of a right circular cylinder of height h and radius r, forming a double-pointed object. For a given surface area A, what are the dimensions r and h that maximize the volume of the object?

67. Slowest shortcut Suppose you are standing in a field near a straight section of railroad tracks just as the locomotive of a train passes the point nearest to you, which is $1/4$ mi away. The train, with length $1/3$ mi, is traveling at 20 mi/hr. If you start running in a straight line across the field, how slowly can you run and still catch the train? In which direction should you run?

68. Rectangles in triangles Find the dimensions and area of the rectangle of maximum area that can be inscribed in the following figures.

a. A right triangle with a given hypotenuse length L

b. An equilateral triangle with a given side length L

c. A right triangle with a given area A

d. An arbitrary triangle with a given area A (The result applies to any triangle, but first consider triangles for which all the angles are less than or equal to $90°$.)

69. Cylinder in a cone A right circular cylinder is placed inside a cone of radius R and height H so that the base of the cylinder lies on the base of the cone.

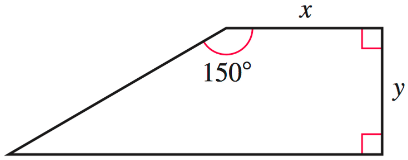
a. Find the dimensions of the cylinder with maximum volume. Specifically, show that the volume of the maximum-volume cylinder is $4/9$ the volume of the cone.

b. Find the dimensions of the cylinder with maximum lateral surface area (area of the curved surface).

70. Another pen problem A rancher is building a horse pen on the corner of her property using 1000 ft of fencing. Because of the unusual shape of her property, the pen must be built in the shape of a trapezoid (see figure).

a. Determine the lengths of the sides that maximize the area of the pen.

b. Suppose there is already a fence along the side of the property opposite the side of length y. Find the lengths of the sides that maximize the area of the pen, using 1000 ft of fencing.

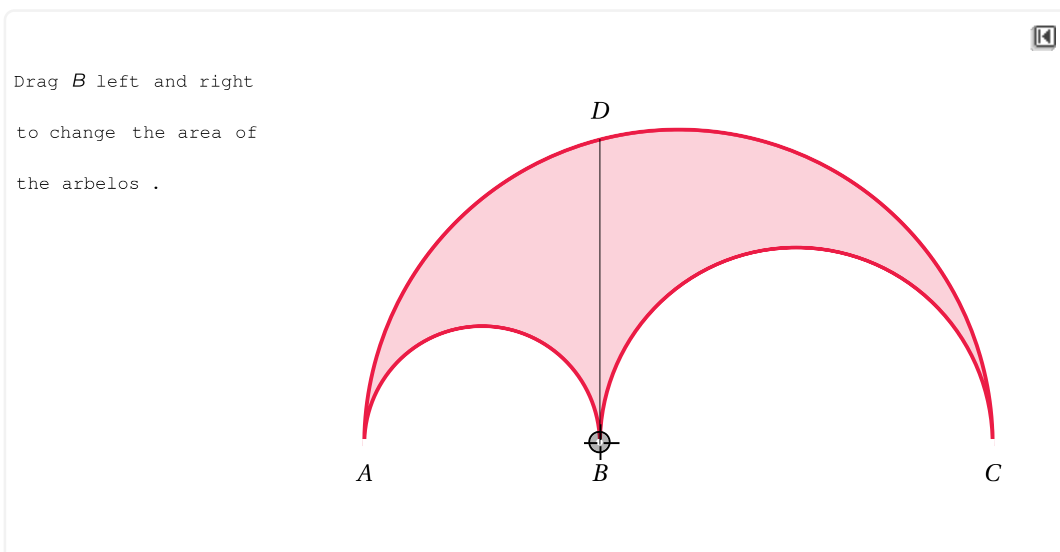


71. Minimum-length roads A house is located at each corner of a square with side lengths of 1 mi. What is the length of the shortest road system with straight roads that connects all of the houses by roads (that is, a road system that allows one to drive from any house to any other house)? (*Hint:* Place two points inside the square at which roads meet.) (*Source*: Paul Halmos, *Problems for Mathematicians Young and Old*, MAA 1991)

72. The arbelos An arbelos is the region enclosed by three mutually tangent semicircles; it is the region inside the larger semicircle and outside the two smaller semicircles (see figure).

a. Given an arbelos in which the diameter of the largest circle is 1, what positions of point B maximize the area of the arbelos?

b. Show that the area of the arbelos is the area of a circle whose diameter is the distance BD in the figure.



73. Proximity questions Find the point on the graph of $y = \sqrt{x}$ that is nearest the point $(p, 0)$ if (i) $p > \frac{1}{2}$; and (ii) $0 < p < \frac{1}{2}$. Express the answer in terms of p.

74. Turning a corner with a pole

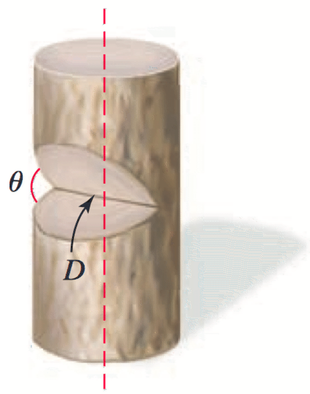
a. What is the length of the longest pole that can be carried horizontally around a corner at which a 3-ft corridor and a 4-ft corridor meet at right angles?

b. What is the length of the longest pole that can be carried horizontally around a corner at which a corridor that is a ft wide and a corridor that is b ft wide meet at right angles?

c. What is the length of the longest pole that can be carried horizontally around a corner at which a corridor that is $a = 5$ ft wide and a corridor that is $b = 5$ ft wide meet at an angle of $120°$?

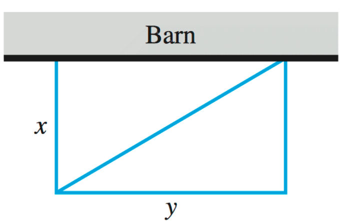
d. What is the length of the longest pole that can be carried around a corner at which a corridor that is a ft wide and a corridor that is b feet wide meet at right angles, assuming there is an 8-foot ceiling and that you may tilt the pole at any angle?

75. Tree notch (Putnam Exam 1938, rephrased) A notch is cut in a cylindrical vertical tree trunk (see figure). The notch penetrates to the axis of the cylinder and is bounded by two half-planes that intersect on a diameter D of the tree. The angle between the two half planes is θ. Prove that for a given tree and fixed angle θ, the volume of the notch is minimized by taking the bounding planes at equal angles to the horizontal plane that also passes through D.



76. Circle in a triangle What are the radius and area of the circle of maximum area that can be inscribed in an isosceles triangle whose two equal sides have length 1?

T **77. A challenging pen problem** A farmer uses 200 meters of fencing to build two triangular pens against a barn (see figure); the pens are constructed with three sides and a diagonal dividing fence. What dimensions maximize the area of the pen?



78. Folded boxes Squares with sides of length x are cut out of each corner of a rectangular piece of cardboard with sides of length ℓ and L. The resulting piece of cardboard is then folded into a box without a lid. Holding ℓ fixed, find the size of the corner squares x that maximizes the volume of the box as $L \to \infty$. (*Source: Mathematics Teacher*, Nov 2002)