4.4 Graphing Functions

We have now collected the tools required for a comprehensive approach to graphing functions. These *analytical methods* are indispensable, even with the availability of powerful graphing utilities, as illustrated by the following example.

Calculators and Analysis »

Suppose you want to graph the harmless-looking function $f(x) = \frac{x^3}{3} - 400x$. The result of plotting f using a graphing calculator with a default window of $[-10, 10] \times [-10, 10]$ is shown in **Figure 4.42a**; one vertical line appears on the screen. Zooming out to the window $[-100, 100] \times [-100, 100]$ produces three vertical lines (**Figure 4.42b**); it is still difficult to understand the behavior of the function using only this graph. Expanding the window even more to $[-1000, 1000] \times [-1000, 1000]$ is no better. So, what do we do?

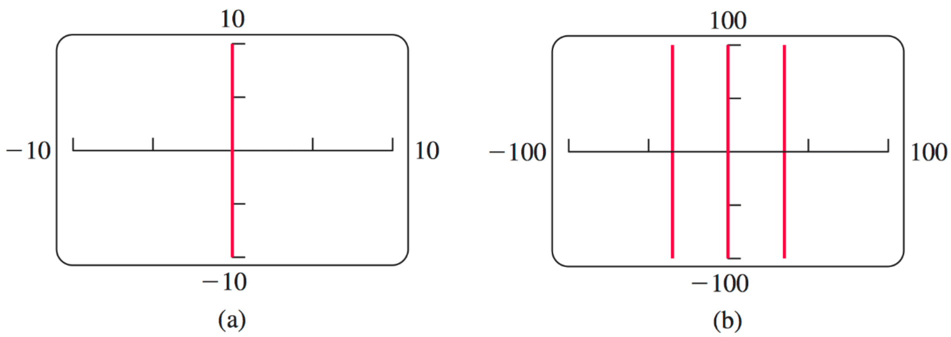


Figure 4.42

Quick Check 1 Graph $f(x) = \frac{x^3}{3} - 400x$ using various windows on a graphing utility. Find a window that gives a better graph of f than those in Figure 4.42. ◆

Answer »

The function $f(x) = \frac{x^3}{3} - 400x$ has a reasonable graph (see Example 2), but it cannot be found automatically by letting technology do all the work. Here is the message of this section: Graphing utilities are valuable for exploring functions, producing preliminary graphs, and checking your work. But they should not be relied on exclusively because they cannot explain *why* a graph has its shape. Rather, graphing utilities should be used in an interactive way with the analytical methods presented in this chapter.

Graphing Guidelines »

The following set of guidelines need not be followed exactly for every function, and you will find that several steps can often be done at once. Depending on the specific problem, some of the steps are best done analytically, while other steps can be done with a graphing utility. Experiment with both approaches and try to find a good balance. We also present a schematic record-keeping procedure to keep track of discoveries as they are made.

Graphing Guidelines for $y = f(x)$

1. Identify the domain or interval of interest. On what interval should the function be graphed? It may be the domain of the function or some subset of the domain.

2. Exploit symmetry. Take advantage of symmetry. For example, is the function *even* ($f(-x) = f(x)$), *odd* ($f(-x) = -f(x)$), or neither?

3. Find the first and second derivatives. They are needed to determine extreme values, concavity, inflection points, and the intervals on which f is increasing and decreasing. Computing derivatives—particularly second derivatives—may not be practical, so some functions may need to be graphed without complete derivative information.

4. Find critical points and possible inflection points. Determine points at which $f'(x) = 0$ or f' is undefined. Determine points at which $f''(x) = 0$ or f'' is undefined.

5. Find intervals on which the function is increasing/decreasing and concave up/down. The first derivative determines the intervals on which f is increasing or decreasing. The second derivative determines the intervals on which the function is concave up or concave down.

6. Identify extreme values and inflection points. Use either the First or the Second Derivative Test to classify the critical points. Both x- and y-coordinates of maxima, minima, and inflection points are needed for graphing.

7. Locate all vertical/horizontal asymptotes and determine end behavior. Vertical asymptotes often occur at zeros of denominators. Horizontal asymptotes require examining limits as $x \to \pm\infty$; these limits determine end behavior. Slant asymptotes occur with rational functions in which the degree of the numerator is one more than the degree of the denominator.

8. Find the intercepts. The y-intercept of the graph is found by setting $x = 0$. The x-intercepts are found by solving $f(x) = 0$; they are the real zeros (or roots) of f.

9. Choose an appropriate graphing window and make a graph. Use the results of the above steps to graph the function. If you use graphing software, check for consistency with your analytical work. Is your graph *complete*—that is, does it show all the essential details of the function?

Note »

Limits at infinity determine the end behavior of many functions, though a modification is required for some functions. For example, the end behavior of $f(x) = \dfrac{1 + x}{x + \sqrt{x}}$ is found by computing $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to\infty} f(x)$ because the domain of f is $(0, \infty)$.

EXAMPLE 1 A warm-up

Given the following information about the first and second derivatives of a function f that is continuous on $(-\infty, \infty)$, summarize the information using a number line, and then sketch a possible graph of f.

$f' < 0,\ f'' > 0$ on $(-\infty, 0)$ $\quad f' > 0,\ f'' > 0$ on $(0, 1)$ $\quad f' > 0,\ f'' < 0$ on $(1, 2)$

$f' < 0,\ f'' < 0$ on $(2, 3)$ $\quad f' < 0,\ f'' > 0$ on $(3, 4)$ $\quad f' > 0,\ f'' > 0$ on $(4, \infty)$

SOLUTION »

Figure 4.43 uses the given information to determine the behavior of f and its graph. For example, on the interval $(-\infty, 0)$, f is decreasing and concave up; so we sketch a segment of a curve with these properties on this interval. Continuing in this manner we obtain a useful summary of the properties of f.

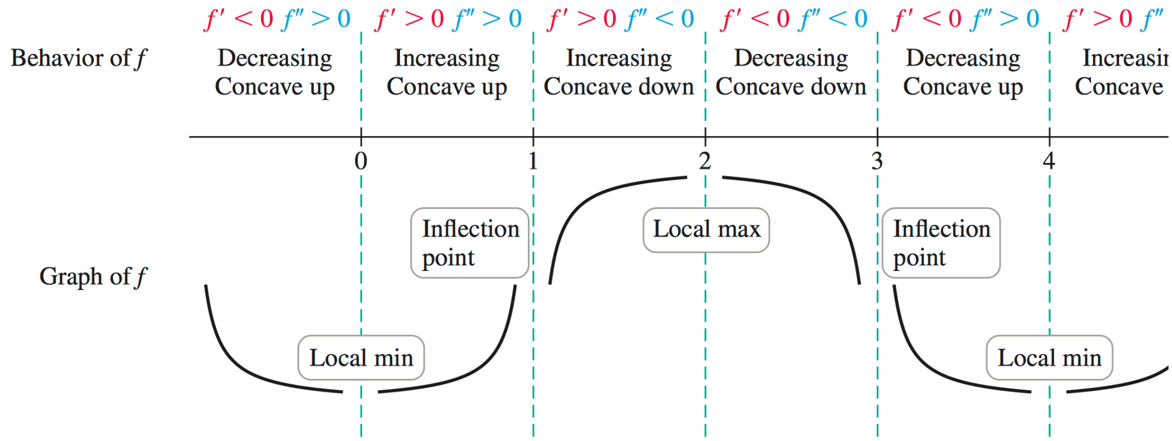


Figure 4.43

Assembling the information shown in Figure 4.43, a possible graph of f is produced (**Figure 4.44**). Notice that derivative information is not sufficient to determine the y-coordinates of points on the curve.

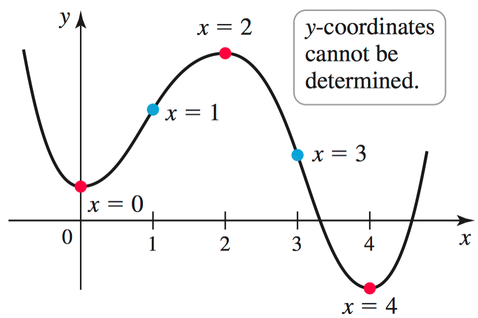


Figure 4.44

Related Exercises 7–8 ◆

Quick Check 2 Explain why the functions f and $f + C$, where C is a constant, have the same derivative properties. ◆

Answer »

EXAMPLE 2 A deceptive polynomial

Use the graphing guidelines to graph $f(x) = \dfrac{x^3}{3} - 400x$ on its domain.

SOLUTION »

1. **Domain** The domain of any polynomial is $(-\infty, \infty)$.

2. **Symmetry** Because f consists of odd powers of the variable, it is an odd function. Its graph is symmetric about the origin.

3. Derivatives The derivatives of f are

$$f'(x) = x^2 - 400 \quad \text{and} \quad f''(x) = 2x.$$

Note »

4. Critical points and possible inflection points Solving $f'(x) = 0$, we find that the critical points are $x = \pm 20$. Solving $f''(x) = 0$, we see that a possible inflection point occurs at $x = 0$.

5. Increasing/decreasing and concavity Note that

$$f'(x) = x^2 - 400 = (x - 20)(x + 20).$$

Solving the inequality $f'(x) < 0$, we find that f is decreasing on the interval $(-20, 20)$. Solving the inequality $f'(x) > 0$ reveals that f is increasing on the intervals $(-\infty, -20)$ and $(20, \infty)$ (**Figure 4.45**). By the First Derivative Test, we have enough information to conclude that f has a local maximum at $x = -20$ and a local minimum at $x = 20$.

Note »

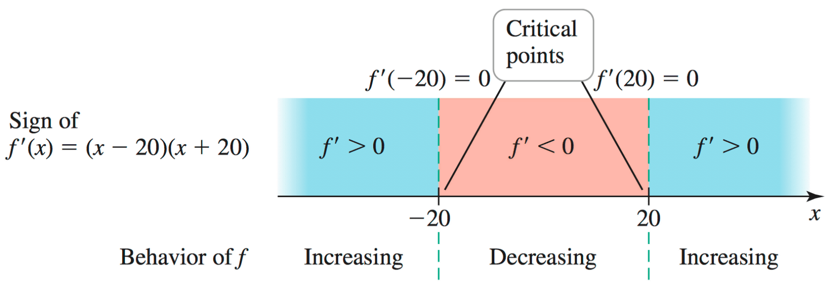


Figure 4.45

Furthermore, $f''(x) = 2x < 0$ on $(-\infty, 0)$, so f is concave down on this interval. Also, $f''(x) > 0$ on $(0, \infty)$, so f is concave up on $(0, \infty)$ (**Figure 4.46**).

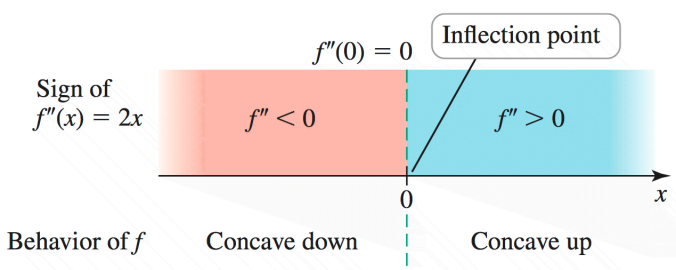


Figure 4.46

The evidence obtained so far is summarized in **Figure 4.47**.

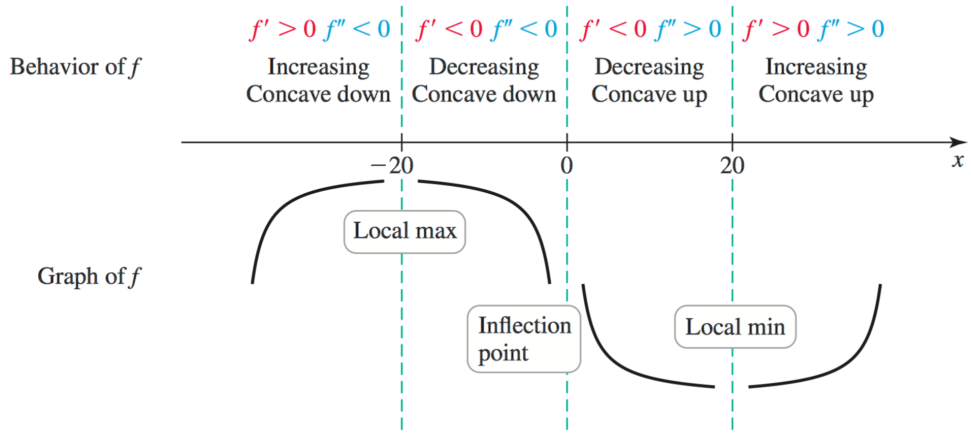


Figure 4.47

6. Extreme values and inflection points In this case, the Second Derivative Test is easily applied and it confirms what we have already learned. Because $f''(-20) < 0$ and $f''(20) > 0$, f has a local maximum at $x = -20$ and a local minimum at $x = 20$. The corresponding function values are $f(-20) = \frac{16{,}000}{3} = 5333\frac{1}{3}$ and $f(20) = -f(-20) = -5333\frac{1}{3}$. Finally, we see that f'' changes sign at $x = 0$, making $(0, 0)$ an inflection point.

7. Asymptotes and end behavior Polynomials have neither vertical nor horizontal asymptotes. Because the highest-power term in the polynomial is x^3 (an odd power) and the leading coefficient is positive, we have the end behavior

$$\lim_{x\to\infty} f(x) = \infty \quad \text{and} \quad \lim_{x\to-\infty} f(x) = -\infty.$$

8. Intercepts The y-intercept is $(0, 0)$. We solve the equation $f(x) = 0$ to find the x-intercepts:

$$\frac{x^3}{3} - 400\,x = x\left(\frac{x^2}{3} - 400\right) = 0.$$

The roots of this equation are $x = 0$ and $x = \pm\sqrt{1200} \approx \pm 34.6$.

9. Graph the function Using the information found in Steps 1–8, we choose the graphing window $[-40, 40] \times [-6000, 6000]$ and produce the graph shown in **Figure 4.48**. Notice that the symmetry detected in Step 2 is evident in this graph.

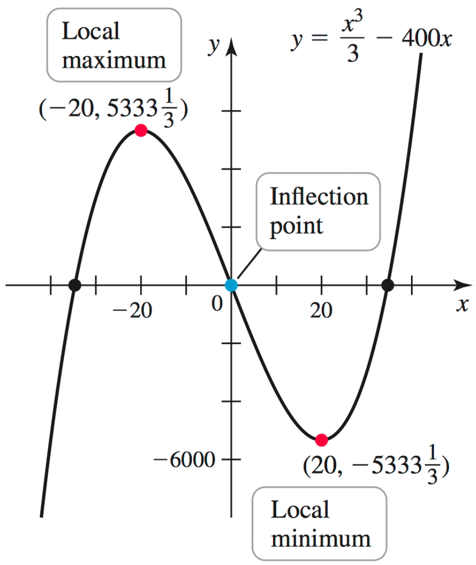


Figure 4.48

Related Exercises 17–18 ◆

EXAMPLE 3 The surprises of a rational function

Use the graphing guidelines to graph $f(x) = \dfrac{10x^3}{x^2 - 1}$ on its domain.

SOLUTION »

1. Domain The zeros of the denominator are $x = \pm 1$, so the domain is $\{x : x \neq \pm 1\}$.

2. Symmetry This function consists of an odd function divided by an even function. The product or quotient of an even function and an odd function is odd. Therefore, the graph is symmetric about the origin.

3. Derivatives The Quotient Rule is used to find the first and second derivatives:

$$f'(x) = \frac{10x^2(x^2 - 3)}{(x^2 - 1)^2} \quad \text{and} \quad f''(x) = \frac{20x(x^2 + 3)}{(x^2 - 1)^3}.$$

4. Critical points and possible inflection points The solutions of $f'(x) = 0$ occur where the numerator equals 0, provided the denominator is nonzero at those points. Solving $10x^2(x^2 - 3) = 0$ gives the critical points $x = 0$ and $x = \pm\sqrt{3}$. The solutions of $f''(x) = 0$ are found by solving $20x(x^2 + 3) = 0$; we see that the only possible inflection point occurs at $x = 0$.

5. Increasing/decreasing and concavity To find the sign of f', first note that the denominator of f' is nonnegative, as is the factor $10x^2$ in the numerator. So the sign of f' is determined by the factor $x^2 - 3$, which is negative on $(-\sqrt{3}, \sqrt{3})$ and positive on $(-\infty, -\sqrt{3})$ and $(\sqrt{3}, \infty)$. Therefore, f is decreasing on $(-\sqrt{3}, \sqrt{3})$ (excluding $x = \pm 1$) and increasing on $(-\infty, -\sqrt{3})$ and $(\sqrt{3}, \infty)$.

The sign of f'' is a bit trickier. Because $x^2 + 3$ is positive, the sign of f'' is determined by the sign of x in the numerator and $(x^2 - 1)^3$ in the denominator. When x and $(x^2 - 1)^3$ have the same sign, $f''(x) > 0$; when x

and $(x^2-1)^3$ have opposite signs, $f''(x)<0$ (Table 4.1). The results of this analysis are shown in **Figure 4.49**.

Note »

Table 4.1

	$20x$	x^2+3	$(x^2-1)^3$	Sign of f''
$(-\infty, -1)$	$-$	$+$	$+$	$-$
$(-1, 0)$	$-$	$+$	$-$	$+$
$(0, 1)$	$+$	$+$	$-$	$-$
$(1, \infty)$	$+$	$+$	$+$	$+$

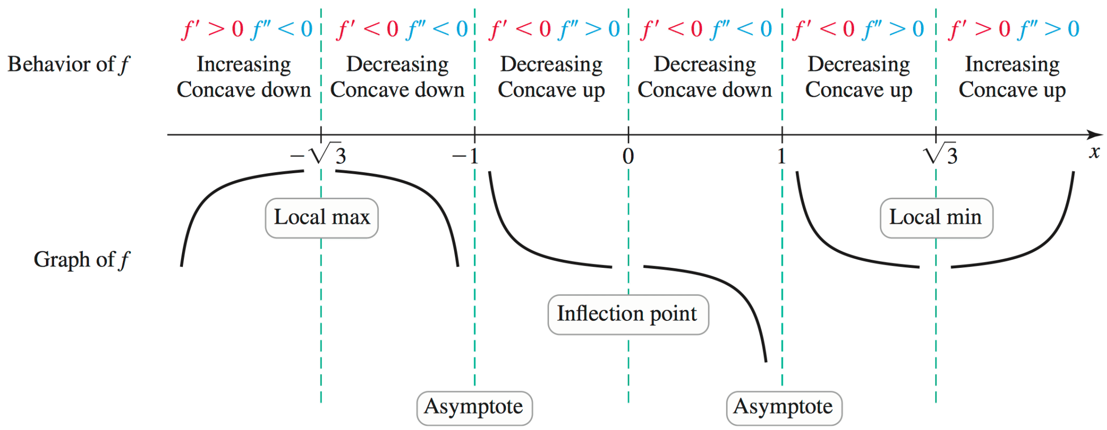


Figure 4.49

6. Extreme values and inflection points The First Derivative Test is easily applied by looking at Figure 4.49. The function is increasing on $(-\infty, -\sqrt{3})$ and decreasing on $(-\sqrt{3}, -1)$; therefore, f has a local maximum at $x=-\sqrt{3}$, where $f(-\sqrt{3})=-15\sqrt{3}$. Similarly, f has a local minimum at $x=\sqrt{3}$, where $f(\sqrt{3})=15\sqrt{3}$. (These results could also be obtained with the Second Derivative Test.) There is no local extreme value at the critical point $x=0$, only a horizontal tangent line.

Using Table 4.1 from Step 5, we see that f'' changes sign at $x=\pm 1$ and at $x=0$. The points $x=\pm 1$ are not in the domain of f, so they cannot correspond to inflection points. However, there is an inflection point at (0, 0).

7. Asymptotes and end behavior Recall from Section 2.4 that zeros of the denominator, which in this case are $x=\pm 1$, are candidates for vertical asymptotes. Checking the sign of f on either side of $x=\pm 1$, we find

$$\lim_{x\to -1^-} f(x)=-\infty, \qquad \lim_{x\to -1^+} f(x)=\infty,$$
$$\lim_{x\to 1^-} f(x)=-\infty, \qquad \lim_{x\to 1^+} f(x)=\infty.$$

It follows that f has vertical asymptotes at $x=\pm 1$. The degree of the numerator is greater than the degree of the denominator, so there are no horizontal asymptotes. Using long division, it can be shown that

$$f(x)=10x+\frac{10x}{x^2-1}.$$

Therefore, as $x\to\pm\infty$, the graph of f approaches the line $y=10x$. This is a slant asymptote (Section 2.5).

8. **Intercepts** The zeros of a rational function coincide with the zeros of the numerator, provided those points are not also zeros of the denominator. In this case, the zeros of f satisfy $10x^3 = 0$, or $x = 0$ (which is not a zero of the denominator). Therefore, $(0, 0)$ is both the x- and y-intercept.

9. **Graphing** We now assemble an accurate graph of f, as shown in **Figure 4.50**. A window of $[-3, 3] \times [-40, 40]$ gives a complete graph of the function. Notice that the symmetry about the origin deduced in Step 2 is apparent in the graph.

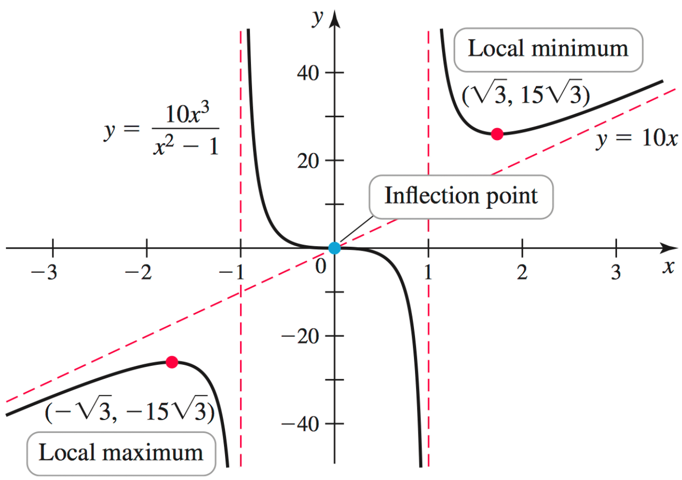


Figure 4.50

Related Exercises 30–31 ◆

Quick Check 3 Verify that the function f in Example 3 is symmetric about the origin by showing that $f(-x) = -f(x)$. ◆

Answer »

In the next example, we show how the guidelines may be streamlined to some extent.

EXAMPLE 4 Roots and cusps

Graph $f(x) = \frac{1}{8}x^{2/3}(9x^2 - 8x - 16)$ on its domain.

SOLUTION »

The domain of f is $(-\infty, \infty)$. The polynomial factor in f consists of both even and odd powers, so f has no special symmetry. Computing the first derivative is straightforward if you first expand f as a sum of three terms:

$$f'(x) = \frac{d}{dx}\left(\frac{9x^{8/3}}{8} - x^{5/3} - 2x^{2/3}\right) \quad \text{Expand } f.$$

$$= 3x^{5/3} - \frac{5}{3}x^{2/3} - \frac{4}{3}x^{-1/3} \quad \text{Differentiate.}$$

$$= \frac{(x-1)(9x+4)}{3x^{1/3}}. \quad \text{Simplify.}$$

The critical points are now identified: f' is undefined at $x = 0$ (because $x^{-1/3}$ is undefined there) and

$f'(x) = 0$ at $x = 1$ and $x = -\frac{4}{9}$. So we have three critical points to analyze. Table 4.2 tracks the signs of the three factors in f' and shows the sign of f' on the relevant intervals; this information is recorded in **Figure 4.51**.

Table 4.2

	$3x^{1/3}$	$9x+4$	$x-1$	Sign of f'
$\left(-\infty, -\frac{4}{9}\right)$	−	−	−	−
$\left(-\frac{4}{9}, 0\right)$	−	+	−	+
$(0, 1)$	+	+	−	−
$(1, \infty)$	+	+	+	+

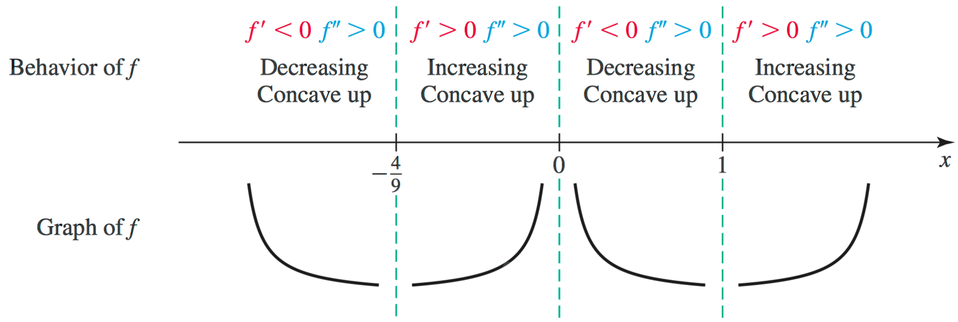


Figure 4.51

We use the second line in the calculation of f' to compute the second derivative:

$$\begin{aligned} f''(x) &= \frac{d}{dx}\left(3x^{5/3} - \frac{5}{3}x^{2/3} - \frac{4}{3}x^{-1/3}\right) \\ &= 5x^{2/3} - \frac{10}{9}x^{-1/3} + \frac{4}{9}x^{-4/3} && \text{Differentiate.} \\ &= \frac{45x^2 - 10x + 4}{9x^{4/3}}. && \text{Simplify.} \end{aligned}$$

Solving $f''(x) = 0$, we discover that $f''(x) > 0$ for all x, except $x = 0$, where it is undefined. Therefore, f is concave up on $(-\infty, 0)$ and $(0, \infty)$ (Figure 4.51).

Note »

By the Second Derivative Test, because $f''(x) > 0$ for $x \neq 0$, the critical points $x = -\frac{4}{9}$ and $x = 1$ correspond to local minima; their y-coordinates are $f\left(-\frac{4}{9}\right) \approx -0.78$ and $f(1) = -\frac{15}{8} = -1.875$.

What about the third critical point $x = 0$? Note that $f(0) = 0$, and f is increasing just to the left of 0 and decreasing just to the right. By the First Derivative Test, f has a local maximum at $x = 0$. Furthermore, $f'(x) \to \infty$ as $x \to 0^-$ and $f'(x) \to -\infty$ as $x \to 0^+$, so the graph of f has a cusp at $x = 0$.

As $x \to \pm\infty$, f is dominated by its highest-power term, which is $\frac{9x^{8/3}}{8}$. This term becomes large and positive as $x \to \pm\infty$; therefore, f has no absolute maximum. Its absolute minimum occurs at $x = 1$ because, comparing the two local minima, $f(1) < f\left(-\frac{4}{9}\right)$.

The roots of f satisfy $\frac{1}{8}x^{2/3}(9x^2 - 8x - 16) = 0$, which gives $x = 0$ and

$$x = \frac{4}{9}\left(1 \pm \sqrt{10}\right) \approx -0.96 \text{ or } 1.85. \quad \text{Use the quadratic formula.}$$

With the information gathered in this analysis, we obtain the graph shown in **Figure 4.52**.

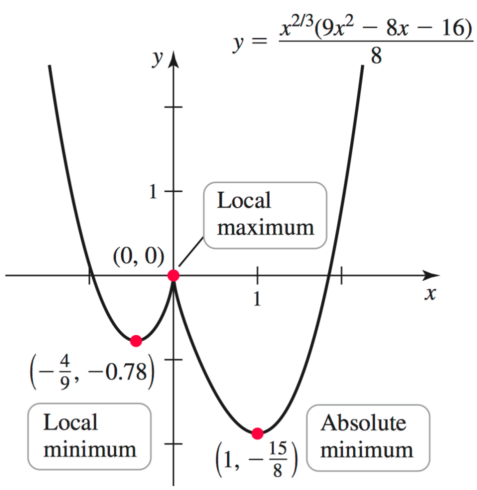


Figure 4.52

Related Exercises 38–40 ◆

Exercises »

Getting Started »

Practice Exercises »

13. Let $f(x) = (x - 3)(x + 3)^2$.

- **a.** Verify that $f'(x) = 3(x - 1)(x + 3)$ and $f''(x) = 6(x + 1)$.
- **b.** Find the critical points and possible inflection points of f.
- **c.** Find the intervals on which f is increasing and decreasing.
- **d.** Determine the intervals on which f is concave up or concave down.
- **e.** Identify the local extreme values and inflections points of f.
- **f.** State the x- and y-intercepts of the graph of f.
- **g.** Use your work in parts (a) through (f) to sketch a graph of f.

14. If $f(x) = \dfrac{1}{3\,x^4 + 5}$, it can be shown that $f\,'\,(x) = -\dfrac{12\,x^3}{\left(3\,x^4 + 5\right)^2}$ and

$f\,''\,(x) = \dfrac{180\,x^2\left(x^2 + 1\right)(x + 1)(x - 1)}{\left(3\,x^4 + 5\right)^3}$. Use these functions to complete the following steps.

- **a.** Find the critical points and possible inflection points of f.
- **b.** Find the intervals on which f is increasing and decreasing.
- **c.** Determine the intervals on which f is concave up or concave down.
- **d.** Identify the local extreme values and inflections points of f.
- **e.** State the x- and y-intercepts of the graph of f.
- **f.** Find the asymptotes of f.
- **g.** Use your work in parts (a) through (f) to sketch a graph of f.

15–46. Graphing functions *Use the guidelines of this section to make a complete graph of f.*

15. $f(x) = x^2 - 6\,x$

16. $f(x) = x - x^2$

17. $f(x) = x^3 - 6\,x^2 + 9\,x$

18. $f(x) = 3\,x - x^3$

19. $f(x) = x^4 - 6\,x^2$

20. $f(x) = x^4 + 4\,x^3$

21. $f(x) = (x - 6)(x + 6)^2$

22. $f(x) = 27\,(x - 2)^2\,(x + 2)$

23. $f(x) = x^3 - 6\,x^2 - 135\,x$

24. $f(x) = x^4 + 8\,x^3 - 270\,x^2 + 1$

25. $f(x) = x^3 - 3\,x^2 - 144\,x - 140$

26. $f(x) = x^3 - 147\,x + 286$

27. $f(x) = x - 2\,\sqrt{x}$

28. $f(x) = 3\,\sqrt{x} - x^{3/2}$

29. $f(x) = \dfrac{3\,x}{x^2 - 1}$

30. $f(x) = \dfrac{2\,x - 3}{2\,x - 8}$

31. $f(x) = \dfrac{x^2}{x-2}$

32. $f(x) = \dfrac{x^2}{x^2-4}$

33. $f(x) = \dfrac{x^2+12}{2x+1}$

34. $f(x) = \dfrac{4x}{x^2+3}$

35. $f(x) = \sqrt{x}\,(x-3)$

36. $f(x) = x^{1/3}(4-x)$

37. $f(x) = x + 2\cos x$ on $[-2\pi, 2\pi]$

38. $f(x) = x - 3x^{2/3}$

39. $f(x) = x - 3x^{1/3}$

40. $f(x) = 2 - 2x^{2/3} + x^{4/3}$

41. $f(x) = \sin x - x$ on $[0, 2\pi]$

42. $f(x) = x\sqrt{x+3}$

43. $f(x) = \dfrac{1+x}{\sqrt{x}}$

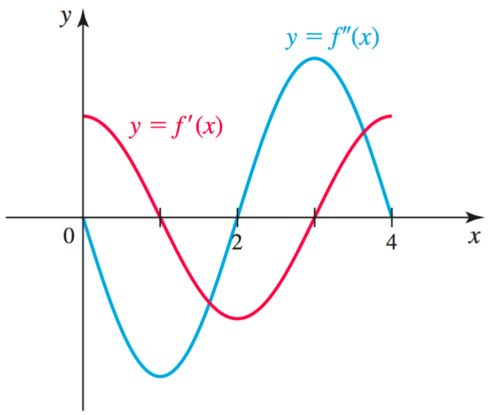
44. $f(x) = x + \cos 2x$

45. $f(x) = x + \tan x$ on $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

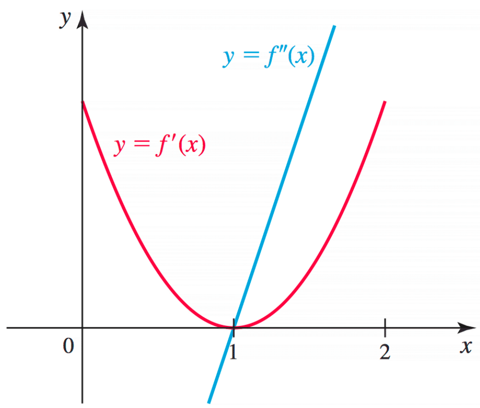
46. $f(x) = \dfrac{3}{x^2} - \dfrac{54}{x^4}$

47–48. *Use the graphs of f' and f'' to find the critical points and inflection points of f, the intervals on which f is increasing or decreasing, and the intervals of concavity. Then graph f assuming $f(0) = 0$.*

47.



48.



T **49–54. Graphing with technology** *Make a complete graph of the following functions. A graphing utility is useful in locating intercepts, local extreme values, and inflection points.*

49. $f(x) = \frac{1}{3}x^3 - 2x^2 - 5x + 2$

50. $f(x) = \frac{1}{15}x^3 - x + 1$

51. $f(x) = 3x^4 + 4x^3 - 12x^2$

52. $f(x) = x^3 - 33x^2 + 216x - 2$

53. $f(x) = \frac{3x - 5}{x^2 - 1}$

54. $f(x) = x^{1/3}(x - 2)^2$

55. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. If the zeros of f' are -3, 1, and 4, then the local extrema are located at these points.

b. If the zeros of f'' are -2 and 4, then the inflection points are located at these points.

c. If the zeros of the denominator of f are -3 and 4, then f has vertical asymptotes at these points.

d. If a rational function has a finite limit as $x \to \infty$, then it must have a finite limit as $x \to -\infty$.

56–59. Functions from derivatives *Use the derivative f' to determine the x-coordinates of the local maxima and minima of f, and the intervals on which f is increasing or decreasing. Sketch a possible graph of f (f is not unique).*

56. $f'(x) = (x-1)(x+2)(x+4)$

57. $f'(x) = 10 \sin 2x$ on $[-2\pi, 2\pi]$

58. $f'(x) = \frac{1}{6}(x+1)(x-2)^2(x-3)$

59. $f'(x) = x^2(x+2)(x-1)$

Explorations and Challenges »

60. Derivative information Suppose a continuous function f is concave up on $(-\infty, 0)$ and $(0, \infty)$. Assume f has a local maximum at $x = 0$. What, if anything, do you know about $f'(0)$? Explain with an illustration.

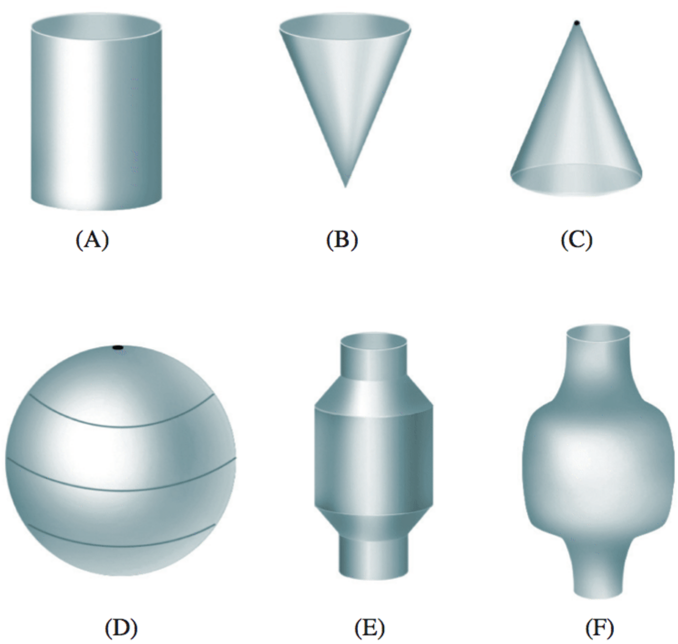
61. Height functions The figure shows six containers, each of which is filled from the top. Assume water is poured into the containers at a constant rate and each container is filled in 10 s. Assume also that the horizontal cross sections of the containers are always circles. Let $h(t)$ be the depth of water in the container at time t, for $0 \le t \le 10$.

a. For each container, sketch a graph of the function $y = h(t)$, for $0 \le t \le 10$.

b. Explain why h is an increasing function.

c. Describe the concavity of the function. Identify inflection points when they occur.

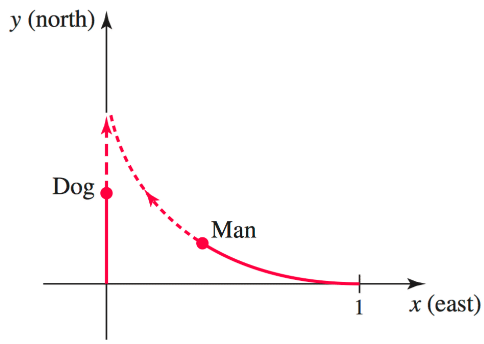
d. For each container, where does h' (the derivative of h) have an absolute maximum on $[0, 10]$?



T **62. A pursuit curve** A man stands 1 mi east of a crossroads. At noon, a dog starts walking north from the crossroads at 1 mi/hr (see figure). At the same instant, the man starts walking and at all times walks directly toward the dog at $s > 1$ mi/hr. The path in the xy-plane followed by the man as he pursues the dog is given by the function

$$y = f(x) = \frac{s}{2}\left(\frac{x^{(s+1)/s}}{s+1} - \frac{x^{(s-1)/s}}{s-1}\right) + \frac{s}{s^2-1}.$$

Select various values of $s > 1$ and graph this pursuit curve. Comment on the changes in the curve as s increases.



T **63–66. Combining technology with analytical methods** *Use a graphing utility together with analytical methods to create a complete graph of the following functions. Be sure to find and label the intercepts, local extrema, inflection points, and asymptotes, and find the intervals on which the function is increasing or decreasing, and the intervals on which the function is concave up or concave down.*

63. $f(x) = \dfrac{x \sin x}{x^2+1}$ on $[-2\pi, 2\pi]$

64. $f(x) = 3\sqrt[4]{x} - \sqrt{x} - 2$

65. $f(x) = 3x^4 - 44x^3 + 60x^2$ (*Hint:* Two different graphing windows may be needed.)

66. $f(x) = \dfrac{\sqrt{4x^2+1}}{x^2+1}$

T **67–71. Special curves** *The following classical curves have been studied by generations of mathematicians. Use analytical methods (including implicit differentiation) and a graphing utility to graph the curves. Include as much detail as possible.*

67. $x^{2/3} + y^{2/3} = 1$ (Astroid or hypocycloid with four cusps)

68. $y = \dfrac{8}{x^2+4}$ (Witch of Agnesi)

69. $x^3 + y^3 = 3xy$ (Folium of Descartes)

70. $y^2 = x^3(1-x)$ (Pear curve)

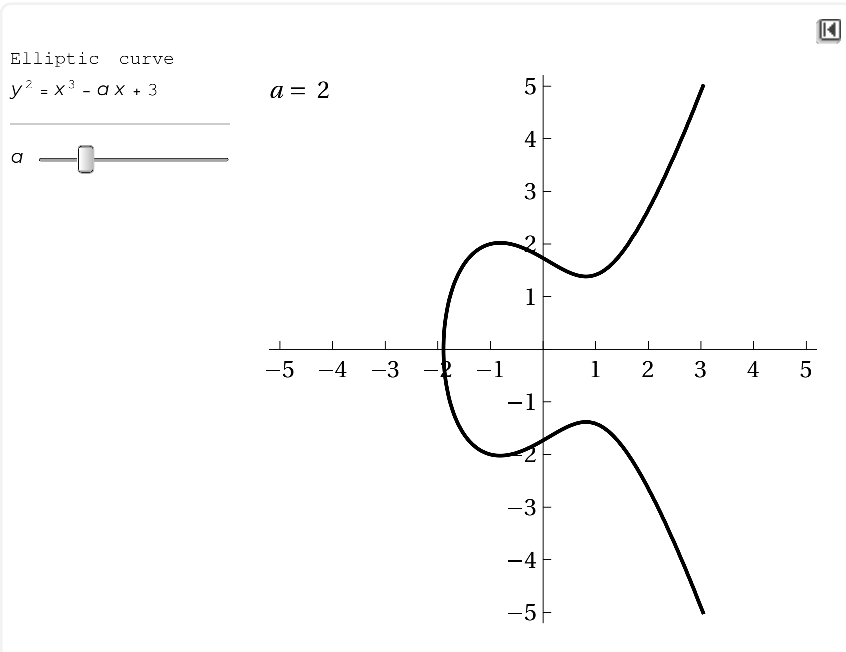
71. $x^4 - x^2 + y^2 = 0$ (Figure-8 curve)

T **72.** **Elliptic curves** The equation $y^2 = x^3 - ax + 3$, where a is a parameter, defines a well-known family of *elliptic curves*.

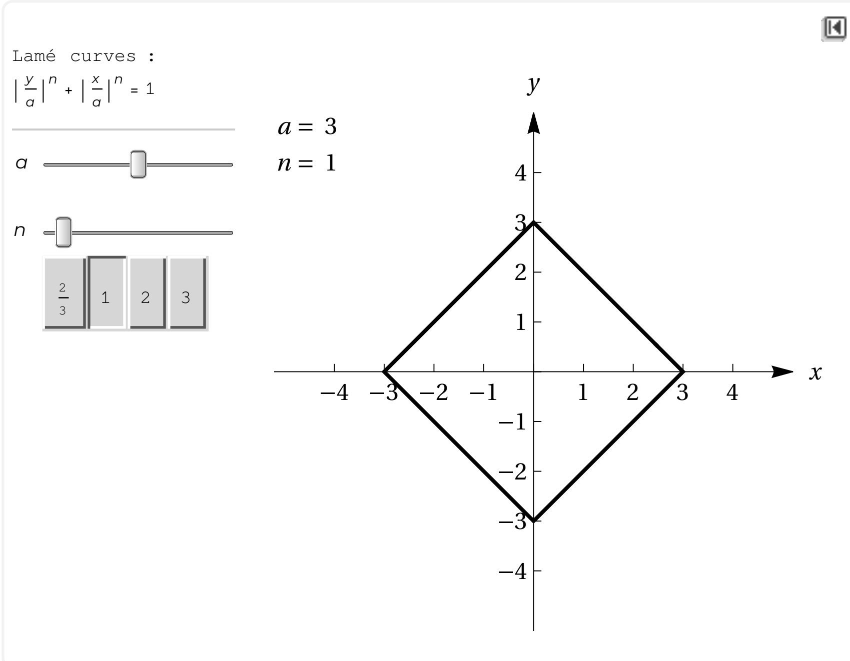
a. Plot a graph of the curve when $a = 3$.

b. Plot a graph of the curve when $a = 4$.

c. By experimentation, determine the approximate value of a $(3 < a < 4)$ at which the graph separates into two curves.



T **73.** **Lamé curves** The equation $\left|\frac{y}{a}\right|^n + \left|\frac{x}{a}\right|^n = 1$, where n and a are positive real numbers, defines the family of Lamé curves. Make a complete graph of this function with $a = 1$, for $n = \frac{2}{3}$, 1, 2, and 3. Describe the progression that you observe as n increases.


Lamé curves :
$\left|\frac{y}{a}\right|^n + \left|\frac{x}{a}\right|^n = 1$
a
n
2/3
1
2
3
a = 3
n = 1
y
x
4
3
2
1
−1
−2
−3
−4
−4
−3
−2
−1
1
2
3
4