4.3 What Derivatives Tell Us

In Section 4.1, we saw that the derivative is a tool for finding critical points, which are related to local maxima and minima. As we show in this section, derivatives (first *and* second derivatives) tell us much more about the behavior of functions.

Increasing and Decreasing Functions »

We have used the terms *increasing* and *decreasing* informally in earlier sections to describe a function or its graph. For example, the graph in **Figure 4.20a** rises as x increases, so the corresponding function is increasing. In **Figure 4.20b**, the graph falls as x increases, so the corresponding function is decreasing.

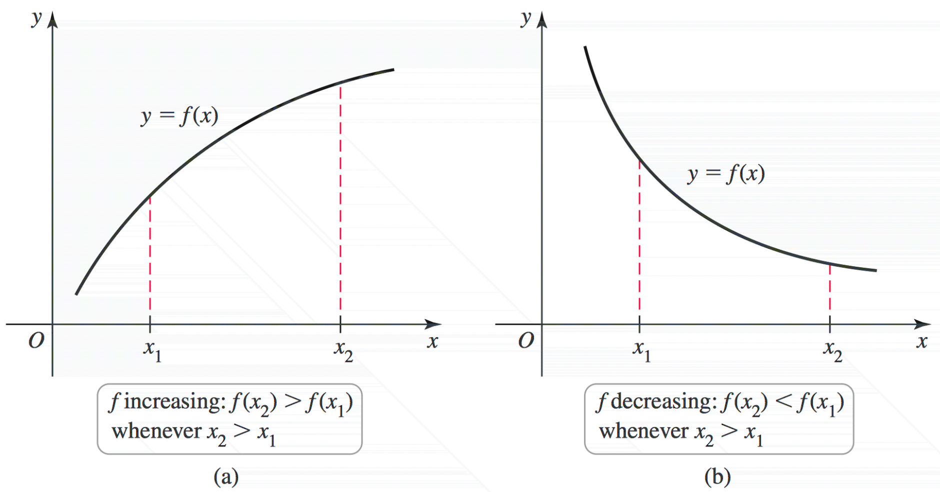


Figure 4.20

The following definition makes these ideas precise.

> **DEFINITION** **Increasing and Decreasing Functions**
>
> Suppose a function f is defined on an interval I. We say that f is **increasing** on I if $f(x_2) > f(x_1)$ whenever x_1 and x_2 are in I and $x_2 > x_1$. We say that f is **decreasing** on I if $f(x_2) < f(x_1)$ whenever x_1 and x_2 are in I and $x_2 > x_1$.

Note »

> A function is called **monotonic** if it is either increasing or decreasing. Some books make a further distinction by defining **nondecreasing** ($f(x_2) \geq f(x_1)$ whenever $x_2 > x_1$) and **nonincreasing** ($f(x_2) \leq f(x_1)$ whenever $x_2 > x_1$).

Intervals of Increase and Decrease

The graph of a function f gives us an idea of the intervals on which f is increasing and decreasing. But how do we determine those intervals precisely? This question is answered by making a connection to the derivative.

Recall that the derivative of a function gives the slopes of tangent lines. If the derivative is positive on an interval, the tangent lines on that interval have positive slopes, and the function is increasing on the interval (**Figure 4.21a**). Said differently, positive derivatives on an interval imply positive rates of change on the interval, which, in turn, indicate an increase in function values.

Similarly, if the derivative is negative on an interval, the tangent lines on that interval have negative slopes, and the function is decreasing on that interval (**Figure 4.21b**). These observations lead to Theorem 4.7, whose proof relies on the Mean Value Theorem.

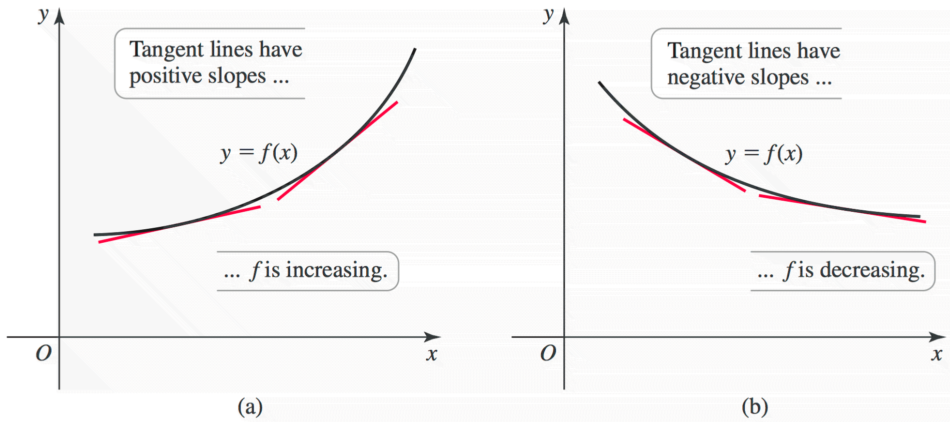


Figure 4.21

> **THEOREM 4.7 Test for Intervals of Increase and Decrease**
>
> Suppose f is continuous on an interval I and differentiable at every interior point of I. If $f'(x) > 0$ at all interior points of I, then f is increasing on I. If $f'(x) < 0$ at all interior points of I, then f is decreasing on I.

Proof: Let a and b be any two distinct points in the interval I with $b > a$. By the Mean Value Theorem,

$$\frac{f(b) - f(a)}{b - a} = f'(c),$$

for some c between a and b. Equivalently,

$$f(b) - f(a) = f'(c)(b - a).$$

Notice that $b - a > 0$ by assumption. So, if $f'(c) > 0$, then $f(b) - f(a) > 0$. Therefore, for all a and b in I with $b > a$, we have $f(b) > f(a)$, which implies that f is increasing on I. Similarly, if $f'(c) < 0$, then $f(b) - f(a) < 0$ or $f(b) < f(a)$. It follows that f is decreasing on I. ◆

Note »

> The converse of Theorem 4.7 may not be true. According to the definition, $f(x) = x^3$ is increasing on $(-\infty, \infty)$, but it is not true that $f'(x) > 0$ on $(-\infty, \infty)$ (because $f'(0) = 0$).

Quick Check 1 Explain why a positive derivative on an interval implies that the function is increasing on the interval. ◆

Answer »

> Positive derivatives on an interval mean the curve is rising on the interval, which means the function is increasing on the interval.

EXAMPLE 1 Sketching a function

Sketch a function f continuous on its domain $(-\infty, \infty)$ satisfying the following conditions.

1. $f' > 0$ on $(-\infty, 0)$, $(4, 6)$, and $(6, \infty)$.

2. $f' < 0$ on $(0, 4)$.

3. $f'(0)$ is undefined.

4. $f'(4) = f'(6) = 0$.

SOLUTION »

By condition (1), f is increasing on the intervals $(-\infty, 0)$, $(4, 6)$, and $(6, \infty)$. By condition (2), f is decreasing on $(0, 4)$. Continuity of f and condition (3) imply that f has a cusp or corner at $x = 0$, and by condition (4), the graph has a horizontal tangent line at $x = 4$ and $x = 6$. It is useful to summarize these results using a *sign graph* (**Figure 4.22**).

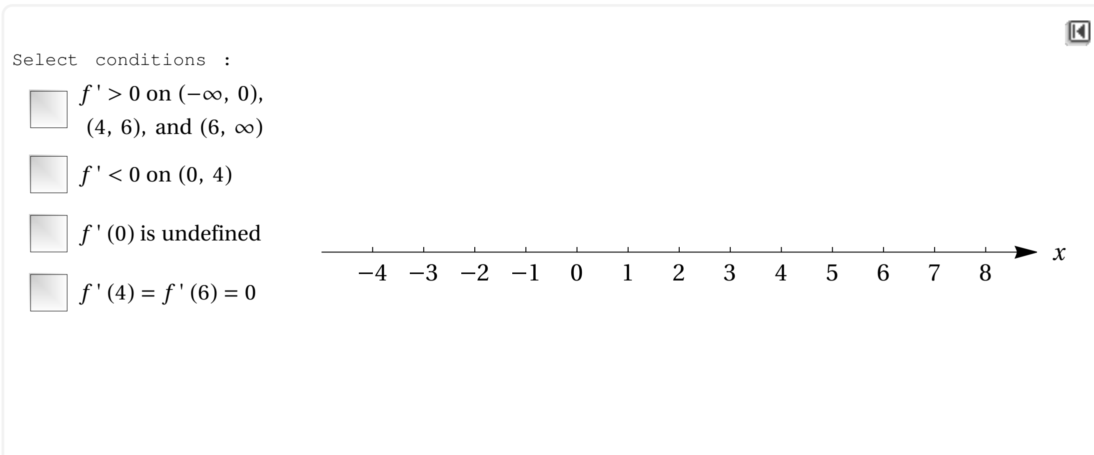


Figure 4.22

One possible graph satisfying these conditions is shown in **Figure 4.23**. Notice that the graph has a cusp at $x = 0$. Furthermore, although $f'(4) = f'(6) = 0$, f has a local minimum at $x = 4$ but no local extremum at $x = 6$.

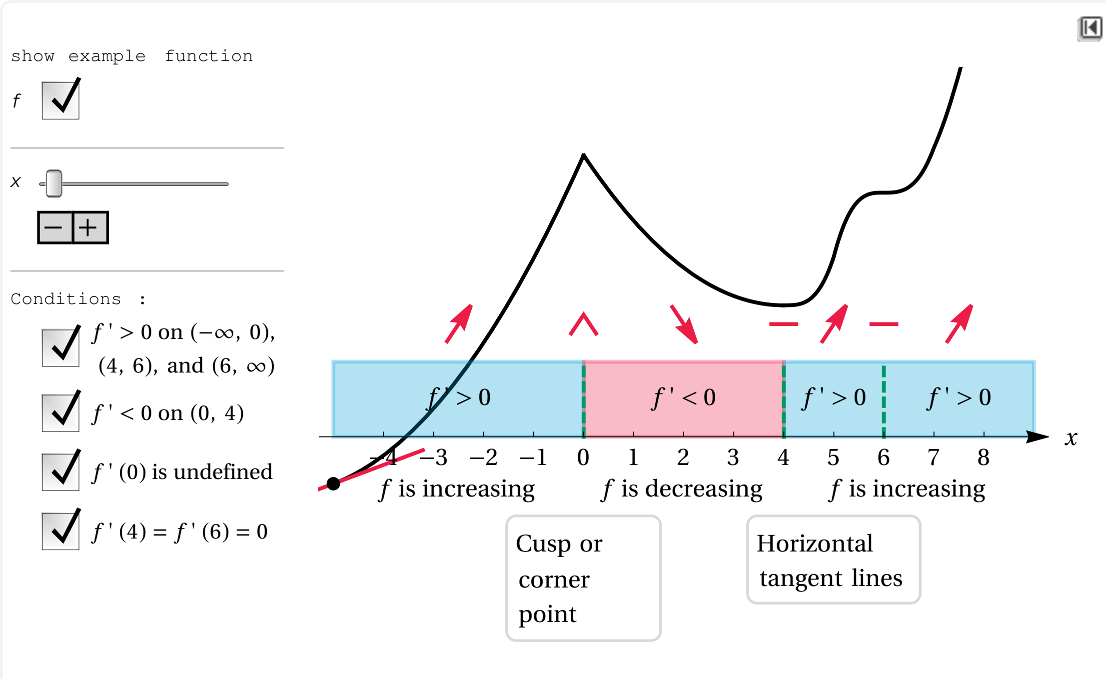


Figure 4.23

Related Exercises 9–10 ◆

EXAMPLE 2 Intervals of increase and decrease

Find the intervals on which the function $f(x) = 2x^3 + 3x^2 + 1$ is increasing and the intervals on which it is decreasing.

SOLUTION

Note that $f'(x) = 6x^2 + 6x = 6x(x+1)$. To find the intervals of increase, we first solve $6x(x+1) = 0$ and determine that the critical points are $x = 0$ and $x = -1$. If f' changes sign, then it does so at these points and nowhere else; that is, f' has the same sign throughout each of the intervals $(-\infty, -1)$, $(-1, 0)$, and $(0, \infty)$. Evaluating f' at a selected point of each interval determines the sign of f' on that interval.

- At $x = -2$, $f'(-2) = 12 > 0$, so $f' > 0$ and f is increasing on $(-\infty, -1)$.

The graph of f has a horizontal tangent line at $x = -1$ and $x = 0$. **Figure 4.24** shows the graph of f superimposed on the graph of f', confirming our conclusions.

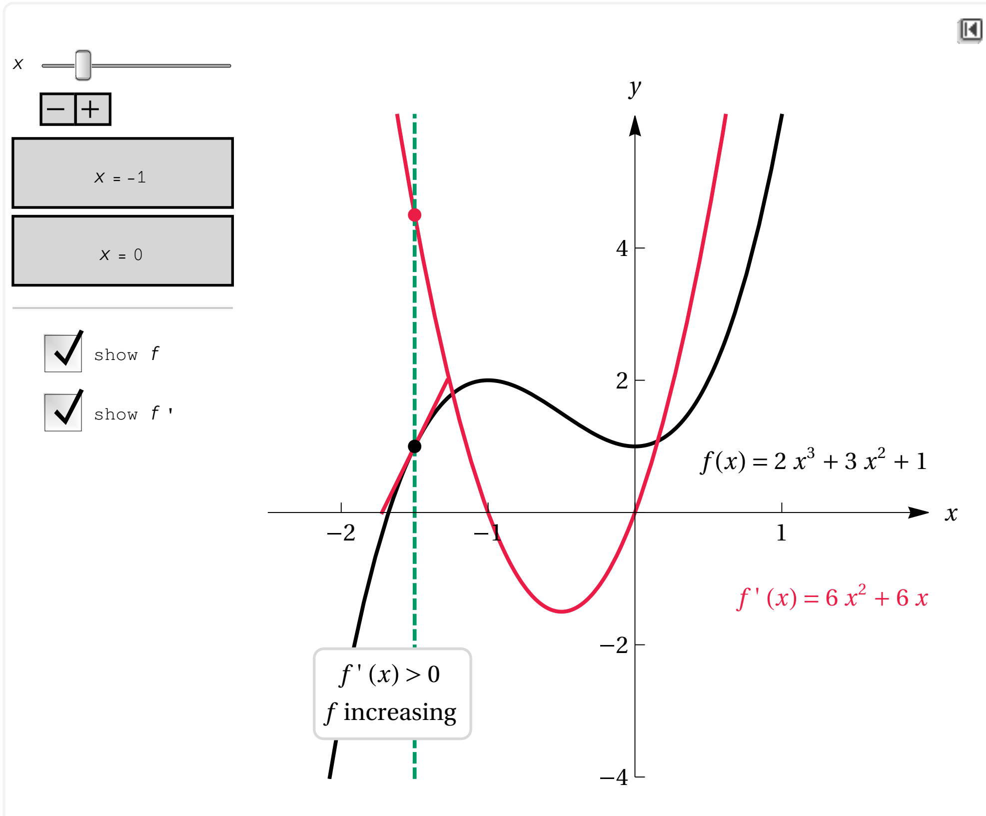


Figure 4.24

Related Exercises 22–27 ◆

Identifying Local Maxima and Minima »

Using what we know about increasing and decreasing functions, we can now identify local extrema. Suppose $x = c$ is a critical point of f, where $f'(c) = 0$. Suppose also that f' changes sign at c with $f'(x) < 0$ on an interval (a, c) to the left of c and $f'(x) > 0$ on an interval (c, b) to the right of c. In this case f' is decreasing to the left of c and increasing to the right of c, which means that f has a local minimum at c, as shown in **Figure 4.25a** .

Similarly, suppose f' changes sign at c with $f'(x) > 0$ on an interval (a, c) to the left of c and $f'(x) < 0$ on an interval (c, b) to the right of c. Then f is increasing to the left of c and decreasing to the right of c, so f has a local maximum at c, as shown in **Figure 4.25b** .

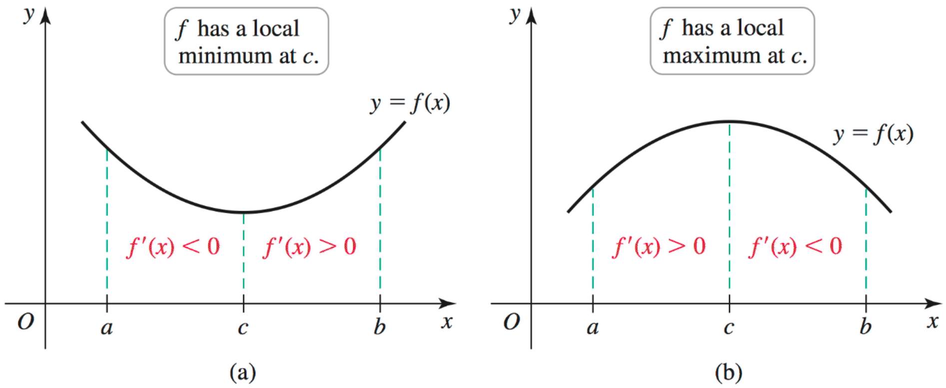


Figure 4.25

Figure 4.26 shows typical features of a function on an interval $[a, b]$. At local maxima or minima (c_2, c_3, and c_4), f' changes sign. Although c_1 and c_5 are critical points, f' does not change sign at these points, so there is no local maximum or minimum at these points. As emphasized earlier, *critical points do not always correspond to local extreme values*.

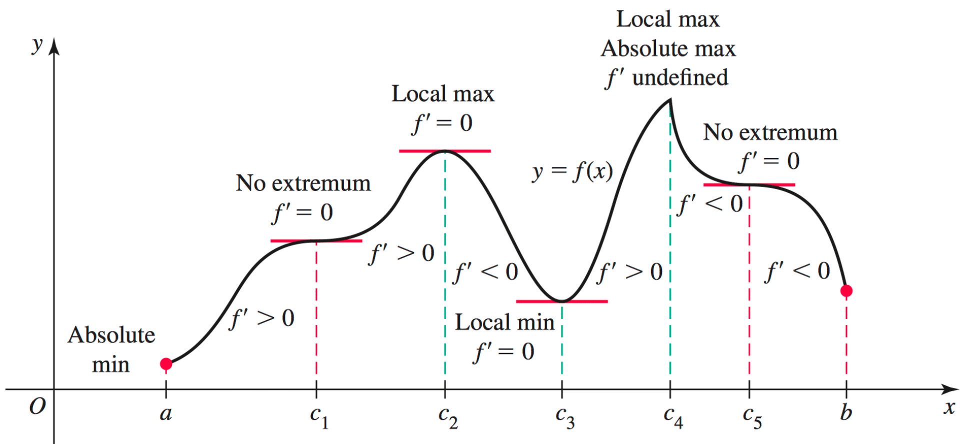
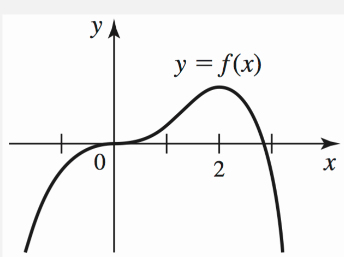


Figure 4.26

Quick Check 2 Sketch a function f that is differentiable on $(-\infty, \infty)$ with the following properties: (i) $x = 0$ and $x = 2$ are critical points; (ii) f is increasing on $(-\infty, 2)$; (iii) f is decreasing on $(2, \infty)$. ◆

Answer »



First Derivative Test

The observations used to interpret Figure 4.26 are summarized in a powerful test for identifying local maxima and minima.

THEOREM 4.8 First Derivative Test

Assume that f is continuous on an interval that contains a critical point c and assume f is differentiable on an interval containing c, except perhaps at c itself.

- If f' changes sign from positive to negative as x increases through c, then f has a **local maximum** at c.
- If f' changes sign from negative to positive as x increases through c, then f has a **local minimum** at c.
- If f' does not change sign at c (from positive to negative or vice versa), then f has no local extreme value at c.

Proof: Suppose $f'(x) > 0$ on an interval (a, c). By Theorem 4.7, we know that f is increasing on (a, c), which implies that $f(x) < f(c)$ for all x in (a, c). Similarly, suppose $f'(x) < 0$ on an interval (c, b). This time Theorem 4.7 says that f is decreasing on (c, b), which implies that $f(x) < f(c)$ for all x in (c, b). Therefore, $f(x) \le f(c)$ for all x in (a, b) and f has a local maximum at c. The proofs of the other two cases are similar. ◆

EXAMPLE 3 Using the First Derivative Test

Consider the function

$$f(x) = 3x^4 - 4x^3 - 6x^2 + 12x + 1.$$

a. Find the intervals on which f is increasing and decreasing.

b. Identify the local extrema of f.

SOLUTION »

a. Differentiating f, we find that

$$\begin{aligned} f'(x) &= 12x^3 - 12x^2 - 12x + 12 \\ &= 12\left(x^3 - x^2 - x + 1\right) \\ &= 12(x+1)(x-1)^2. \end{aligned}$$

Solving $f'(x) = 0$ gives the critical points $x = -1$ and $x = 1$. The critical points determine the intervals $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$ on which f' does not change sign. Choosing a test point in each interval, a sign graph of f' is constructed (**Figure 4.27**), which summarizes the behavior of f.

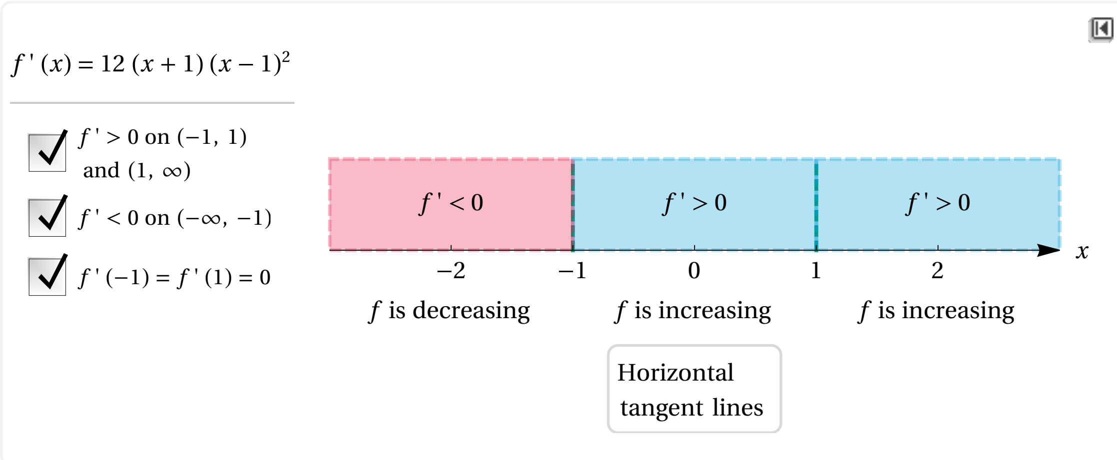


Figure 4.27

b. Note that f is a polynomial, so it is continuous on $(-\infty, \infty)$. Because f' changes sign from negative to positive as x passes through the critical point $x = -1$, it follows by the First Derivative Test that f has a local minimum value of $f(-1) = -10$ at $x = -1$. Observe that f' is positive on both sides near $x = 1$, so f does not have a local extreme value at $x = 1$ (**Figure 4.28**).

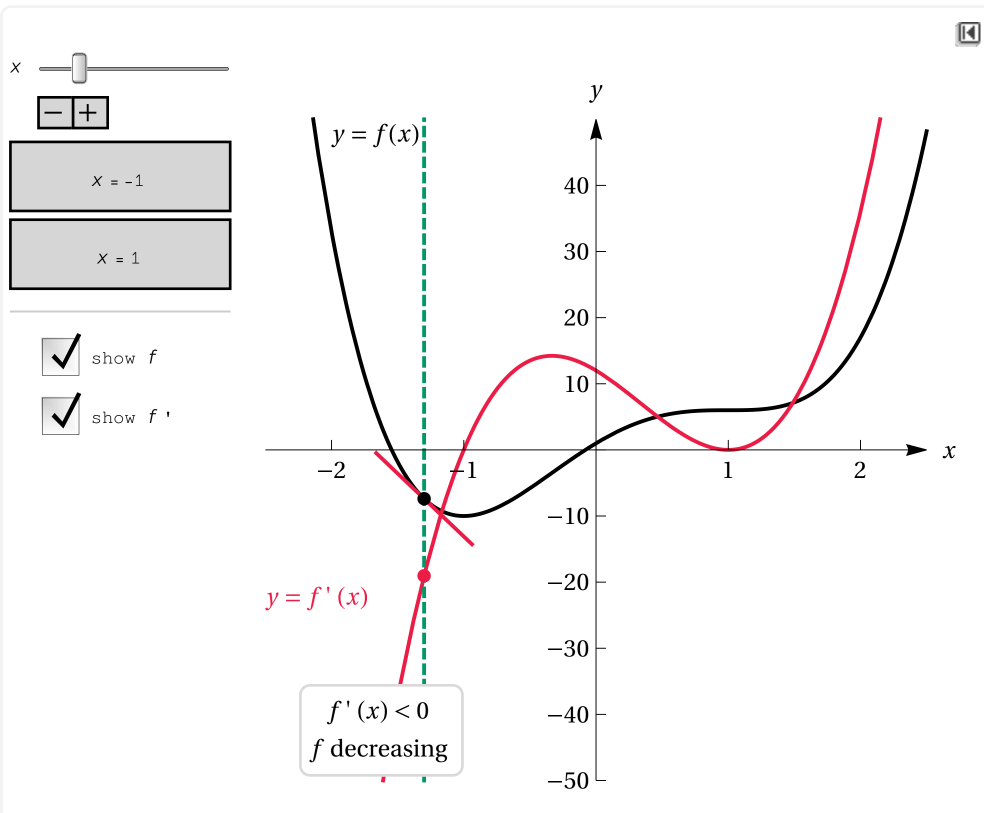


Figure 4.28

Related Exercises 45–46 ◆

EXAMPLE 4 **Extreme values**

Find the local extrema of the function $g(x) = x^{2/3}(2 - x)$.

SOLUTION »

In Example 4b of Section 4.1, we found that

$$g'(x) = \frac{4}{3}x^{-1/3} - \frac{5}{3}x^{2/3} = \frac{4 - 5x}{3x^{1/3}}$$

and that the critical points of g are $x = 0$ and $x = \frac{4}{5}$. These two critical points are *candidates* for local extrema, and Theorem 4.8 is used to classify each as a local minimum, local maximum, or neither.

On the interval $(-\infty, 0)$, the numerator of g' is positive and the denominator is negative (**Figure 4.29**). Therefore, $g'(x) < 0$ on $(-\infty, 0)$. On the interval $\left(0, \frac{4}{5}\right)$, the numerator of g' is positive, as is the denominator. Therefore, $g'(x) > 0$ on $\left(0, \frac{4}{5}\right)$. We see that as x passes through 0, g' changes sign from negative to positive, which means g has a local minimum at 0. A similar argument shows that g' changes sign from positive to negative as x passes through $\frac{4}{5}$, so g has a local maximum at $\frac{4}{5}$.

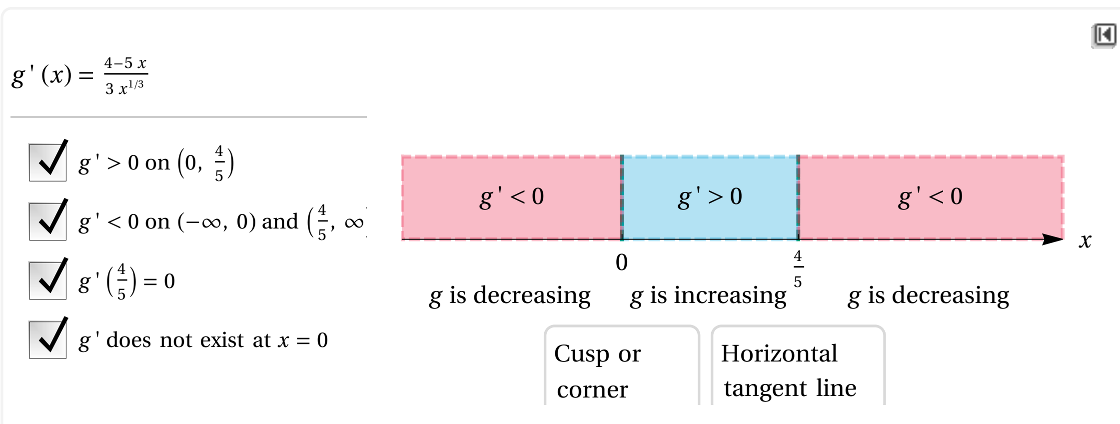


Figure 4.29

These observations are confirmed by the graphs of g and g' (**Figure 4.30**).

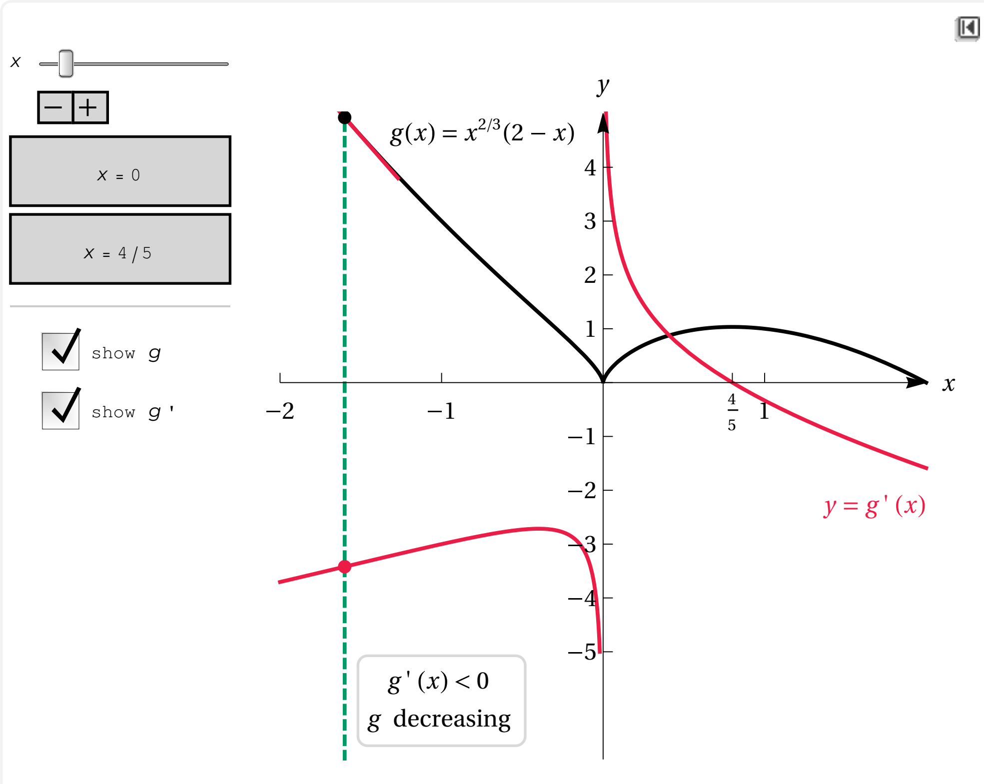


Figure 4.30

Related Exercise 47 ◆

Quick Check 3 Explain how the First Derivative Test determines whether $f(x) = x^2$ has a local maximum or local minimum at $x = 0$. ◆

Answer »

$f'(x) < 0$ on $(-\infty, 0)$ and $f'(x) > 0$ on $(0, \infty)$. Therefore, f has a local minimum at $x = 0$ by the First Derivative Test.

Absolute Extreme Values on Any Interval

Theorem 4.1 guarantees the existence of absolute extreme values only on closed intervals. What can be said about absolute extrema on intervals that are not closed? The following theorem provides a valuable test.

THEOREM 4.9 One Local Extremum Implies Absolute Extremum

Suppose f is continuous on an interval I that contains exactly one local extremum at c.

- If a local minimum occurs at c, then $f(c)$ is the absolute minimum of f on I.
- If a local maximum occurs at c, then $f(c)$ is the absolute maximum of f on I.

The proof of Theorem 4.9 is beyond the scope of this text, although **Figure 4.31** illustrates why the theorem is plausible. Assume f has exactly one local minimum on I at c. Notice that there can be no other point on the graph at which f has a value less than $f(c)$. If such a point did exist, the graph would have to bend downward to drop below $f(c)$, which, by the continuity of f, cannot happen as it implies additional local extreme values on I. A similar argument applies to a solitary local maximum.

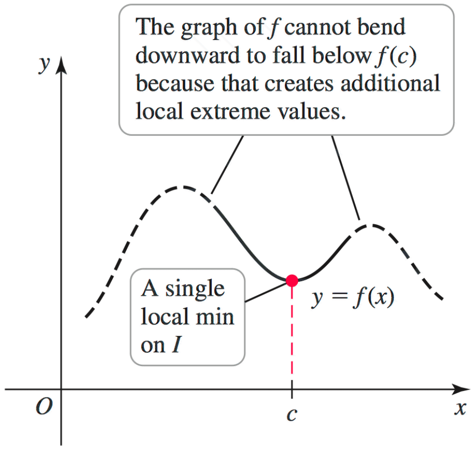


Figure 4.31

EXAMPLE 5 Finding an absolute extremum

Verify that $f(x) = \frac{1}{4}x^4 - x^3 + \frac{3}{2}x^2 - 9x + 2$ has an absolute extreme value on its domain.

SOLUTION

As a polynomial, f is differentiable on its domain $(-\infty, \infty)$ with

$$f'(x) = x^3 - 3x^2 + 3x - 9 = (x-3)\left(x^2+3\right).$$

Solving $f'(x) = 0$ and noting that $x^2 + 3 > 0$ for all x gives the single critical point $x = 3$. It may be verified that $f'(x) < 0$ for $x < 3$ and $f'(x) > 0$ for $x > 3$. Therefore, by Theorem 4.8, f has a local minimum at $x = 3$. Because it is the only local extremum on $(-\infty, \infty)$, it follows from Theorem 4.9 that the absolute minimum value of f occurs at $x = 3$, where $f(3) = -\frac{73}{4}$ (**Figure 4.32**).

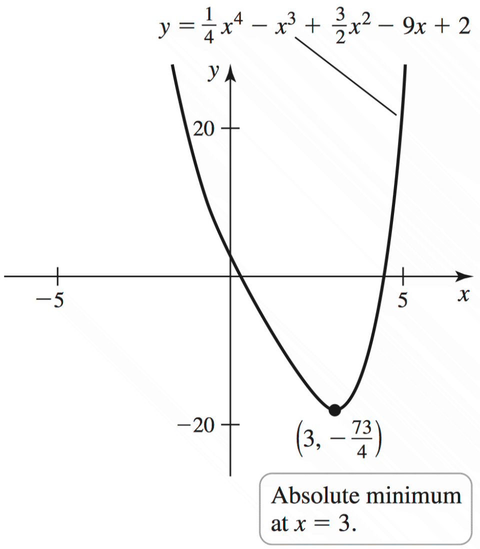


Figure 4.32

Related Exercises 49–51 ◆

Concavity and Inflection Points »

Just as the first derivative is related to the slope of tangent lines, the second derivative also has geometric meaning. Consider $f(x) = \sin x$, shown in **Figure 4.33**. Its graph bends upward for $-\pi < x < 0$, reflecting the fact that the slopes of the tangent lines increase as x increases. It follows that the first derivative is increasing for $-\pi < x < 0$. A function with the property that f' is increasing on an interval is *concave up* on that interval.

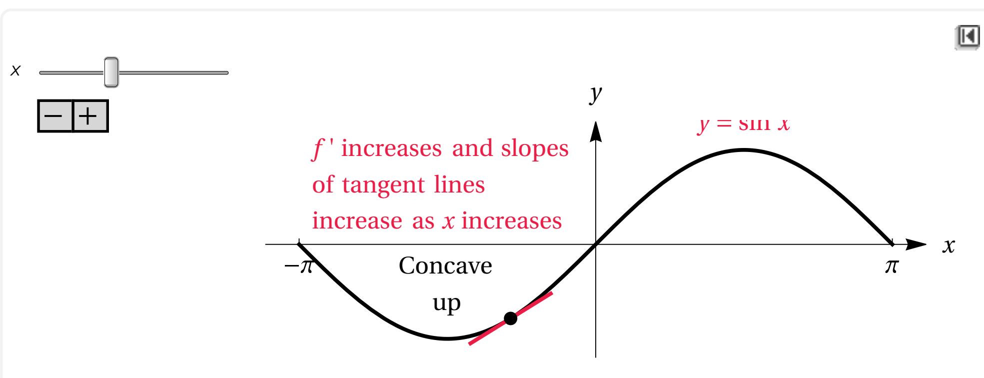


Figure 4.33

Similarly, $f(x) = \sin x$ bends downward for $0 < x < \pi$ because it has a decreasing first derivative on that interval. A function with the property that f' is decreasing as x increases on an interval is *concave down* on that interval.

Here is another useful characterization of concavity. If a function is concave up at a point (any point in $(-\pi, 0)$, Figure 4.33), then its graph near that point lies *above* the tangent line at that point. Similarly, if a function is concave down at a point (any point in $(0, \pi)$, Figure 4.33), then its graph near that point lies *below* the tangent line at that point (Exercise 102).

Finally, imagine a function that changes concavity (from up to down, or vice versa) at a point c. For example, $f(x) = \sin x$ in Figure 4.33 changes from concave up to concave down as x passes through $x = 0$. A point on the graph of f at which f changes concavity is called an *inflection point*.

DEFINITION **Concavity and Inflection Point**

Let f be differentiable on an open interval I. If f' is increasing on I, then f is **concave up** on I. If f' is decreasing on I, then f is **concave down** on I.

If f is continuous at c and f changes concavity at c (from up to down, or vice versa), then f has an **inflection point** at c.

Applying Theorem 4.7 to f' leads to a test for concavity in terms of the second derivative. Specifically, if $f'' > 0$ on an interval I, then f' is increasing on I and f is concave up on I. Similarly, if $f'' < 0$ on I, then f is concave down on I. If the values of f'' pass through zero at a point c (from positive to negative, or vice versa), then the concavity of f changes at c and f has an inflection point at c (**Figure 4.34a**). We now have a useful interpretation of the second derivative: It measures *concavity*.

THEOREM 4.10 **Test for Concavity**

Suppose f'' exists on an open interval I.

- If $f'' > 0$ on I, then f is concave up on I.
- If $f'' < 0$ on I, then f is concave down on I.
- If c is a point of I at which f'' changes sign at c (from positive to negative, or vice versa), then f has an inflection point at c.

There are a few important but subtle points here. The fact that $f''(c) = 0$ does not necessarily imply that f has an inflection point at c. A good example is $f(x) = x^4$. Although $f''(0) = 0$, the concavity does not change at $x = 0$ (a similar function is shown in **Figure 4.34b**).

Typically, if f has an inflection point at c, then $f''(c) = 0$, reflecting the smooth change in concavity. However, an inflection point may also occur at a point where f'' does not exist. For example, the function $f(x) = x^{1/3}$ has a vertical tangent line and an inflection point at $x = 0$ (a similar function is shown in **Figure 4.34c**). Finally, note that the function shown in **Figure 4.34d**, with behavior similar to that of $f(x) = x^{2/3}$, does not have an inflection point at c despite the fact that $f''(c)$ does not exist. In summary, if $f''(c) = 0$ or $f''(c)$ does not exist, then $(c, f(c))$ is a candidate for an inflection point. To be certain an inflection point occurs at c, we must show that the concavity of f changes at c.

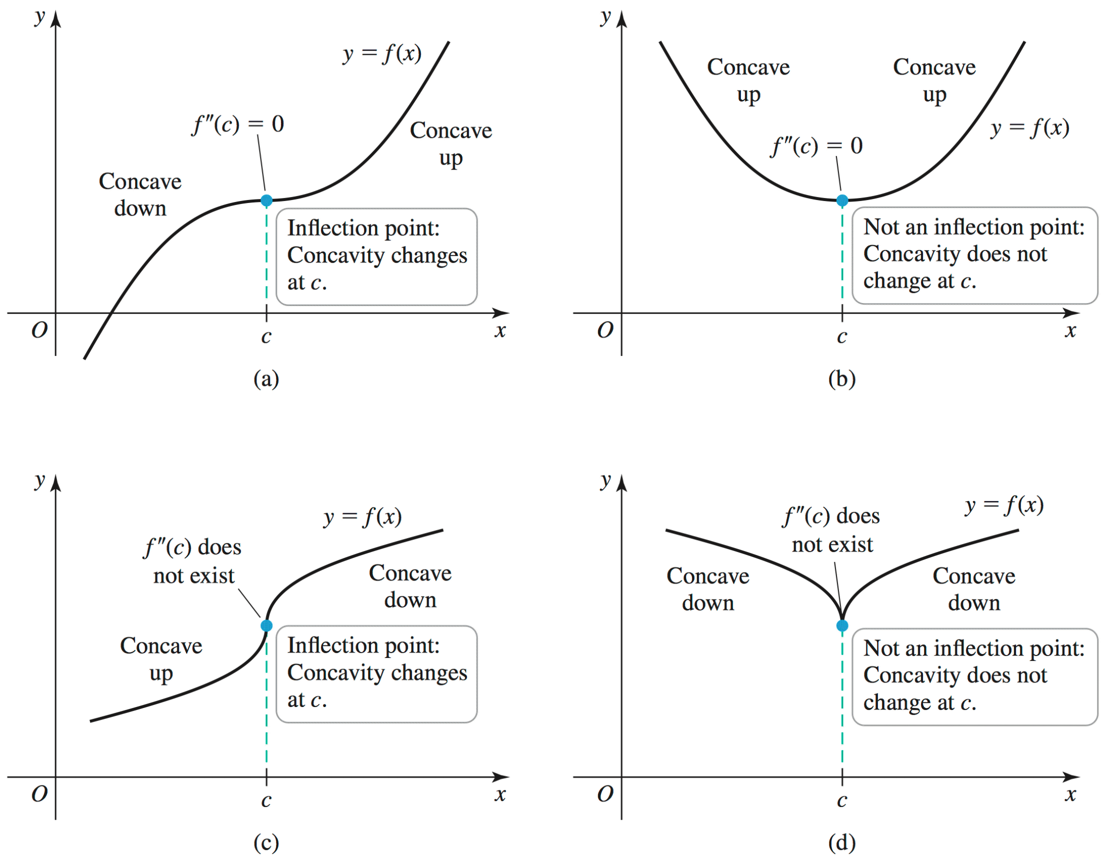


Figure 4.34

Quick Check 4 Verify that the function $f(x) = x^4$ is concave up for $x > 0$ and for $x < 0$. Is $x = 0$ an inflection point? Explain. ◆

Answer »

$f''(x) = 12x^2$, so $f''(x) > 0$ for $x < 0$ and for $x > 0$. There is no inflection point at $x = 0$ because the second derivative does not change sign.

EXAMPLE 6 Interpreting concavity

Sketch a function satisfying each set of conditions on some interval.

a. $f'(t) > 0$ and $f''(t) > 0$

b. $g'(t) > 0$ and $g''(t) < 0$

c. Which of the functions, f or g, could describe a population that increases and approaches a steady state as $t \to \infty$?

SOLUTION »

a. **Figure 4.35a** shows the graph of a function that is increasing $(f'(t) > 0)$ and concave up $(f''(t) > 0)$.

b. **Figure 4.35b** shows the graph of a function that is increasing $(g'(t) > 0)$ and concave down $(g''(t) < 0)$.

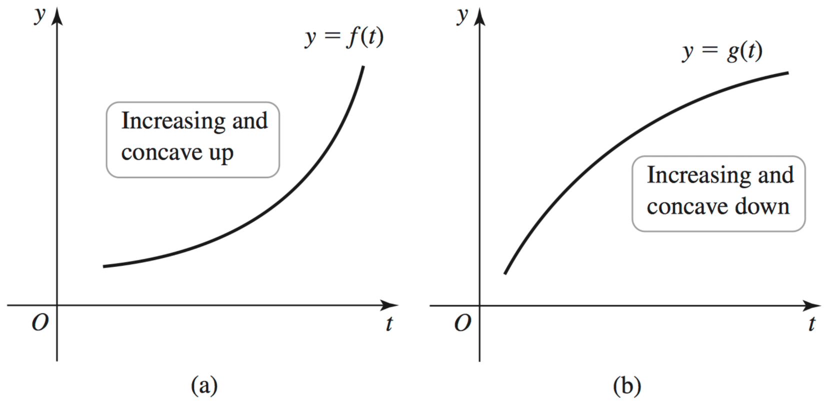


Figure 4.35

c. Because f increases at an *increasing* rate, the graph of f could not approach a horizontal asymptote, so f could not describe a population that approaches a steady state. On the other hand, g increases at a *decreasing* rate, so its graph could approach a horizontal asymptote, depending on the rate at which g increases.

Related Exercises 53–56 ◆

EXAMPLE 7 Detecting concavity

Identify the intervals on which the function $f(x) = 3x^4 - 4x^3 - 6x^2 + 12x + 1$ is concave up or concave down. Then locate the inflection points.

SOLUTION »

Second Derivative Test

It is now a short step to a test that uses the second derivative to identify local maxima and minima.

THEOREM 4.11 **Second Derivative Test for Local Extrema**

Suppose f'' is continuous on an open interval containing c with $f'(c) = 0$.

- If $f''(c) > 0$, then f has a local minimum at c (**Figure 4.38a**).
- If $f''(c) < 0$, then f has a local maximum at c (**Figure 4.38b**).
- If $f''(c) = 0$, then the test is inconclusive; f may have a local maximum, local minimum, or neither at c.

Note »

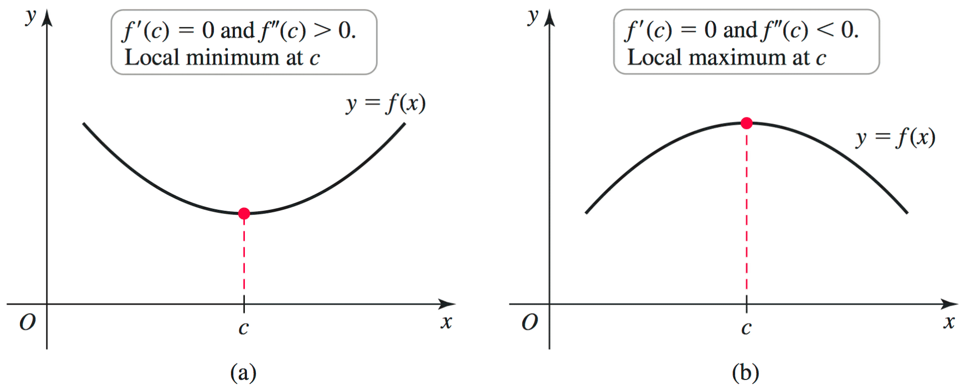


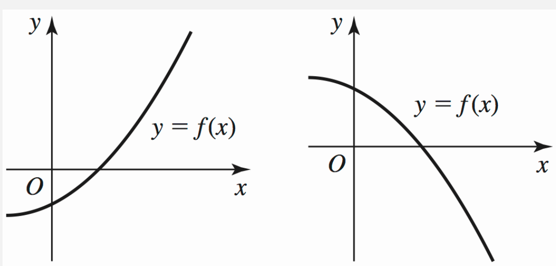
Figure 4.38

Proof: Assume $f''(c) > 0$. Because f'' is continuous on an interval I containing c, it follows that $f'' > 0$ on some open interval I containing c and that f' is increasing on I. Because $f'(c) = 0$, it follows that f' changes sign at c from negative to positive, which, by the First Derivative Test, implies that f has a local minimum at c. The proofs of the second and third statements are similar. ◆

Quick Check 5 Make a sketch of a function with $f'(x) > 0$ and $f''(x) > 0$ on an interval. Make a sketch of a function with $f'(x) < 0$ and $f''(x) < 0$ on an interval. ◆

Answer »

The first curve should be rising and concave up. The second curve should be falling and concave down.



EXAMPLE 8 The Second Derivative Test

Use the Second Derivative Test to locate the local extrema of the following functions.

a. $f(x) = 3x^4 - 4x^3 - 6x^2 + 12x + 1$ on $[-2, 2]$

b. $f(x) = \sin^2 x$

SOLUTION »

a. This function was considered in Examples 3 and 7, where we found that

$$f'(x) = 12(x+1)(x-1)^2 \quad \text{and} \quad f''(x) = 12(x-1)(3x+1).$$

Therefore, the critical points of f are $x = -1$ and $x = 1$. Evaluating f'' at the critical points, we find that $f''(-1) = 48 > 0$. By the Second Derivative Test, f has a local minimum at $x = -1$. At the other critical point, $f''(1) = 0$, so the test is inconclusive. You can check that the first derivative does not change sign at $x = 1$, which means f does not have a local maximum or minimum at $x = 1$ (**Figure 4.39**).

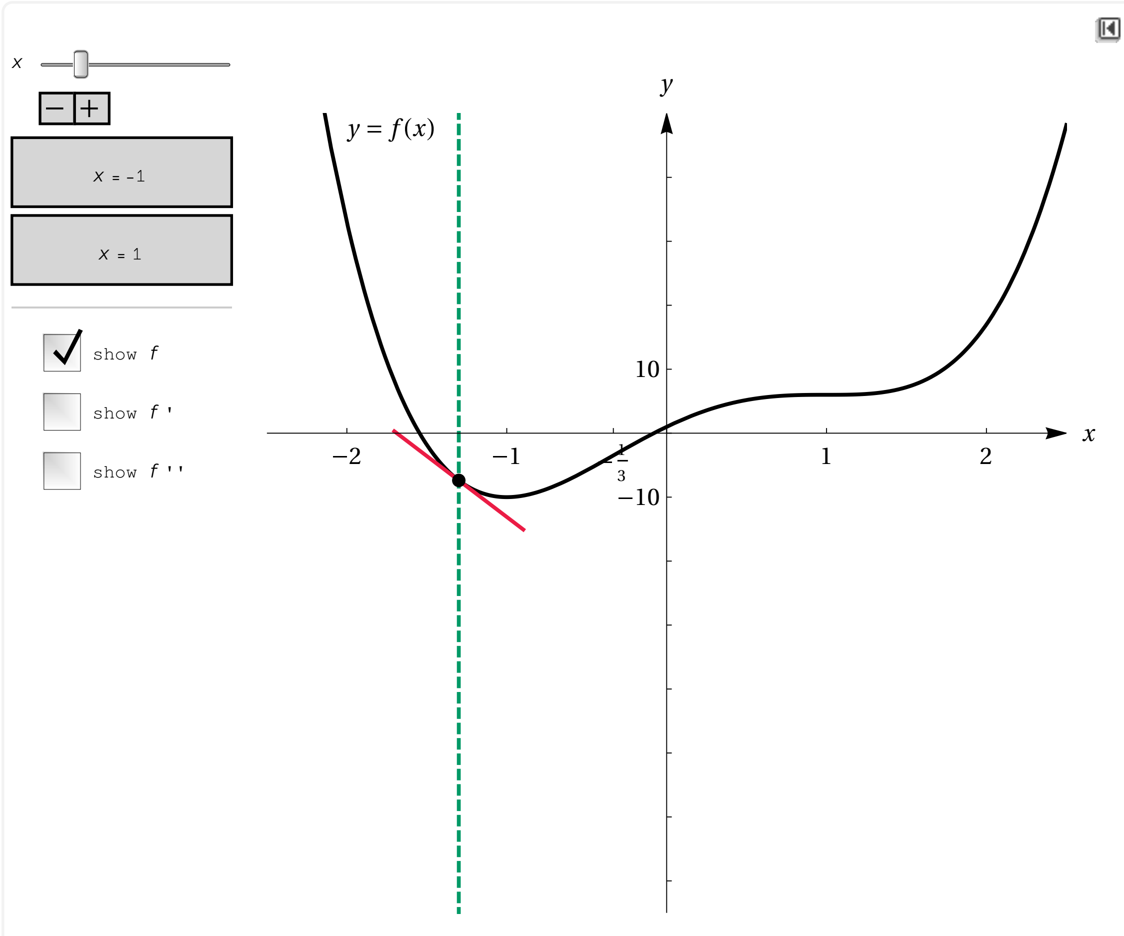


Figure 4.39

b. Using the Chain Rule and a trigonometric identity, we have $f'(x) = 2\sin x \cos x = \sin 2x$ and $f''(x) = 2\cos 2x$. The critical points occur when $f'(x) = \sin 2x = 0$, or when $x = 0, \pm\frac{\pi}{2}, \pm\pi, \ldots$. To apply the Second Derivative Test, we evaluate f'' at the critical points:

- $f''(0) = 2 > 0$, so f has a local minimum at $x = 0$.

This pattern continues, and we see that f has alternating local maxima and minima, evenly spaced every $\frac{\pi}{2}$ units (**Figure 4.40**).

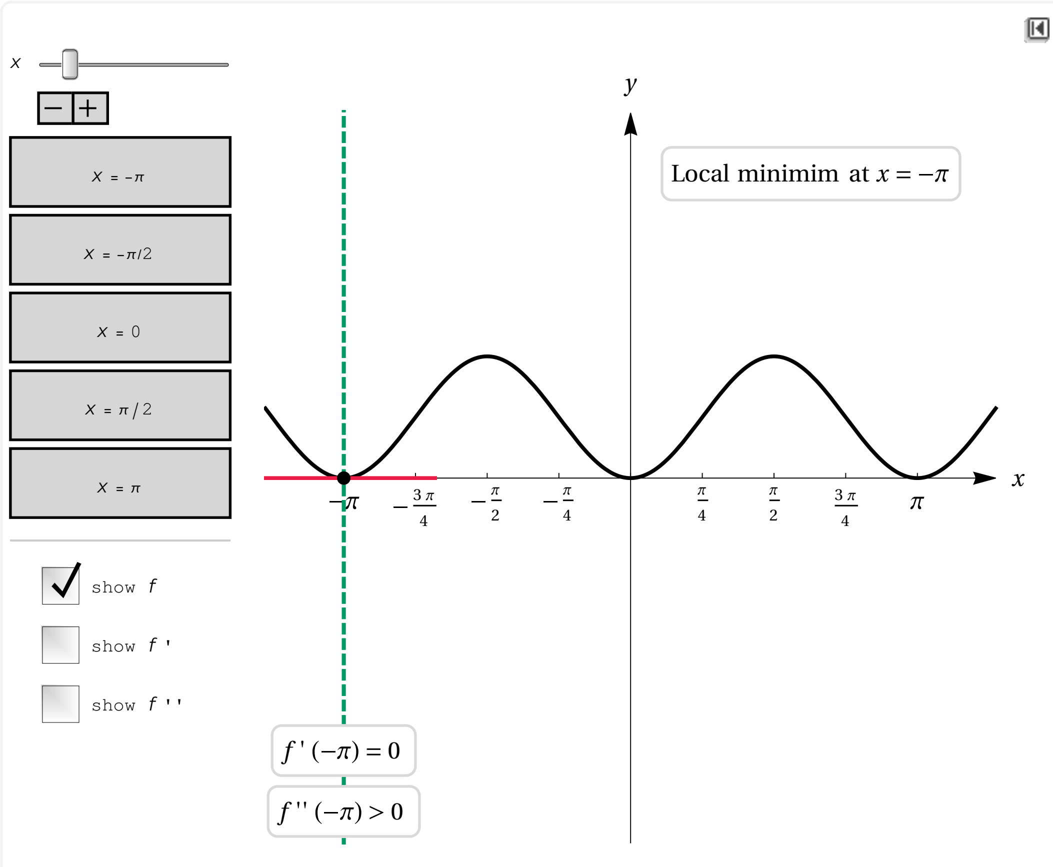


Figure 4.40

Related Exercises 70, 72 ◆

Recap of Derivative Properties »

This section has demonstrated that the first and second derivatives of a function provide valuable information about its graph. The relationships among a function's derivatives and its extreme points and concavity are summarized in **Figure 4.41** .

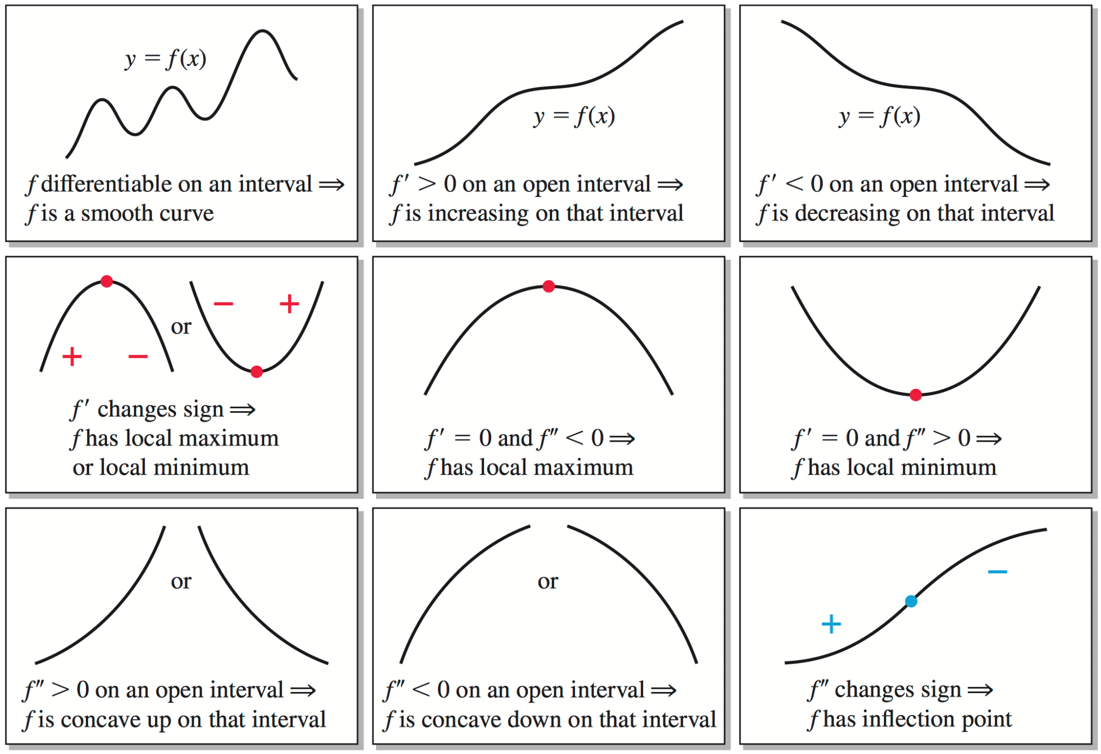


Figure 4.41

Exercises »

Getting Started »

Practice Exercises »

19–40. Increasing and decreasing functions *Find the intervals on which f is increasing and the intervals on which it is decreasing.*

19. $f(x) = 4 - x^2$

20. $f(x) = x^2 - 16$

21. $f(x) = (x - 1)^2$

22. $f(x) = x^3 + 4x$

23. $f(x) = \dfrac{x^3}{3} - \dfrac{5x^2}{2} + 4x$

24. $f(x) = -\dfrac{x^3}{3} + \dfrac{x^2}{2} + 2x$

25. $f(x) = 12 + x - x^2$

26. $f(x) = x^4 - 4x^3 + 4x^2$

27. $f(x) = -\dfrac{x^4}{4} + x^3 - x^2$

28. $f(x) = 2\,x^5 - \dfrac{15\,x^4}{4} + \dfrac{5\,x^3}{3}$

29. $f(x) = -2\cos x - x$ on $[0, 2\pi]$

30. $f(x) = \sqrt{2}\,\sin x - x$ on $[0, 2\pi]$

31. $f(x) = 3\cos 3\,x$ on $[-\pi, \pi]$

32. $f(x) = \cos^2 x$ on $[-\pi, \pi]$

33. $f(x) = x^{2/3}\left(x^2 - 4\right)$

34. $f(x) = x^2\,\sqrt{9 - x^2}$ on $(-3, 3)$

35. $f(x) = -12\,x^5 + 75\,x^4 - 80\,x^3$

36. $f(x) = 3\,x^4 - 16\,x^3 + 24\,x^2$

37. $f(x) = -2\,x^4 + x^2 + 10$

38. $f(x) = \dfrac{x^4}{4} - \dfrac{8\,x^3}{3} + \dfrac{15\,x^2}{2} + 8$

39. $f(x) = \sin x - x\cos x$ on $(0, 2\pi)$

40. $f(x) = x^2\sin x - 2\sin x + 2\,x\cos x$ on $(0, 2\pi)$

41–48. First Derivative Test

a. *Locate the critical points of* f.

b. *Use the First Derivative Test to locate the local maximum and minimum values.*

c. *Identify the absolute maximum and minimum values of the function on the given interval (when they exist).*

41. $f(x) = x^2 + 3$ on $[-3, 2]$

42. $f(x) = -x^2 - x + 2$ on $[-4, 4]$

43. $f(x) = x\,\sqrt{4 - x^2}$ on $[-2, 2]$

44. $f(x) = 2\,x^3 + 3\,x^2 - 12\,x + 1$ on $[-2, 4]$

45. $f(x) = -x^3 + 9\,x$ on $[-4, 3]$

46. $f(x) = 2\,x^5 - 5\,x^4 - 10\,x^3 + 4$ on $[-2, 4]$

47. $f(x) = x^{2/3}(x - 5)$ on $[-5, 5]$

48. $f(x) = \dfrac{x^2}{x^2 - 1}$ on $[-4, 4]$

49–52. Absolute extreme values *Verify that the following functions satisfy the conditions of Theorem 4.9 on their domains. Then find the location and value of the absolute extrema guaranteed by the theorem.*

49. $f(x) = -3x^2 + 2x - 5$

50. $f(x) = 4x + \dfrac{1}{\sqrt{x}}$

51. $A(r) = \dfrac{24}{r} + 2\pi r^2,\ r > 0$

52. $f(x) = x\sqrt{3 - x}$

53–56. Sketching curves *Sketch a graph of a function* f *that is continuous on* $(-\infty, \infty)$ *and has the following properties.*

53. $f'(x) > 0,\ f''(x) > 0$

54. $f'(x) < 0$ and $f''(x) > 0$ on $(-\infty, 0)$; $f'(x) > 0$ and $f''(x) > 0$ on $(0, \infty)$

55. $f'(x) < 0$ and $f''(x) < 0$ on $(-\infty, 0)$; $f'(x) < 0$ and $f''(x) > 0$ on $(0, \infty)$

56. $f'(x) < 0$ and $f''(x) > 0$ on $(-\infty, 0)$; $f'(x) < 0$ and $f''(x) < 0$ on $(0, \infty)$

57–68. Concavity *Determine the intervals on which the following functions are concave up or concave down. Identify any inflection points.*

57. $f(x) = x^4 - 2x^3 + 1$

58. $f(x) = -x^4 - 2x^3 + 12x^2$

59. $f(x) = 5x^4 - 20x^3 + 10$

60. $f(x) = \dfrac{1}{1 + x^2}$

61. $g(t) = \dfrac{t - 2}{t + 3}$

62. $g(x) = \sqrt[3]{x - 4}$

63. $f(\theta) = \theta \sin\theta + 2\cos\theta$ on $(0, 2\pi)$

64. $f(x) = x^2 + 4\sin x$ on $[0, \pi]$

65. $f(x) = 4(x + 1)^{5/2}(3x - 4)$ on $[-1, \infty)$

66. $h(t) = 2 + \cos 2t$ on $[0, \pi]$

67. $g(t) = 3\,t^5 - 30\,t^4 + 80\,t^3 + 100$

68. $f(x) = 2\,x^4 + 8\,x^3 + 12\,x^2 - x - 2$

69–84. Second Derivative Test *Locate the critical points of the following functions. Then use the Second Derivative Test to determine (if possible) whether they correspond to local maxima or local minima.*

69. $f(x) = x^3 - 3\,x^2$

70. $f(x) = 6\,x^2 - x^3$

71. $f(x) = 4 - x^2$

72. $f(x) = x^3 - \dfrac{3}{2}\,x^2 - 36\,x$

73. $f(x) = 2\,x^3 - 3\,x^2 + 12$

74. $p(x) = \dfrac{x-4}{x^2+20}$

75. $f(x) = 3\,x^4 - 4\,x^3 + 2$

76. $g(x) = \dfrac{x^4}{2 - 12\,x^2}$

T **77.** $f(x) = x^3 + \cos x$

78. $f(x) = \sqrt{x}\left(\dfrac{12}{7}\,x^3 - 4\,x^2\right)$

79. $p(t) = 2\,t^3 + 3\,t^2 - 36\,t$

80. $f(x) = \dfrac{x^4}{4} - \dfrac{5\,x^3}{3} - 4\,x^2 + 48\,x$

81. $h(x) = (x+a)^4$; a constant

82. $f(x) = x^3 - 13\,x^2 - 9\,x$

83. $f(\theta) = 2\sin 3\,\theta - 3\,\theta$ on $[0, \pi]$

84. $f(x) = 2\,x^{-3} - x^{-2}$

85. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.

a. If $f'(x) > 0$ and $f''(x) < 0$ on an interval, then f is increasing at a decreasing rate on the interval.

b. If $f'(c) > 0$ and $f''(c) = 0$, then f has a local maximum at c.

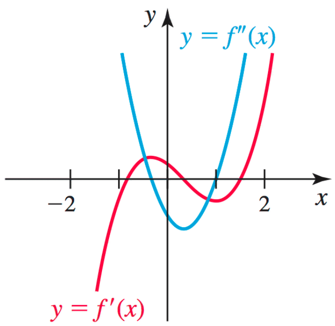
c. Two functions that differ by an additive constant both increase and decrease on the same intervals.

d. If f and g increase on an interval, then the product $f\,g$ also increases on that interval.

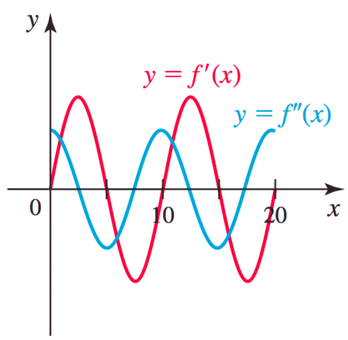
e. There exists a function f that is continuous on $(-\infty, \infty)$ with exactly three critical points, all of which correspond to local maxima.

86–87. Functions from derivatives *Consider the following graphs of f' and f''. On the same set of axes, sketch the graph of a possible function f. The graphs of f are not unique.*

86.



87.



88. Is it possible? Determine whether the following properties can be satisfied by a function that is continuous on $(-\infty, \infty)$. If such a function is possible, provide an example or a sketch of the function. If such a function is not possible, explain why.

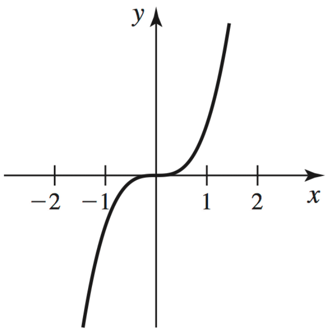
a. A function f is concave down and positive everywhere.

b. A function f is increasing and concave down everywhere.

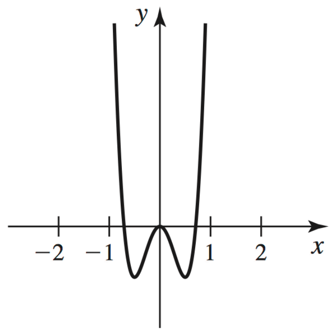
c. A function f has exactly two local extrema and three inflection points.

d. A function f has exactly four zeros and two local extrema.

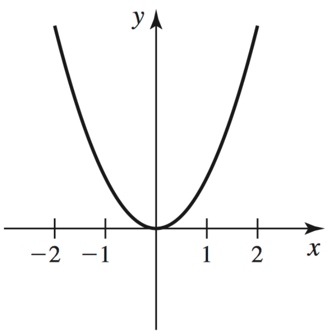
89. Matching derivatives and functions The following figures show the graphs of three functions (graphs a–c). Match each function with its first derivative (graphs A–C) *and* its second derivative (graphs i–iii).



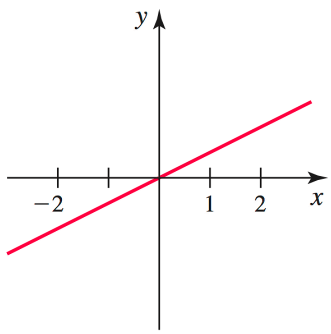
(a)



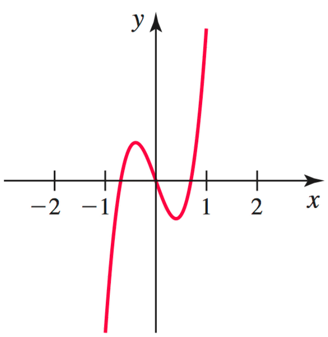
(b)



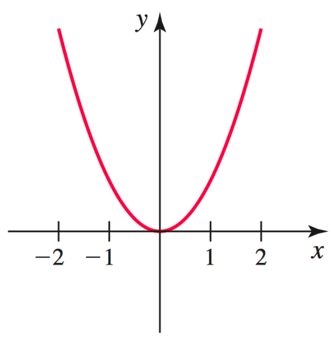
(c)



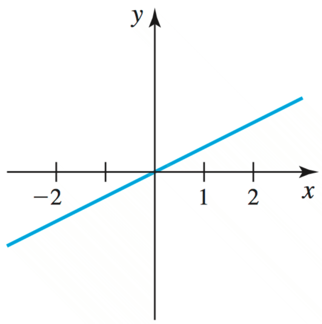
(A)



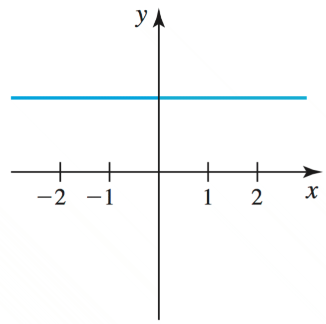
(B)



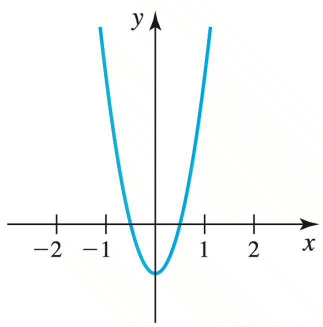
(C)



(i)

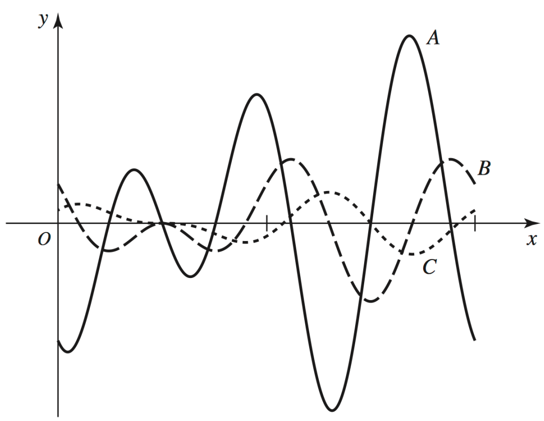


(ii)



(iii)

90. Graphical analysis The figure shows the graphs of f, f', and f''. Which curve is which?



91. Sketching graphs Sketch the graph of a function f that is continuous on $[a, b]$ such that f, f', and f'' have the signs indicated in the following table on $[a, b]$. There are eight different cases lettered A-H that correspond to eight different graphs.

Case	A	B	C	D	E	F	G	H
f	+	+	+	+	−	−	−	−
f'	+	+	−	−	+	+	−	−
f''	+	−	+	−	+	−	+	−

92–95. Designer functions *Sketch the graph of a function f that is continuous on* $(-\infty, \infty)$ *and satisfies the following sets of conditions.*

92. $f''(x) > 0$ on $(-\infty, -2)$; $f''(-2) = 0$; $f'(-1) = f'(1) = 0$; $f''(2) = 0$; $f'(3) = 0$; $f''(x) > 0$ on $(4, \infty)$

93. $f(-2) = f''(-1) = 0$; $f'\left(-\frac{3}{2}\right) = 0$; $f(0) = f'(0) = 0$; $f(1) = f'(1) = 0$

94. $f'(x) > 0$, for all x in the domain of f'; $f'(-2)$ and $f'(1)$ do not exist; $f''(0) = 0$

95. $f''(x) > 0$ on $(-\infty, -2)$; $f''(x) < 0$ on $(-2, 1)$; $f''(x) > 0$ on $(1, 3)$; $f''(x) < 0$ on $(3, \infty)$

T **96.** **Graph carefully** Graph the function $f(x) = 60x^5 - 901x^3 + 27x$ in the window $[-4, 4] \times [-10{,}000, 10{,}000]$. How many extreme values do you see? Locate *all* the extreme values by analyzing f'.

97. **Interpreting the derivative** The graph of f' on the interval $[-3, 2]$ is shown in the figure.

a. On what interval(s) is f increasing? Decreasing?

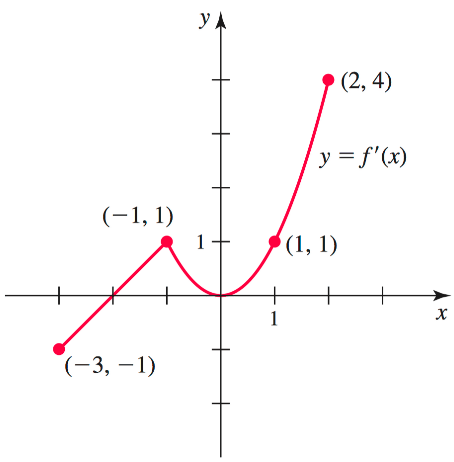
b. Find the critical points of f. Which critical points correspond to local maxima? Local minima? Neither?

c. At what point(s) does f have an inflection point?

d. On what interval(s) is f concave up? Concave down?

e. Sketch the graph of f''.

f. Sketch one possible graph of f.



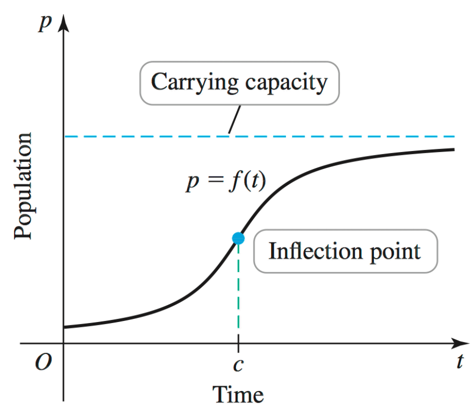
98. **Concavity of parabolas** Consider the general parabola described by the function $f(x) = ax^2 + bx + c$. For what values of a, b, and c is f concave up? For what values of a, b, and c, is f concave down?

Explorations and Challenges »

T **99. Demand functions and elasticity** Economists use *demand functions* to describe how much of a commodity can be sold at varying prices. For example, the demand function $D(p) = 500 - 10p$ says that at a price of $p = 10$, a quantity of $D(10) = 400$ units of the commodity can be sold. The elasticity $E = \frac{dD}{dp}\frac{p}{D}$ of the demand gives the approximate percent change in the demand for every 1% change in the price. (See Section 3.6 or the Guided Project *Elasticity in Economics* for more on demand functions and elasticity.)

- **a.** Compute the elasticity of the demand function $D(p) = 500 - 10p$.
- **b.** If the price is \$12 and increases by 4.5%, what is the approximate percent change in the demand?
- **c.** Show that for the linear demand function $D(p) = a - bp$, where a and b are positive real numbers, the elasticity is a decreasing function, for $p \geq 0$ and $p \neq a/p$.
- **d.** Show that the demand function $D(p) = a/p^b$, where a and b are positive real numbers, has a constant elasticity for all positive prices.

100. Population models A typical population curve is shown in the figure. The population is small at $t = 0$ and increases toward a steady-state level called the *carrying capacity*. Explain why the maximum growth rate occurs at an inflection point of the population curve.



101. Population models The population of a species is given by the function $P(t) = \frac{Kt^2}{t^2 + b}$, where $t \geq 0$ is measured in years and K and b are positive real numbers.

- **a.** With $K = 300$ and $b = 30$, what is $\lim_{t\to\infty} P(t)$, the carrying capacity of the population?
- **b.** With $K = 300$ and $b = 30$, when does the maximum growth rate occur?
- **c.** For arbitrary positive values of K and b, when does the maximum growth rate occur (in terms of K and b)?

102. Tangent lines and concavity Give an argument to support the claim that if a function is concave up at a point, then the tangent line at that point lies below the curve near that point.

103. General quartic Show that the general quartic (fourth-degree) polynomial $f(x) = x^4 + ax^3 + bx^2 + cx + d$, where a, b, c, and d are real numbers, has either zero or two inflection points, and the latter case occurs provided $b < \frac{3a^2}{8}$.

104. First Derivative Test is not exhaustive Sketch the graph of a (simple) nonconstant function f that has a local maximum at $x = 1$, with $f'(1) = 0$, where f' does not change sign from positive to negative as x increases through 1. Why can't the First Derivative Test be used to classify the critical point at $x = 1$ as a local maximum? How could the test be rephrased to account for such a critical point?

105. Properties of cubics Consider the general cubic polynomial $f(x) = x^3 + a\,x^2 + b\,x + c$, where a, b, and c are real numbers.

a. Prove that f has exactly one local maximum and one local minimum provided $a^2 > 3\,b$.

b. Prove that f has no extreme values if $a^2 < 3\,b$.