4.2 Mean Value Theorem

In Section 4.1, we learned how to find the absolute extrema of a function. This information is needed to produce an accurate graph of a function (Sections 4.3 and 4.4) and to solve optimization problems (Section 4.5). The procedures used to solve these types of problems depend on several results developed over the next two sections.

We begin with the *Mean Value Theorem*, a cornerstone in the theoretical framework of calculus. Several critical theorems rely on the Mean Value Theorem; the theorem also appears in practical applications. The proof of the Mean Value Theorem relies on a preliminary result known as Rolle's Theorem.

Rolle's Theorem »

Consider a function f that is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b). Furthermore, assume f has the special property that $f(a) = f(b)$ (**Figure 4.13**). The statement of Rolle's Theorem is not surprising: It says that somewhere between a and b, there is at least one point at which f has a horizontal tangent line.

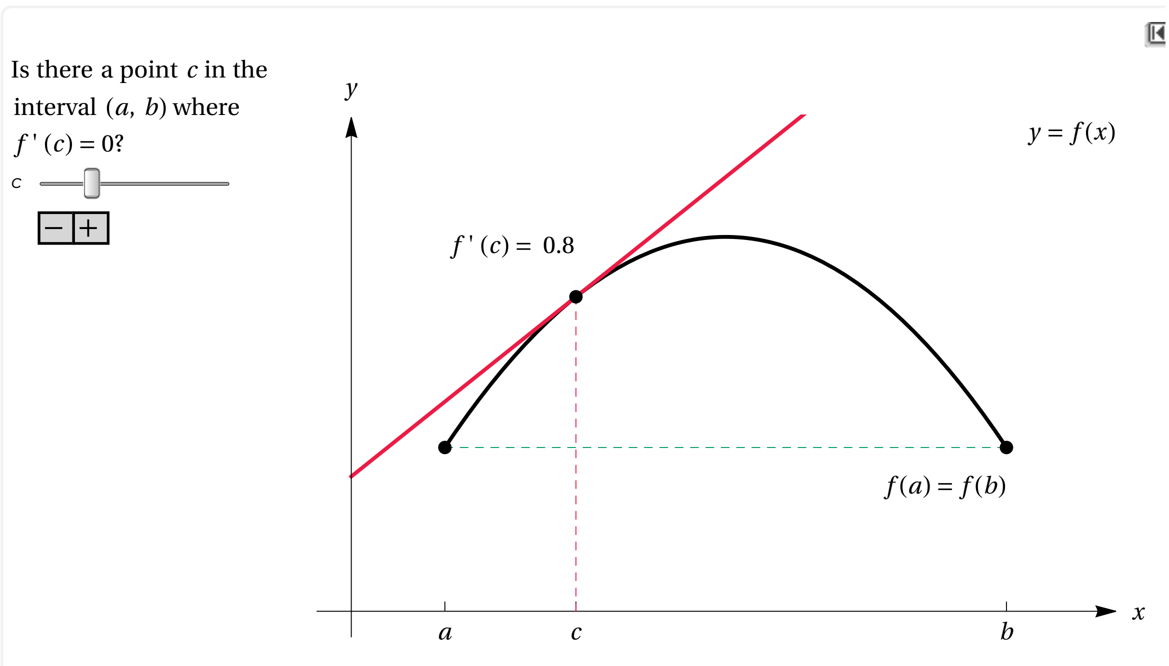


Figure 4.13

THEOREM 4.3 Rolle's Theorem

Let f be continuous on a closed interval $[a, b]$ and differentiable on (a, b) with $f(a) = f(b)$. There is at least one point c in (a, b) such that $f'(c) = 0$.

Proof: The function f satisfies the conditions of Theorem 4.1 (Extreme Value Theorem) and thus it attains its absolute maximum and minimum values on $[a, b]$. Those values are attained either at an endpoint or at an interior point c.

Note »

> The Extreme Value Theorem, discussed in Section 4.1, states that a function that is continuous on a closed bounded interval attains its absolute maximum and minimum values on that interval.

Case 1: First suppose f attains both its absolute maximum and minimum values at the endpoints. Because $f(a) = f(b)$, the maximum and minimum values are equal, and it follows that f is a constant function on $[a, b]$. Therefore, $f'(x) = 0$ for all x in (a, b), and the conclusion of the theorem holds.

Case 2: Assume at least one of the absolute extreme values of f does not occur at an endpoint. Then, f must attain an absolute extreme value at an interior point of $[a, b]$; therefore, f must have either a local maximum or a local minimum at a point c in (a, b). Because f is differentiable on (a, b) we know from Theorem 4.2 that at a local extremum the derivative is zero. Thus, $f'(c) = 0$ for at least one point c of (a, b), and again the conclusion of the theorem holds. ◆

Why does Rolle's Theorem require continuity? A function that is not continuous on $[a, b]$ may have identical values at both endpoints and still not have a horizontal tangent line at any point on the interval (**Figure 4.14a**). Similarly, a function that is continuous on $[a, b]$ but not differentiable at a point of (a, b) may also fail to have a horizontal tangent line (**Figure 4.14b**).

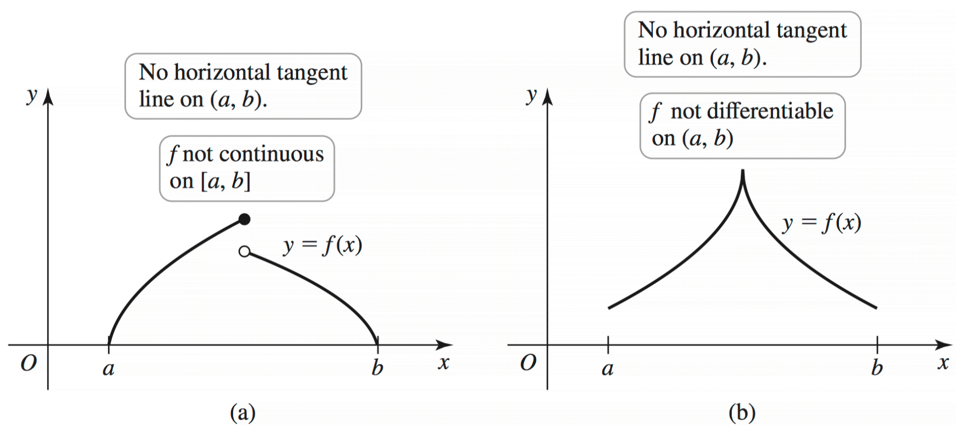


Figure 4.14

Quick Check 1 Where on the interval $[0, 4]$ does $f(x) = 4x - x^2$ have a horizontal tangent line? ◆

Answer »

> $x = 2$

EXAMPLE 1 Verifying Rolle's Theorem

Find an interval I on which Rolle's Theorem applies to $f(x) = x^3 - 7x^2 + 10x$. Then find all points c in I at which $f'(c) = 0$.

SOLUTION »

Because f is a polynomial, it is everywhere continuous and differentiable. We need an interval $[a, b]$ with the property that $f(a) = f(b)$. Noting that $f(x) = x(x-2)(x-5)$, we choose the interval $[0, 5]$, because $f(0) = f(5) = 0$ (other intervals are possible). The goal is to find points c in the interval $(0, 5)$ at which $f'(c) = 0$, which amounts to the familiar task of finding the critical points of f. The critical points satisfy

$$f'(x) = 3x^2 - 14x + 10 = 0.$$

Using the quadratic formula, the roots are

$$x = \frac{7 \pm \sqrt{19}}{3}, \quad \text{or} \quad x \approx 0.88 \quad \text{and} \quad x \approx 3.79.$$

As shown in **Figure 4.15**, the graph of f has two points at which the tangent line is horizontal.

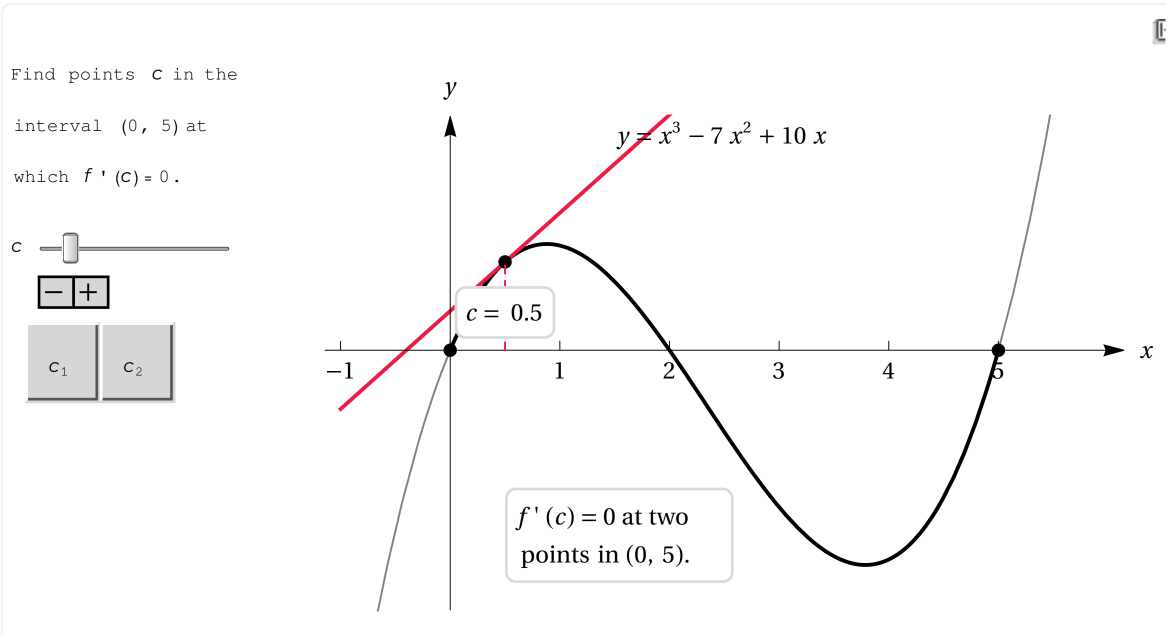


Figure 4.15

Related Exercises 11, 16 ◆

Mean Value Theorem »

The Mean Value Theorem is easily understood with the aid of a picture. **Figure 4.16** shows a function f differentiable on (a, b) with a secant line passing through $(a, f(a))$ and $(b, f(b))$; its slope is the average rate of change of f over $[a, b]$. The Mean Value Theorem claims that there exists a point c in (a, b) at which the slope of the tangent line at c is equal to the slope of the secant line. In other words, we can find a point on the graph of f where the tangent line is parallel to the secant line.

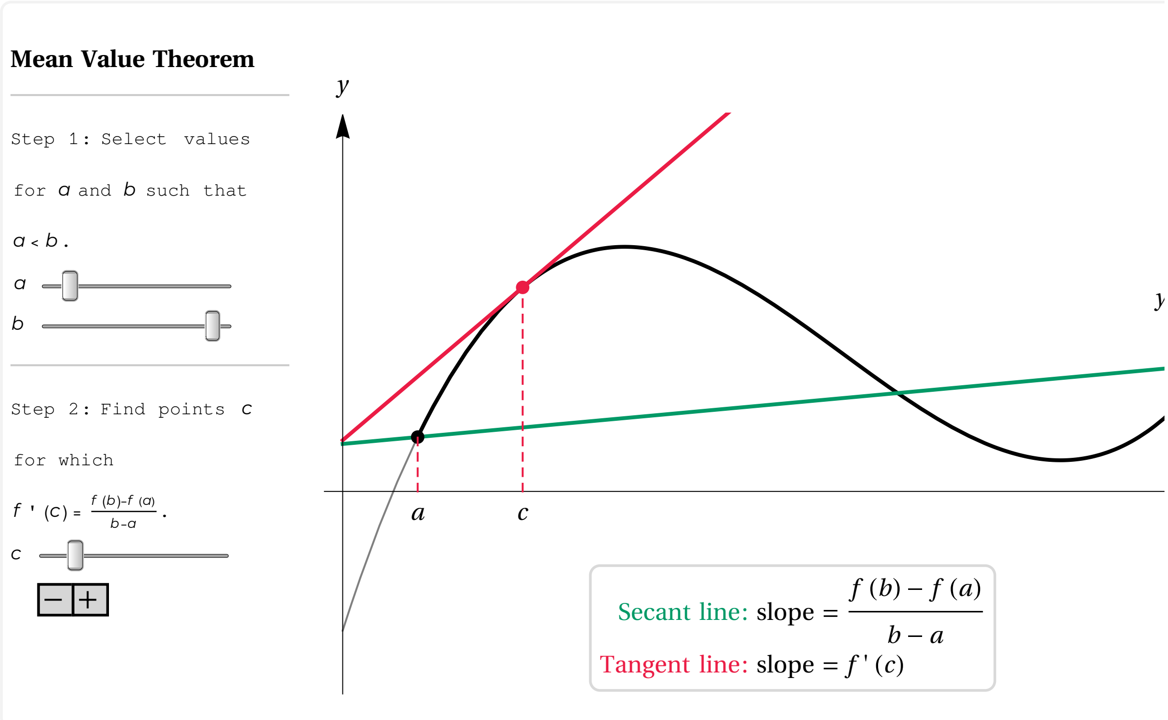


Figure 4.16

THEOREM 4.4 Mean Value Theorem

If f is continuous on a closed interval $[a, b]$ and differentiable on (a, b), then there is at least one point c in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Proof: The strategy of the proof is to use the function f of the Mean Value Theorem to form a new function g that satisfies Rolle's Theorem. Notice that the continuity and differentiability conditions of Rolle's Theorem and the Mean Value Theorem are the same. We devise g so that it satisfies the conditions $g(a) = g(b) = 0$.

As shown in **Figure 4.17**, the secant line joining $(a, f(a))$ and $(b, f(b))$ is described by a function ℓ. We now define a new function g that measures the difference between the given function f and the line ℓ. This function is simply $g(x) = f(x) - \ell(x)$. Because f and ℓ are continuous on $[a, b]$ and differentiable on (a, b), it follows that g is also continuous on $[a, b]$ and differentiable on (a, b). Furthermore, because the graphs of f and ℓ intersect at $x = a$ and $x = b$, we have $g(a) = f(a) - \ell(a) = 0$ and $g(b) = f(b) - \ell(b) = 0$.

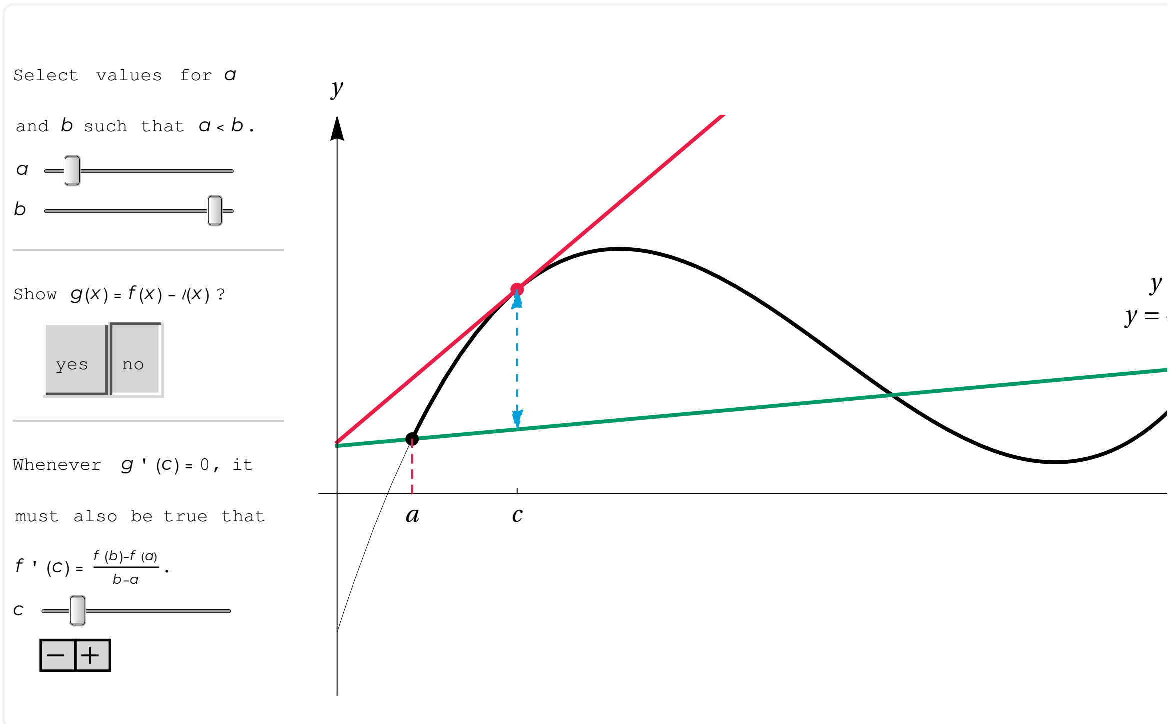


Figure 4.17

We now have a function g that satisfies the conditions of Rolle's Theorem. By that theorem, we are guaranteed the existence of at least one point c in the interval (a, b) such that $g'(c) = 0$. By the definition of g, this condition implies that $f'(c) - \ell'(c) = 0$, or $f'(c) = \ell'(c)$.

We are almost finished. What is $\ell'(c)$? It is just the slope of the secant line, which is

$$\frac{f(b) - f(a)}{b - a}.$$

Therefore, $f'(c) = \ell'(c)$ implies that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

◆

Note »

The proofs of Rolle's Theorem and the Mean Value Theorem are nonconstructive: The theorems claim that a certain point exists, but their proofs do not say how to find it.

Quick Check 2 Sketch the graph of a function that illustrates why the continuity condition of the Mean Value Theorem is needed. Sketch the graph of a function that illustrates why the differentiability condition of the Mean Value Theorem is needed. ◆

Answer »

The functions shown in Figure 4.14 provide examples.

The following situation offers an interpretation of the Mean Value Theorem. Imagine taking 2 hours to drive to a town 100 miles away. While your average speed is $\frac{100 \text{ mi}}{2 \text{ hr}} = 50 \text{ mi/hr}$, your instantaneous speed (measured by the speedometer) almost certainly varies. The Mean Value Theorem says that at some point during the trip, your instantaneous speed equals your average speed, which is 50 mi/hr. In Example 2, we apply these ideas to the science of weather forecasting.

EXAMPLE 2 Mean Value Theorem in action

The *lapse rate* is the rate at which the temperature T decreases in the atmosphere with respect to increasing altitude z. It is typically reported in units of °C/km and is defined by $\gamma = -\dfrac{dT}{dz}$. When the lapse rate rises above 7°C/km in a certain layer of the atmosphere, it indicates favorable conditions for thunderstorm and tornado formation, provided other atmospheric conditions are also present.

Suppose the temperature at $z = 2.9$ km is $T = 7.6$°C, and the temperature at $z = 5.6$ km is $T = -14.3$°C. Assume also that the temperature function is continuous and differentiable at all altitudes of interest. What can a meteorologist conclude from these data?

Note »

> Meteorologists look for "steep" lapse rates in the layer of the atmosphere where the pressure is between 700 and 500 hPa (hectopascals). This range of pressure typically corresponds to altitudes between 3 km and 5.5 km. The data in Example 2 were recorded in Denver at nearly the same time a tornado struck 50 mi to the north.

SOLUTION »

Figure 4.18 shows the two data points plotted on a graph of altitude and temperature. The slope of the line joining these points is

$$\frac{-14.3\,°\text{C} - 7.6\,°\text{C}}{5.6\text{ km} - 2.9\text{ km}} = -8.1\,°\text{C/km},$$

which means, on average, the temperature is decreasing at 8.1°C/km in the layer of air between 2.9 km and 5.6 km. With only two data points, we cannot know the entire temperature profile. The Mean Value Theorem, however, guarantees that there is at least one altitude at which $\dfrac{dT}{dz} = -8.1\,°\text{C/km}$. At each such altitude, the lapse rate is $\gamma = -\dfrac{dT}{dz} = 8.1\,°\text{C/km}$. Because this lapse rate is above the 7°C/km threshold associated with unstable weather, the meteorologist might expect an increased likelihood of severe storms.

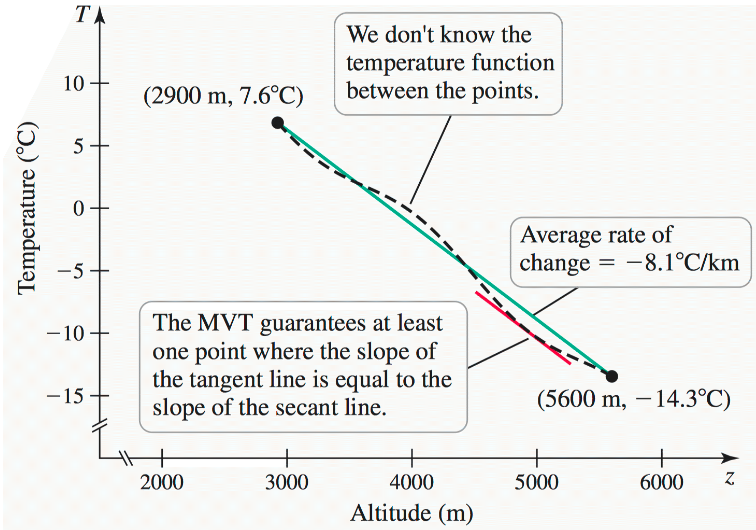


Figure 4.18

Related Exercises 19, 40 ◆

EXAMPLE 3 Verifying the Mean Value Theorem

Determine whether the function $f(x) = 2x^3 - 3x + 1$ satisfies the conditions of the Mean Value Theorem on the interval $[-2, 2]$. If so, find the point(s) guaranteed to exist by the theorem.

SOLUTION »

The polynomial f is everywhere continuous and differentiable, so it satisfies the conditions of the Mean Value Theorem. The average rate of change of the function on the interval $[-2, 2]$ is

$$\frac{f(2) - f(-2)}{2 - (-2)} = \frac{11 - (-9)}{4} = 5.$$

The goal is to find points in $(-2, 2)$ at which the line tangent to the curve has a slope of 5 – that is, to find points at which $f'(x) = 5$. Differentiating f, this condition becomes

$$f'(x) = 6x^2 - 3 = 5 \text{ or } x^2 = \frac{4}{3}.$$

Therefore, the points guaranteed to exist by the Mean Value Theorem are $x = \pm\frac{2}{\sqrt{3}} \approx \pm 1.15$. The tangent lines have slope 5 at the corresponding points on the curve (**Figure 4.19**).

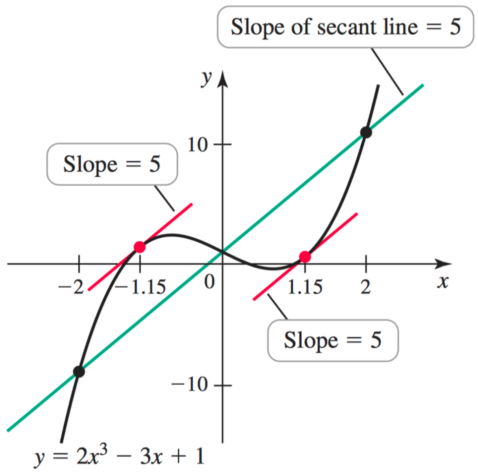


Figure 4.19

Related Exercises 21–22 ◆

Consequences of the Mean Value Theorem »

We close with two results that follow from the Mean Value Theorem.

We already know that the derivative of a constant function is zero; that is, if $f(x) = C$, then $f'(x) = 0$ (Theorem 3.2). Theorem 4.5 states the converse of this result.

> **THEOREM 4.5 Zero Derivative Implies Constant Function**
>
> If f is differentiable and $f'(x) = 0$ at all points of an open interval I, then f is a constant function on I.

Proof: Suppose $f'(x) = 0$ on $[a, b]$, where a and b are distinct points of I. By the Mean Value Theorem, there exists a point c in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = \underbrace{f'(c) = 0}_{\substack{f'(x) = 0 \text{ for} \\ \text{all } x \text{ in } I}}.$$

Multiplying both sides of this equation by $b - a \neq 0$, it follows that $f(b) = f(a)$, and this is true for every pair of points a and b in I. If $f(b) = f(a)$ for every pair of points in an interval, then f is a constant function on that interval. ◆

Theorem 4.6 builds on the conclusion of Theorem 4.5.

> **THEOREM 4.6 Functions with Equal Derivatives Differ by a Constant**
>
> If two functions have the property that $f'(x) = g'(x)$ for all x of an open interval I, then $f(x) - g(x) = C$ on I, where C is a constant; that is, f and g differ by a constant.

Proof: The fact that $f'(x) = g'(x)$ on I implies that $f'(x) - g'(x) = 0$ on I. Recall that the derivative of a difference of two functions equals the difference of the derivatives, so we can write

$$f'(x) - g'(x) = (f - g)'(x) = 0.$$

Now we have a function $f - g$ whose derivative is zero on I. By Theorem 4.5, $f(x) - g(x) = C$ for all x in I, where C is a constant; that is, f and g differ by a constant. ◆

Quick Check 3 Give two distinct linear functions f and g that satisfy $f'(x) = g'(x)$; that is, the lines have equal slopes. Show that f and g differ by a constant. ◆

Answer »

> The graphs of $f(x) = 3x$ and $g(x) = 3x + 2$ have the same slope. Note that $f(x) - g(x) = -2$, a constant.

The utility of Theorems 4.5 and 4.6 will become apparent in Section 4.9, where we establish a pivotal result that has far-reaching consequences.

Exercises »

Getting Started »

Practice Exercises »

11–18. Rolle's Theorem *Determine whether Rolle's Theorem applies to the following functions on the given interval. If so, find the point(s) guaranteed to exist by Rolle's Theorem.*

11. $f(x) = x(x-1)^2$; $[0, 1]$

12. $f(x) = \sin 2x$; $[0, \pi/2]$

13. $f(x) = \cos 4x$; $[\pi/8, 3\pi/8]$

14. $f(x) = 1 - |x|$; $[-1, 1]$

15. $f(x) = 1 - x^{2/3}$; $[-1, 1]$

16. $f(x) = x^3 - 2x^2 - 8x$; $[-2, 4]$

17. $g(x) = x^3 - x^2 - 5x - 3$; $[-1, 3]$

18. $h(x) = \sqrt{x}$; $[0, a]$, where $a > 0$

19. Lapse rates in the atmosphere Refer to Example 2. Concurrent measurements indicate that at an elevation of 6.1 km, the temperature is $-10.3\,°$C, and at an elevation of 3.2 km, the temperature is $8.0\,°$C. Based on the Mean Value Theorem, can you conclude that the lapse rate exceeds the threshold value of $7\,°\text{C/km}$ at some intermediate elevation? Explain.

20. Drag racer acceleration The fastest drag racers can reach a speed of 330 mi/hr over a quarter-mile strip in 4.45 seconds (from a standing start). Complete the following sentence about such a drag racer: At some point during the race, the maximum acceleration of the drag racer is at least _______mi/hr/s.

21–32. Mean Value Theorem *Consider the following functions on the given interval* $[a, b]$.

a. *Determine whether the Mean Value Theorem applies to the following functions on the given interval* $[a, b]$.

b. *If so, find the point(s) that are guaranteed to exist by the Mean Value Theorem.*

21. $f(x) = 7 - x^2$; $[-1, 2]$

22. $f(x) = x^3 - 2x^2$; $[0, 1]$

23. $f(x) = \begin{cases} -2x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$; $[-1, 1]$

24. $f(x) = \dfrac{1}{(x-1)^2}$; $[0, 2]$

25. $f(x) = \sqrt{x}$; $[1, 4]$

26. $f(x) = |x - 1|$; $[-1, 4]$

T **27.** $f(x) = \sin x$; $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

T **28.** $f(x) = \tan x$; $\left[0, \dfrac{\pi}{4}\right]$

29. $f(x) = x^{-1/3}$; $\left[\dfrac{1}{8}, 8\right]$

30. $f(x) = x + \dfrac{1}{x}$; $[1, 3]$

31. $f(x) = 2x^{1/3}$; $[-8, 8]$

32. $f(x) = \dfrac{x}{x+2}$; $[-1, 2]$

33. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.

a. The continuous function $f(x) = 1 - |x|$ satisfies the conditions of the Mean Value Theorem on the interval $[-1, 1]$.

b. Two differentiable functions that differ by a constant always have the same derivative.

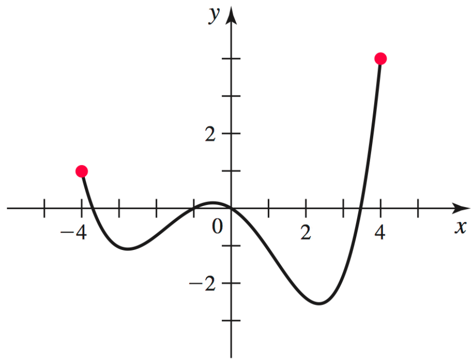
c. If $f'(x) = 0$, then $f(x) = 10$.

34. Without evaluating derivatives, determine which of the functions $f(x) = \sin^2 x$, $g(x) = -\cos^2 x$, $h(x) = 2\sin^2 x$, and $p(x) = \dfrac{1}{\csc^2 x}$ have the same derivative.

35. Without evaluating derivatives, determine which of the functions $g(x) = 2x^{10}$, $h(x) = x^{10} + 2$, and $p(x) = x^{10} - \sin 2$ have the same derivative as $f(x) = x^{10}$.

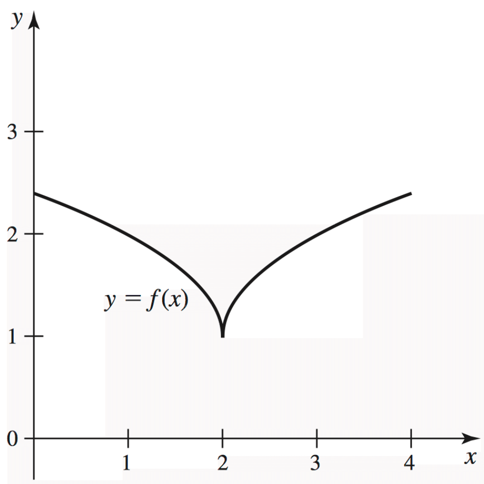
36. Find all functions f whose derivative is $f'(x) = x + 1$.

37. Mean Value Theorem and graphs By visual inspection, locate all points on the interval $(-4, 4)$ at which the slope of the tangent line equals the average rate of change of the function on the interval $[-4, 4]$.

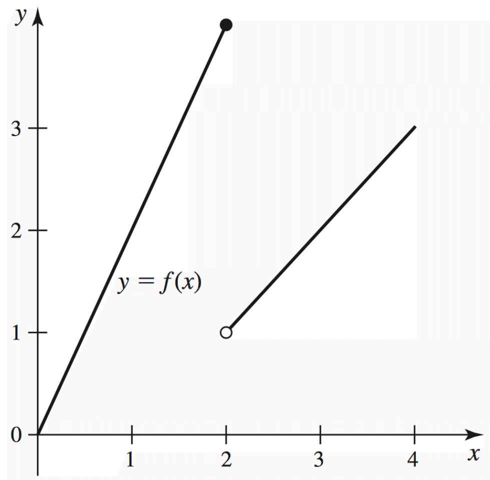


38–39. Mean Value Theorem and graphs *Find all points on the interval* $(1, 3)$ *at which the slope of the tangent line equals the average rate of change of* f *on* $[1, 3]$. *Reconcile your results with the Mean Value Theorem.*

38.



39.



40. Avalanche forecasting Avalanche forecasters measure the *temperature gradient* $\frac{dT}{dh}$, which is the rate at which the temperature in a snowpack T changes with respect to its depth h. A large temperature gradient may lead to a weak layer in the snowpack. When these weak layers collapse, avalanches occur. Avalanche forecasters use the following rule of thumb: If $\frac{dT}{dh}$ exceeds $10°\mathrm{C/m}$ anywhere in the snowpack, conditions are favorable for weak-layer formation and the risk of avalanche increases. Assume the temperature function is continuous and differentiable.

a. An avalanche forecaster digs a snow pit and takes two temperature measurements. At the surface ($h = 0$) the temperature is $-16°$C. At a depth of 1.1 m, the temperature is $-2°$C. Using the Mean Value Theorem, what can he conclude about the temperature gradient? Is the formation of a weak layer likely?

b. One mile away, a skier finds that the temperature at a depth of 1.4 m is $-1°$C, and at the surface it is $-12°$C. What can be concluded about the temperature gradient? Is the formation of a weak layer in her location likely?

c. Because snow is an excellent insulator, the temperature of snow-covered ground is near $0°$C. Furthermore, the surface temperature of snow in a particular area does not vary much from one location to the next. Explain why a weak layer is more likely to form in places where the snowpack is not too deep.

d. The term *isothermal* is used to describe the situation where all layers of the snowpack are at the same temperature (typically near the freezing point). Is a weak layer likely to form in isothermal snow? Explain.

41. Mean Value Theorem and the police A state patrol officer saw a car start from rest at a highway on-ramp. She radioed ahead to a patrol officer 30 mi along the highway. When the car reached the location of the second officer 28 min later, it was clocked going 60 mi/hr. The driver of the car was given a ticket for exceeding the 60-mi/hr speed limit. Why can the officer conclude that the driver exceeded the speed limit?

42. Mean Value Theorem and the police again Compare carefully to Exercise 41. A state patrol officer saw a car start from rest at a highway on-ramp. She radioed ahead to another officer 30 mi along the highway. When the car reached the location of the second officer 30 min later, it was clocked going 60 mi/hr. Can the patrol officer conclude that the driver exceeded the speed limit?

43. **Running pace** Explain why if a runner completes a 6.2-mi (10-km) race in 32 min, then he must have been running at exactly 11 mi/hr at least twice in the race. Assume the runner's speed at the finish line is zero.

44. **Mean Value Theorem for linear functions** Interpret the Mean Value Theorem when it is applied to any linear function.

45. **Mean Value Theorem for quadratic functions** Consider the quadratic function $f(x) = A\,x^2 + B\,x + C$, where A, B, and C are real numbers with $A \neq 0$. Show that when the Mean Value Theorem is applied to f on the interval $[a, b]$, the number c guaranteed by the theorem is the midpoint of the interval.

46. **Means**

a. Show that the point c guaranteed to exist by the Mean Value Theorem for $f(x) = x^2$ on $[a, b]$ is the arithmetic mean of a and b; that is, $c = \dfrac{a+b}{2}$.

b. Show that the point c guaranteed to exist by the Mean Value Theorem for $f(x) = \dfrac{1}{x}$ on $[a, b]$, where $0 < a < b$, is the geometric mean of a and b; that is, $c = \sqrt{a\,b}$.

47. **Equal derivatives** Verify that the functions $f(x) = \tan^2 x$ and $g(x) = \sec^2 x$ have the same derivative. What can you say about the difference $f - g$? Explain.

48. **Equal derivatives** Verify that the functions $f(x) = \sin^2 x$ and $g(x) = -\cos^2 x$ have the same derivative. What can you say about the difference $f - g$? Explain.

49. **100-m speed** The Jamaican sprinter Usain Bolt set a world record of 9.58 s in the 100-meter dash in the summer of 2009. Did his speed ever exceed 37 km/hr during the race? Explain.

Explorations and Challenges »

50. Suppose $f'(x) > 1$, for all $x > 0$, and $f(0) = 0$. Show that $f(x) > x$, for all $x > 0$.

51. Suppose $f'(x) < 2$, for all $x \geq 2$, and $f(2) = 7$. Show that $f(4) < 11$.

52. Use the Mean Value Theorem to prove that $1 + \dfrac{a}{2} > \sqrt{1+a}$, for $a > 0$. (*Hint:* For a given value of $a > 0$, let $f(x) = \sqrt{1+x}$ on $[0, a]$ and use the fact that $\sqrt{1+c} > 1$, for $c > 0$.)

53. Prove the following statements.

a. $|\sin a - \sin b| \leq |a - b|$, for any real numbers a and b

b. $|\sin a| \leq |a|$, for any real number a

54. **Condition for nondifferentiability** Suppose $f'(x) < 0 < f''(x)$, for $x < a$ and $f'(x) > 0 > f''(x)$, for $x > a$. Prove that f is not differentiable at a. (*Hint:* Assume f is differentiable at a, and apply the Mean Value Theorem to f'.) More generally show that if f' and f'' change sign at the same point, then f is not differentiable at that point.

55. **Generalized Mean Value Theorem** Suppose the functions f and g are continuous on $[a, b]$ and differentiable on (a, b), where $g(a) \neq g(b)$. Then there is a point c in (a, b) at which

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

This result is known as the **Generalized (or Cauchy's) Mean Value Theorem**.

a. If $g(x) = x$, then show that the Generalized Mean Value Theorem reduces to the Mean Value Theorem.

b. Suppose $f(x) = x^2 - 1$, $g(x) = 4x + 2$, and $[a, b] = [0, 1]$. Find a value of c satisfying the Generalized Mean Value Theorem.