4 Applications of the Derivative

Chapter Preview Much of the previous chapter was devoted to the basic mechanics of derivatives: evaluating them and interpreting them as rates of change. We now apply derivatives to a variety of mathematical questions about the properties of functions and their graphs. One outcome of this work is a set of analytical curve-sketching methods that produce accurate graphs of functions. Equally important, derivatives allow us to formulate and solve a wealth of practical problems. For example, a weather probe dropped from an airplane accelerates until it reaches its terminal velocity: When is the acceleration the greatest? An economist has a mathematical model that relates the demand for a product to its price: What price maximizes the revenue? In this chapter, we develop the tools needed to answer such questions. In addition, we begin an ongoing discussion about approximating functions, we present an important result called the Mean Value Theorem, and we work with a powerful method that enables us to evaluate a new kind of limit. The chapter concludes with two important topics: a numerical approach to approximating roots of functions, called Newton's method; and a preview of integral calculus, which is the subject of Chapter 5.

4.1 Maxima and Minima

With a working understanding of derivatives, we now undertake one of the fundamental tasks of calculus: analyzing the behavior and producing accurate graphs of functions. An important question associated with any function concerns its maximum and minimum values: On a given interval (perhaps the entire domain), where does the function assume its largest and smallest values? Questions about maximum and minimum values take on added significance when a function represents a practical quantity, such as the profits of a company, the surface area of a container, or the speed of a space vehicle.

Absolute Maxima and Minima »

Imagine taking a long hike through varying terrain from west to east. Your elevation changes as you walk over hills, through valleys, and across plains, and you reach several high and low points along the journey. Analogously, when we examine a function over an interval on the x-axis, its values increase and decrease, reaching high points and low points (**Figure 4.1**). You can view our study of functions in this chapter as an exploratory hike along the x-axis.

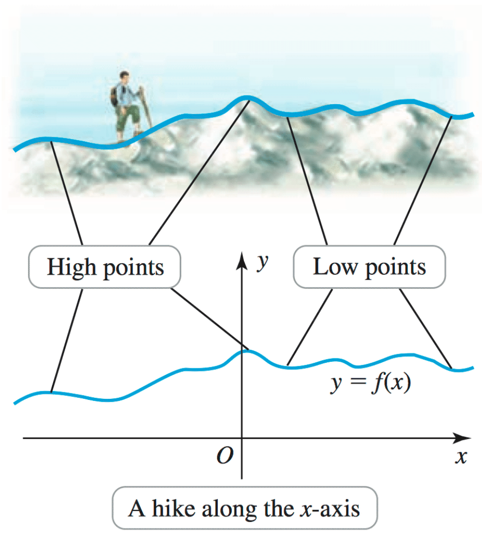


Figure 4.1

DEFINITION **Absolute Maximum and Minimum**

Let f be defined on an interval I containing c. If $f(c) \geq f(x)$ for every x in I, then f has an **absolute maximum** value of $f(c)$ on I at c. If $f(c) \leq f(x)$ for every x in I, then f has an **absolute minimum** value of $f(c)$ on I at c.

Note »

The existence and location of absolute extreme values depend on both the function and the interval of interest. **Figure 4.2** shows various cases for the function $f(x) = x^2$. Notice that if the interval of interest is not closed, a function may not attain absolute extreme values.

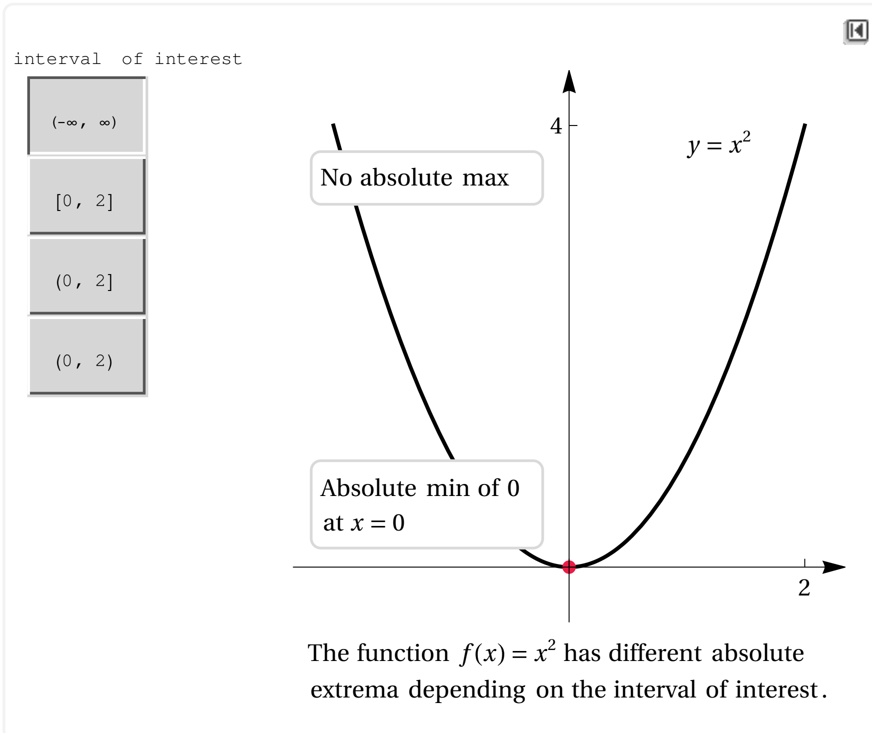


Figure 4.2

However, defining a function on a closed interval is not enough to guarantee the existence of absolute extreme values. Both functions in **Figure 4.3** are defined at every point of a closed interval, but neither function attains an absolute maximum—the discontinuity in each function prevents it from happening.

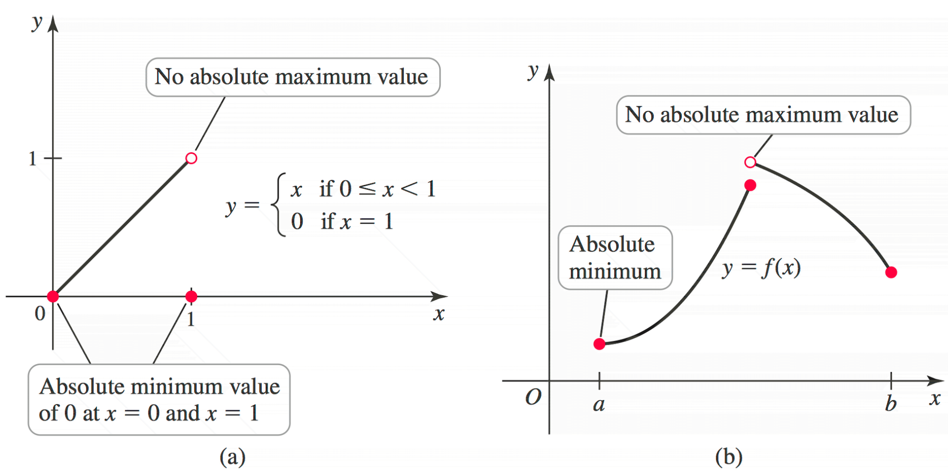


Figure 4.3

It turns out that *two* conditions ensure the existence of absolute minimum and maximum values on an interval: The function must be continuous on an interval and the interval must be closed and bounded.

THEOREM 4.1 Extreme Value Theorem

A function that is continuous on a closed interval $[a, b]$ has an absolute maximum value and an absolute minimum value on that interval.

Note »

The proof of the Extreme Value Theorem relies on some deep properties of the real numbers, found in advanced books.

Quick Check 1 Sketch the graph of a function that is continuous on an interval but does not have an absolute minimum value. Sketch the graph of a function that is defined on a closed interval but does not have an absolute minimum value. ◆

Answer »

The continuous function $f(x) = x$ does not have an absolute minimum on the open interval $(0, 1)$. The function $f(x) = -x$ on $\left[0, \frac{1}{2}\right)$ and $f(x) = 0$ on $\left[\frac{1}{2}, 1\right]$ does not have an absolute minimum on $[0, 1]$.

EXAMPLE 1 Locating absolute maximum and minimum values

For the functions in **Figure 4.4**, identify the location of the absolute maximum value and the absolute minimum value on the interval $[a, b]$. Do the functions meet the conditions of the Extreme Value Theorem?

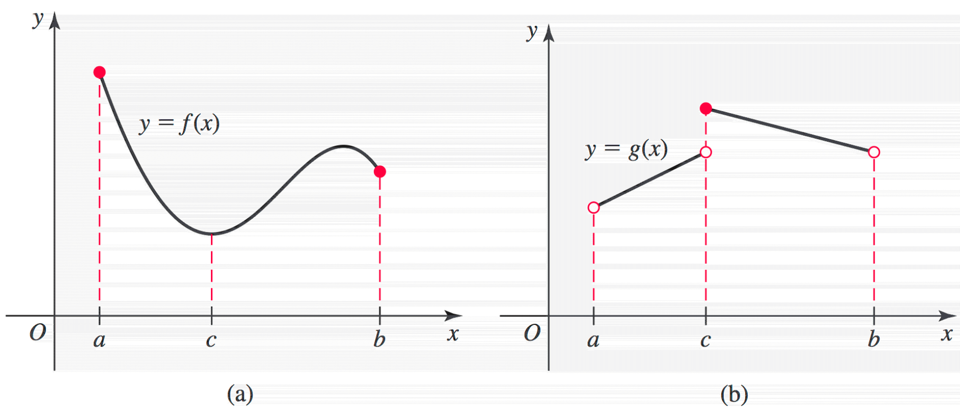


Figure 4.4

SOLUTION »

a. The function f is continuous on the closed interval $[a, b]$, so the Extreme Value Theorem guarantees an absolute maximum (which occurs at a) and an absolute minimum (which occurs at c).

b. The function g does not satisfy the conditions of the Extreme Value Theorem because it is not continuous and it is defined only on the open interval (a, b). It does not have an absolute minimum value. It does, however, have an absolute maximum at c. Therefore, a function may violate the conditions of the Extreme Value Theorem and still have an absolute minimum or maximum (or both).

Related Exercises 11–14 ◆

Local Maxima and Minima »

Figure 4.5 shows a function defined on the interval $[a, b]$. It has an absolute minimum at the endpoint b and an absolute maximum at the interior point s. In addition, the function has special behavior at q, where its value is greatest *among values at nearby points*, and at p and r, where its value is least *among values at nearby points*. A point at which a function takes on the maximum or minimum value among nearby points is important.

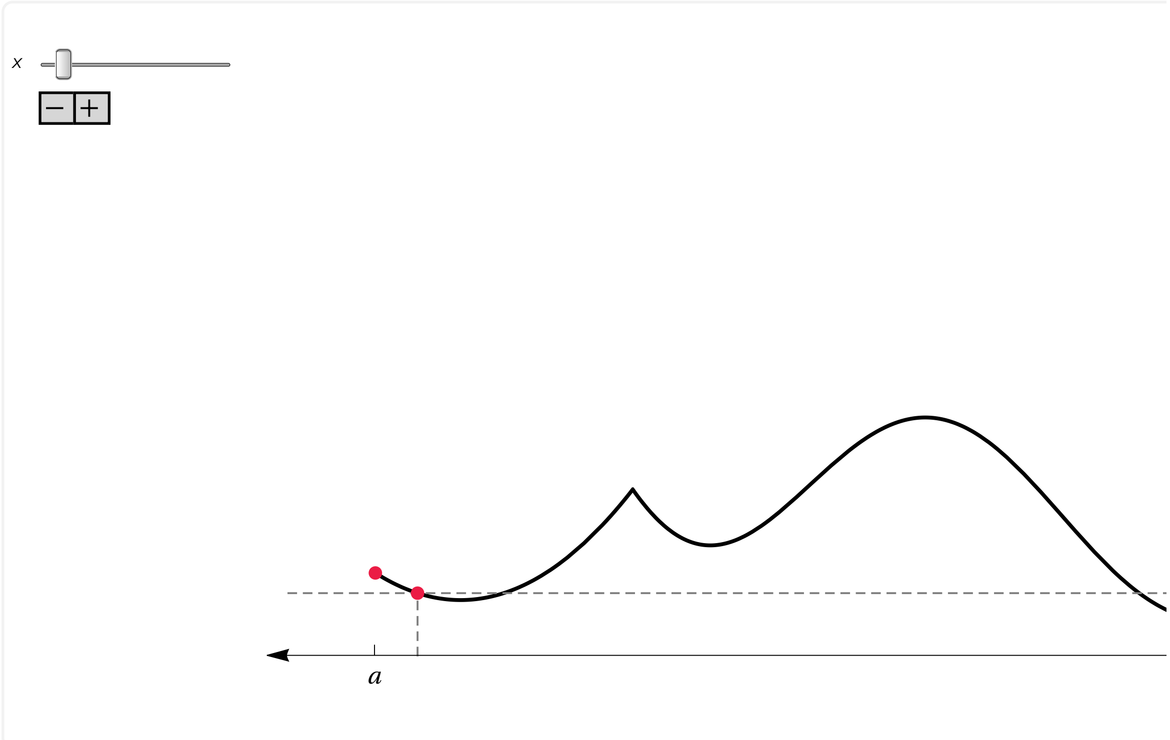


Figure 4.5

DEFINITION **Local Maximum and Minimum Values**

Suppose c is an interior point of some interval I on which f is defined. If $f(c) \geq f(x)$ for all x in I, then $f(c)$ is a **local maximum** value of f. If $f(c) \leq f(x)$ for all x in I, then $f(c)$ is a **local minimum** value of f.

Note »

Local maximum and minimum values are also called *relative maximum and minimum values*. *Local extrema* (plural) and *local extremum* (singular) refer to either local maxima or local minima.

In this book, we adopt the convention that local maximum values and local minimum values occur only at interior points of the interval(s) of interest. For example, in Figure 4.5, the minimum value that occurs at the endpoint b is not a local minimum. However, it is the absolute minimum of the function on $[a, b]$.

EXAMPLE 2 Locating various maxima and minima

Figure 4.6 shows the graph of a function defined on $[a, b]$. Identify the location of the various maxima and minima using the terms *absolute* and *local*.

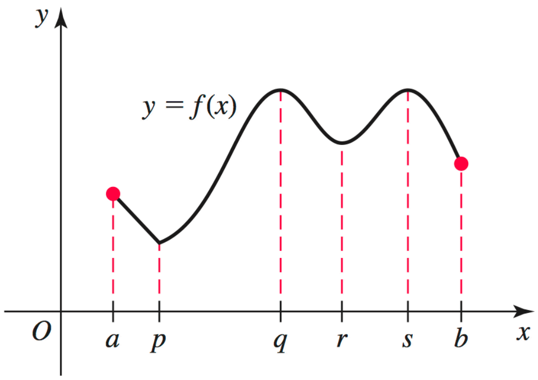


Figure 4.6

SOLUTION »

The function f is continuous on a closed interval; by Theorem 4.1, it has absolute maximum and minimum values on $[a, b]$. The function has a local minimum value and its absolute minimum value at p. It has another local minimum value at r. The absolute maximum value of f occurs at both q and s (which are also local maximum values).

Related Exercises 15–18 ◆

Critical Points

Another look at Figure 4.6 shows that local maxima and minima occur at points in the open interval (a, b) where the derivative is zero ($x = q$, r, and s) and at points where the derivative fails to exist ($x = p$). We now make this observation precise.

Figure 4.7 illustrates a function that is differentiable at c with a local maximum at c. For x near c with $x < c$, the secant lines between the points $(x, f(x))$ and $(c, f(c))$ have nonnegative slopes. For x near c with $x > c$, the secant lines between the points $(x, f(x))$ and $(c, f(c))$ have nonpositive slopes. As $x \to c$, the slopes of these secant lines approach the slope of the tangent line at $(c, f(c))$.

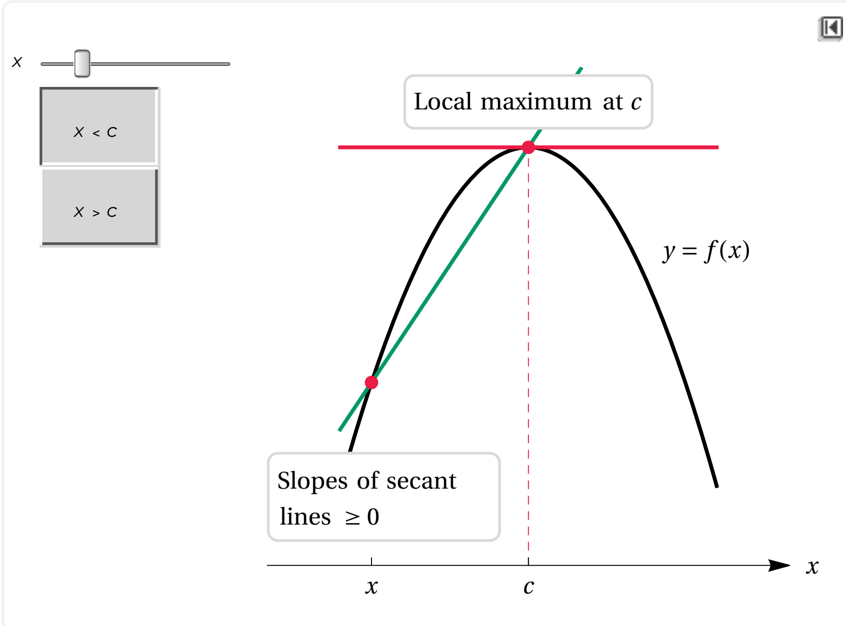


Figure 4.7

These observations imply that the slope of the tangent line must be both nonnegative and nonpositive, which happens only if $f'(c) = 0$. Similar reasoning leads to the same conclusion for a function with a local minimum at c: if $f'(c)$ exists, then $f'(c)$ must be zero. This argument is an outline of the proof (Exercise 85) of the following theorem.

THEOREM 4.2 Local Extreme Point Theorem

If f has a local minimum or maximum at c and $f'(c)$ exists, then $f'(c) = 0$.

Note »

Theorem 4.2, often attributed to Fermat, is one of the clearest examples in mathematics of a necessary, but not sufficient, condition. A local maximum (or minimum) at c necessarily implies a critical point at c, but a critical point at c is not sufficient to imply a local maximum (or minimum) exists there.

Local extrema can also occur at points c where $f'(c)$ does not exist. **Figure 4.8** shows two such cases, one in which c is a point of discontinuity and one in which f has a corner point at c.

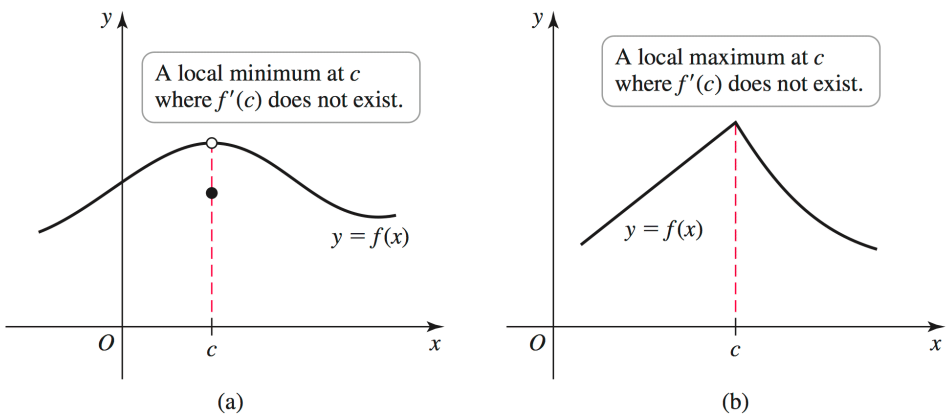


Figure 4.8

Because local extrema may occur at points c where $f'(c) = 0$ or where $f'(c)$ does not exist, we make the following definition.

> **DEFINITION** **Critical Point**
>
> An interior point c of the domain of f at which $f'(c) = 0$ or $f'(c)$ fails to exist is called a **critical point** of f.

Note that the converse of Theorem 4.2 is not necessarily true. It is possible that $f'(c) = 0$ at a point without a local maximum or local minimum value occurring there (**Figure 4.9a**). It is also possible that $f'(c)$ fails to exist, with no local extreme value occurring at c (**Figure 4.9b**). Therefore, critical points are *candidates* for local extreme points, but you must determine whether they actually correspond to local maxima or minima. This procedure is discussed in Section 4.3.

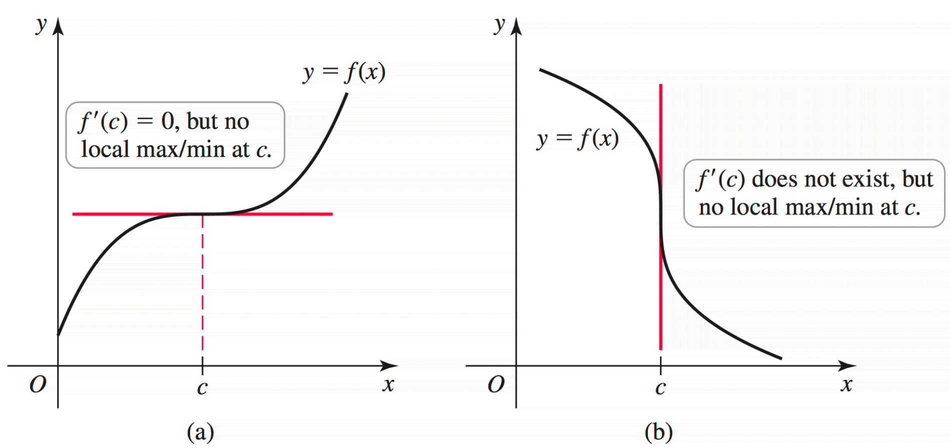


Figure 4.9

EXAMPLE 3 Locating critical points

Find the critical points of $f(x) = \dfrac{x}{x^2 + 1}$.

SOLUTION

Note that f is differentiable on its domain, which is $(-\infty, \infty)$. By the Quotient Rule,

$$f'(x) = \frac{(x^2+1) - 2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}.$$

Setting $f'(x) = 0$ and noting that $x^2 + 1 > 0$ for all x, the critical points satisfy the equation $1 - x^2 = 0$. Therefore, the critical points are $x = 1$ and $x = -1$. The graph of f (**Figure 4.10**) shows that f has a local (and absolute) maximum at $\left(1, \frac{1}{2}\right)$ and a local (and absolute) minimum at $\left(-1, -\frac{1}{2}\right)$.

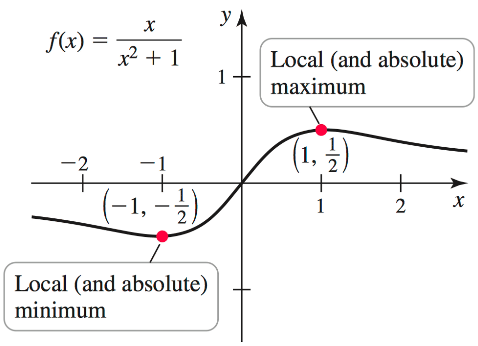


Figure 4.10

Related Exercises 30–31 ◆

Quick Check 2 Consider the function $f(x) = x^3$. Where is the critical point of f? Does f have a local maximum or minimum at the critical point? ◆

Answer »

> The critical point is $x = 0$. Although $f'(0) = 0$, the function has neither a local minimum nor maximum at $x = 0$.

Locating Absolute Maxima and Minima »

Theorem 4.1 guarantees the existence of absolute extreme values of a continuous function on a closed interval $[a, b]$, but it doesn't say where these values are located. Two observations lead to a procedure for locating absolute extreme values.

- An absolute extreme value in the interior of an interval is also a local extreme value, and we know that local extreme values occur at the critical points of f.
- Absolute extreme values may also occur at the endpoints of the interval of interest.

These two facts suggest the following procedure for locating the absolute extreme values of a continuous function on a closed interval.

PROCEDURE **Locating Absolute Maximum and Minimum Values**

Assume the function f is continuous on the closed interval $[a, b]$.

1. Locate the critical points c in (a, b), where $f'(c) = 0$ or $f'(c)$ does not exist. These points are candidates for absolute maxima and minima.

2. Evaluate f at the critical points and at the endpoints of $[a, b]$.

3. Choose the largest and smallest values of f from Step 2 for the absolute maximum and minimum values, respectively.

Note that the preceding procedure box does not address the case in which f is continuous on an open interval. If the interval of interest is an open interval, then absolute extreme values—if they exist—occur at interior points.

EXAMPLE 4 Absolute extreme values

Find the absolute maximum and minimum values of the following functions.

a. $f(x) = x^4 - 2x^3$ on the interval $[-2, 2]$

b. $g(x) = x^{2/3}(2 - x)$ on the interval $[-1, 2]$

SOLUTION »

a. Because f is a polynomial, its derivative exists everywhere. So, if f has critical points, they are points at which $f'(x) = 0$. Computing f' and setting it equal to zero, we have

$$f'(x) = 4x^3 - 6x^2 = 2x^2(2x - 3) = 0.$$

Solving this equation gives the critical points $x = 0$ and $x = \frac{3}{2}$, both of which lie in the interval $[-2, 2]$; these points and the endpoints are *candidates* for the locations of absolute extrema. Evaluating f at each of these points, we have

$$f(-2) = 32, \quad f(0) = 0, \quad f\left(\frac{3}{2}\right) = -\frac{27}{16}, \quad \text{and} \quad f(2) = 0.$$

The largest of these function values is $f(-2) = 32$, which is the absolute maximum of f on $[-2, 2]$. The smallest of these values is $f\left(\frac{3}{2}\right) = -\frac{27}{16}$, which is the absolute minimum of f on $[-2, 2]$. The graph of f (**Figure 4.11**) shows that the critical point $x = 0$ corresponds to neither a local maximum nor a local minimum.

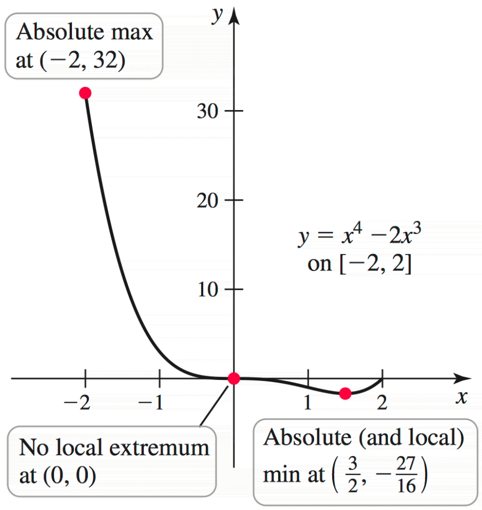


Figure 4.11

b. Differentiating $g(x) = x^{2/3}(2 - x) = 2\,x^{2/3} - x^{5/3}$, we have

$$g\,'(x) = \frac{4}{3}\,x^{-1/3} - \frac{5}{3}\,x^{2/3} = \frac{4 - 5\,x}{3\,\sqrt[3]{x}}.$$

Because $g\,'(0)$ is undefined and 0 is in the domain of g, $x = 0$ is a critical point. In addition, $g\,'(x) = 0$ when $4 - 5\,x = 0$, so $x = \dfrac{4}{5}$ is also a critical point. These two critical points and the endpoints are *candidates* for the location of absolute extrema. The next step is to evaluate g at the critical points and endpoints:

$$g(-1) = 3, \quad g(0) = 0, \quad g\left(\frac{4}{5}\right) \approx 1.03, \quad \text{and} \quad g(2) = 0.$$

The largest of these function values is $g(-1) = 3$, which is the absolute maximum value of g on $[-1, 2]$. The least of these values is 0, which occurs twice. Therefore, g has its absolute minimum value on $[-1, 2]$ at the critical point $x = 0$ and the endpoint $x = 2$ (**Figure 4.12**).

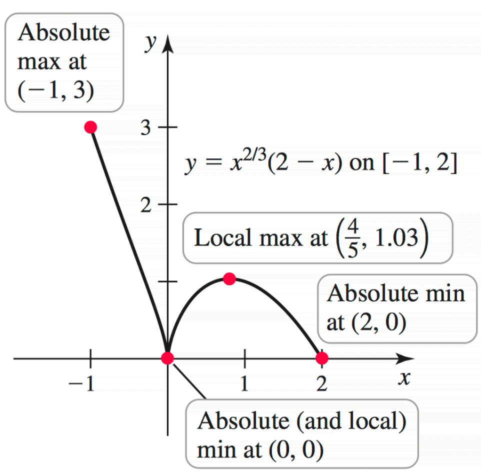


Figure 4.12

Related Exercises 44, 50 ◆

We now apply these ideas to a practical situation.

EXAMPLE 5 Trajectory high point

A stone is launched vertically upward from a bridge 80 ft above the ground at a speed of 64 ft/s. Its height above the ground t seconds after the launch is given by

$$f(t) = -16t^2 + 64t + 80, \text{ for } 0 \le t \le 5.$$

When does the stone reach its maximum height?

Note »

SOLUTION »

We must evaluate the height function at the critical points and at the endpoints. The critical points satisfy the equation

$$f'(t) = -32t + 64 = -32(t-2) = 0,$$

so the only critical point is $t = 2$. We now evaluate f at the endpoints and at the critical point:

$$f(0) = 80, \quad f(2) = 144, \quad \text{and} \quad f(5) = 0.$$

On the interval $[0, 5]$, the absolute maximum occurs at $t = 2$, at which time the stone reaches a height of 144 ft. Because $f'(t)$ is the velocity of the stone, the maximum height occurs at the instant the velocity is zero.

Related Exercise 67 ◆

Exercises »

Getting Started »

Practice Exercises »

T **23–42. Locating critical points** *Find the critical points of the following functions. Assume a is a nonzero constant.*

23. $f(x) = 3x^2 - 4x + 2$

24. $f(x) = \frac{1}{8}x^3 - \frac{1}{2}x$

25. $f(x) = \frac{x^3}{3} - 9x$

26. $f(x) = \frac{x^4}{4} - \frac{x^3}{3} - 3x^2 + 10$

27. $f(x) = 3x^3 + \frac{3x^2}{2} - 2x$

28. $f(x) = \frac{4x^5}{5} - 3x^3 + 5$

29. $f(x) = x^3 - 4\,a^2\,x$

30. $f(x) = \dfrac{(x+1)^2}{x^2+1}$

31. $f(t) = \dfrac{4\,t}{4\,t^3+1}$

32. $f(x) = 12\,x^5 - 20\,x^3$

33. $f(x) = 2\sqrt{x} - x$ on $[0, 4]$

34. $f(x) = \sin x \cos x$

35. $f(x) = x^2\sqrt{x+5}$

36. $f(x) = (x-6)\sqrt{x}$ on $[0, 4]$

37. $f(x) = x\sqrt{x-a}$

38. $f(x) = \dfrac{x}{\sqrt{x-a}}$

39. $f(t) = \dfrac{1}{5}\,t^5 - a^4\,t$

40. $f(x) = x^3 - 3\,a\,x^2 + 3\,a^2\,x - a^3$

41–62. Absolute maxima and minima *Determine the location and value of the absolute extreme values of* f *on the given interval if they exist.*

41. $f(x) = x^2 - 10$ on $[-2, 3]$

42. $f(x) = (x+1)^{4/3}$ on $[-9, 7]$

43. $f(x) = x^3 - 3\,x^2$ on $[-1, 3]$

44. $f(x) = x^4 - 4\,x^3 + 4\,x^2$ on $[-1, 3]$

45. $f(x) = 3\,x^5 - 25\,x^3 + 60\,x$ on $[-2, 3]$

46. $f(x) = \sin 2\,x - \sqrt{3}\,x$ on $[0, \pi]$

47. $f(x) = \cos^2 x$ on $[0, \pi]$

48. $f(x) = \dfrac{x}{\left(x^2+3\right)^2}$ on $[-2, 2]$

49. $f(x) = \sin 3\,x$ on $[-\pi/4, \pi/3]$

50. $f(x) = 3\,x^{2/3} - x$ on $[0, 27]$

51. $f(x) = x + \sin 2x$ on $[0, \pi/2]$

52. $f(x) = x\sqrt{2 - x^2}$ on $\left[-\sqrt{2}, \sqrt{2}\right]$

53. $f(x) = 2x^3 - 15x^2 + 24x$ on $[0, 5]$

54. $f(x) = 4x^3 - 21x^2 + 36x$ on $[1, 3]$

55. $f(x) = \dfrac{4x^3}{3} + 5x^2 - 6x$ on $[-4, 1]$

56. $f(x) = 2x^6 - 15x^4 + 24x^2$ on $[-2, 2]$

57. $f(x) = \dfrac{x}{\left(x^2 + 9\right)^5}$ on $[-2, 2]$

58. $f(x) = x^{1/2}\left(\dfrac{x^2}{5} - 4\right)$ on $[0, 4]$

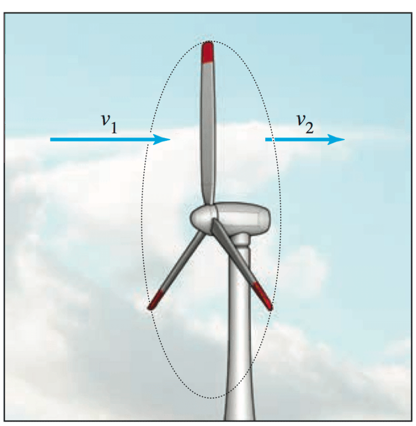
59. $f(x) = \sec x$ on $\left[-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right]$

60. $f(x) = x^{1/3}(x + 4)$ on $[-27, 27]$

61. $f(x) = x^{2/3}\left(4 - x^2\right)$ on $[-2, 2]$

62. $f(t) = \dfrac{3t}{t^2 + 1}$ on $[-2, 2]$

63. **Efficiency of wind turbines** A wind turbine takes wind energy and converts it into electrical power. Let v_1 equal the upstream velocity of the wind before it encounters the wind turbine and let v_2 equal the downstream velocity of the wind after it passes through the area swept out by the turbine blades.



a. Assuming that $v_1 > 0$, give a physical explanation to show that $0 \le \dfrac{v_2}{v_1} \le 1$.

b. The amount of power extracted from the wind depends on the ratio $r = \frac{v_2}{v_1}$, the ratio of the downstream velocity to upstream velocity. Let $R(r)$ equal the fraction of power that is extracted from the total available power in the wind stream, for a given value of r. In about 1920, German Physicist Albert Betz showed that $R(r) = \frac{1}{2}(1+r)\left(1-r^2\right)$, where $0 \le r \le 1$ (a derivation of R is outlined in Exercise 64). Calculate $R(1)$ and explain how you could have arrived at this value without using the formula for R. Give a physical explanation of why it is unlikely or impossible for it to be the case that $r = 1$.

c. Calculate $R(0)$ and give a physical explanation of why it is unlikely or impossible for it to be the case that $r = 0$.

d. The maximum value of R is called the *Betz limit.* It represents the theoretical maximum amount of power that can be extracted from the wind. Find this value and explain its physical meaning.

64. Derivation of wind turbine formula A derivation of the function R in Exercise 63, based upon three equations from physics, is outlined here. Consider again the figure given in Exercise 63, where v_1 equals the upstream velocity of the wind just before the wind stream encounters the wind turbine, and v_2 equals the downstream velocity of the wind just after the wind stream passes through the area swept out by the turbine blades. An equation for the power extracted by the rotor blades, based upon conservation of momentum, is $P = v^2 \rho A (v_1 - v_2)$, where v is the velocity of the wind (in m/s) as it passes through the turbine blades, ρ is the density of air (in kg/m^3), and A is the area (in m^2) of the circular region swept out by the rotor blades.

a. Another expression for the power extracted by the rotor blades, based upon conservation of energy, is $P = \frac{1}{2}\rho v A\left(v_1{}^2 - v_2{}^2\right)$. Equate the two power equations and solve for v.

b. Show that $P = \frac{\rho A}{4}(v_1 + v_2)\left(v_1{}^2 - v_2{}^2\right)$

c. If the wind were to pass through the same area A without being disturbed by rotor blades, the amount of available power would be $P_0 = \frac{\rho A v_1{}^3}{2}$. Let $r = \frac{v_2}{v_1}$ and simplify the ratio $\frac{P}{P_0}$ to obtain the function $R(r)$ given in Exercise 63. (*Source: Journal of Applied Physics*, 105, 2009)

65. Maximum distance from origin Suppose the position of an object moving horizontally after t seconds is given by the function $s(t) = 32t - t^4$, where $0 \le t \le 3$ and s is measured in feet, with $s > 0$ corresponding to positions to the right of the origin. When is the object furthest to the right?

66. Minimum-surface-area box All boxes with a square base and a volume of 50 ft^3 have a surface area given by $S(x) = 2x^2 + \frac{200}{x}$, where x is the length of the sides of the base. Find the absolute minimum of the surface area function on the interval $(0, \infty)$. What are the dimensions of the box with minimum surface area?

67. Trajectory high point A stone is launched vertically upward from a cliff 192 ft above the ground at a speed of 64 ft/s. Its height above the ground t seconds after the launch is given by $s = -16t^2 + 64t + 192$, for $0 \le t \le 6$. When does the stone reach its maximum height?

68. Maximizing revenue A sales analyst determines that the revenue from sales of fruit smoothies is given by $R(x) = -60x^2 + 300x$, where x is the price in dollars charged per item, for $0 \le x \le 5$.

a. Find the critical points of the revenue function.

b. Determine the absolute maximum value of the revenue function and give the price that maximizes the revenue.

69. Maximizing profit Suppose a tour guide has a bus that holds a maximum of 100 people. Assume his profit (in dollars) for taking n people on a city tour is $P(n) = n(50 - 0.5n) - 100$. (Although P is defined only for positive integers, treat it as a continuous function.)

a. How many people should the guide take on a tour to maximize the profit?

b. Suppose the bus holds a maximum of 45 people. How many people should be taken on a tour to maximize the profit?

70. Minimizing rectangle perimeters All rectangles with an area of 64 have a perimeter given by $P(x) = 2x + \frac{128}{x}$, where x is the length of one side of the rectangle. Find the absolute minimum value of the perimeter function on the interval $(0, \infty)$. What are the dimensions of the rectangle with minimum perimeter?

71. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. The function $f(x) = \sqrt{x}$ has a local maximum on the interval $[0, \infty)$.

b. If a function has an absolute maximum on a closed interval, then the function must be continuous on that interval.

c. A function f has the property that $f'(2) = 0$. Therefore, f has a local extreme value at $x = 2$.

d. Absolute extreme values of a function on a closed interval always occur at a critical point or an endpoint of the interval.

Explorations and Challenges »

T 72–73. Absolute maxima and minima

a. *Find the critical points of f on the given interval.*

b. *Determine the absolute extreme values of f on the given interval. Use a graphing utility to confirm your conclusions.*

72. $f(x) = \dfrac{x}{\sqrt{x-4}}$ on $[6, 12]$

73. $f(x) = |2x - x^2|$ on $[-2, 3]$

T 74–77. Critical points and extreme values

a. *Find the critical points of the following functions on the given interval. Use a root finder, if necessary.*

b. *Use a graphing utility to determine whether the critical points correspond to local maxima, local minima, or neither.*

c. *Find the absolute maximum and minimum values on the given interval, if they exist.*

74. $f(x) = 6x^4 - 16x^3 - 45x^2 + 54x + 23$ on $[-5, 5]$

75. $f(\theta) = 2\sin\theta + \cos\theta$ on $[-2\pi, 2\pi]$

76. $g(x) = (x-3)^{5/3}(x+2)$ on $[-4, 4]$

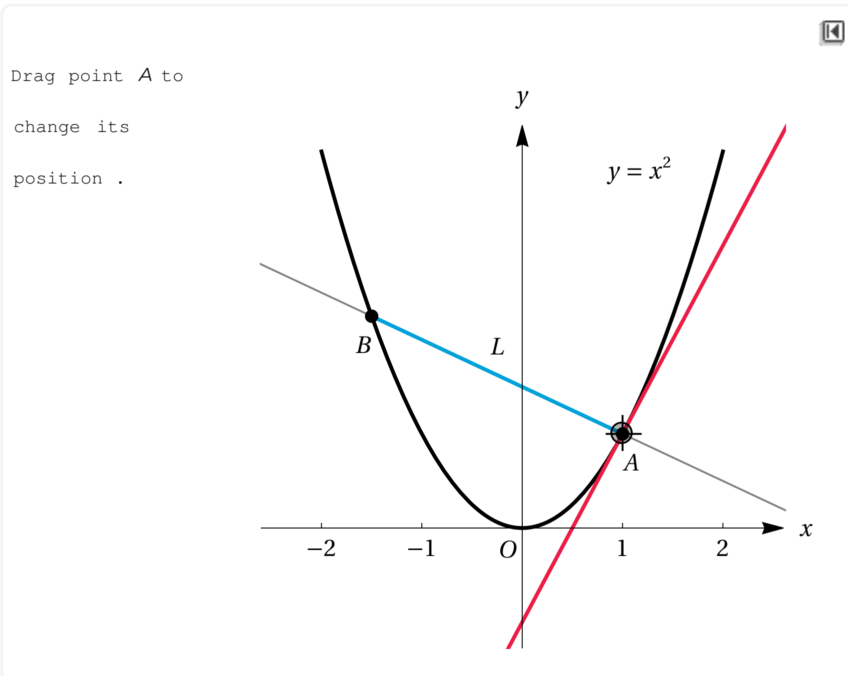
77. $h(x) = \dfrac{5 - x}{x^2 + 2x - 3}$ on $[-10, 10]$

T 78–79. Absolute value functions *Graph the following functions and determine the local and absolute extreme values on the given interval.*

78. $f(x) = |x - 3| + |x + 2|$ on $[-4, 4]$

79. $g(x) = |x - 3| - 2|x + 1|$ on $[-2, 3]$

T 80. Dancing on a parabola Two people, A and B, walk along the parabola $y = x^2$ in such a way that the line segment L between them is always perpendicular to the line tangent to the parabola at A's position. The goal of this exercise is to determine the positions of A and B when L has minimum length. Assume the coordinates of A are (a, a^2).



a. Find the slope of the line tangent to the parabola at A and find the slope of the line that is perpendicular to the tangent line at A.

b. Find the equation of the line joining A and B.

c. Find the position of B on the parabola.

d. Write the function $F(a)$ that gives the *square* of the distance between A and B as it varies with a. (The square of the distance is minimized at the same point that the distance is minimized; it is easier to work with the square of the distance.)

e. Find the critical point of F on the interval $a > 0$.

f. Evaluate F at the critical point and verify that it corresponds to an absolute minimum. What are the positions of A and B that minimize the length of L? What is the minimum length?

g. Graph the function F to check your work.

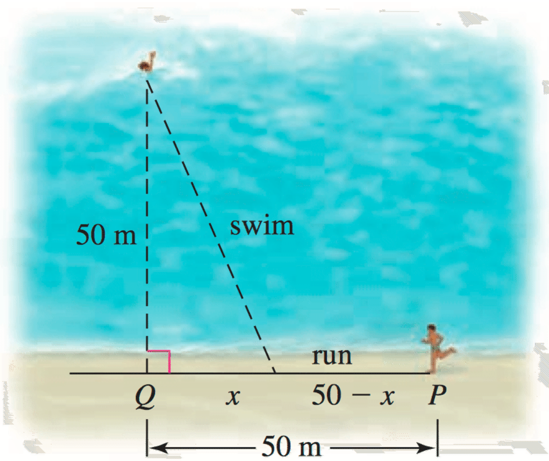
T 81. **Every second counts** You must get from a point P on the straight shore of a lake to a stranded swimmer who is 50 m from a point Q on the shore that is 50 m from you (see figure). Assuming that you can swim at a speed of 2 m/s and run at a speed of 4 m/s, the goal of this exercise is to determine the point along the shore, x meters from Q, where you should you stop running and start swimming to reach the swimmer in the minimum time.

a. Find the function T that gives the travel time as a function of x, where $0 \le x \le 50$.

b. Find the critical point of T on $(0, 50)$.

c. Evaluate T at the critical point and the endpoints ($x = 0$ and $x = 50$) to verify that the critical point corresponds to an absolute minimum. What is the minimum travel time?

d. Graph the function T to check your work.



82. **Extreme values of parabolas** Consider the function $f(x) = a x^2 + b x + c$, with $a \ne 0$. Explain geometrically why f has exactly one absolute extreme value on $(-\infty, \infty)$. Find the critical point to determine the value of x at which f has an extreme value.

83. **Values of related functions** Suppose f is differentiable on $(-\infty, \infty)$ and assume it has a local extreme value at the point $x = 2$, where $f(2) = 0$. Let $g(x) = x f(x) + 1$ and let $h(x) = x f(x) + x + 1$, for all values of x.

a. Evaluate $g(2)$, $h(2)$, $g'(2)$, and $h'(2)$.

b. Does either g or h have a local extreme value at $x = 2$? Explain.

T 84. **A family of double-humped functions** Consider the functions $f(x) = \dfrac{x}{\left(x^2+1\right)^n}$, where n is a positive integer.

a. Show that these functions are odd for all positive integers n.

b. Show that the critical points of these functions are $x = \pm \dfrac{1}{\sqrt{2n-1}}$, for all positive integers n.

(Start with the special cases $n = 1$ and $n = 2$.)

c. Show that as n increases the absolute maximum values of these functions decrease.

d. Use a graphing utility to verify your conclusions.

85. **Proof of the Local Extreme Value Theorem** Prove Theorem 4.2 for a local maximum: If f has a local maximum value at the point c and $f'(c)$ exists, then $f'(c) = 0$. Use the following steps.

a. Suppose f has a local maximum at c. What is the sign of $f(x) - f(c)$ if x is near c and $x > c$? What is the sign of $f(x) - f(c)$ if x is near c and $x < c$?

b. If $f'(c)$ exists, then it is defined by $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$. Examine this limit as $x \to c^+$ and conclude that $f'(c) \le 0$.

c. Examine the limit in part (b) as $x \to c^-$ and conclude that $f'(c) \ge 0$.

d. Combine parts (b) and (c) to conclude that $f'(c) = 0$.

86. Even and odd functions

a. Suppose a nonconstant even function f has a local minimum at c. Does f have a local maximum or minimum at $-c$? Explain. (An even function satisfies $f(-x) = f(x)$.)

b. Suppose a nonconstant odd function f has a local minimum at c. Does f have a local maximum or minimum at $-c$? Explain. (An odd function satisfies $f(-x) = -f(x)$.)