3.9 Related Rates

We now return to the theme of derivatives as rates of change in problems in which the variables change with respect to *time*. The essential feature of these problems is that two or more variables, which are related in a known way, are themselves changing in time. Here are two examples illustrating this type of problem.

- An oil rig springs a leak and the oil spreads in a (roughly) circular patch around the rig. If the radius of the oil patch increases at a known rate, how fast is the area of the patch changing (Example 1)?
- Two airliners approach an airport with known speeds, one flying west and one flying north. How fast is the distance between the airliners changing (Example 2)?

In the first problem, the two related variables are the radius and the area of the oil patch. Both are changing in time. The second problem has three related variables: the positions of the two airliners and the distance between them. Again, the three variables change in time. The goal in both problems is to determine the rate of change of one of the variables at a specific moment of time—hence the name *related rates*.

We present a progression of examples in this section. After the first example, a general procedure is given for solving related-rate problems.

Examples »

EXAMPLE 1 **Spreading oil**

An oil rig springs a leak in calm seas and the oil spreads in a circular patch around the rig. If the radius of the oil patch increases at a rate of 30 m/hr, how fast is the area of the patch increasing when the patch has a radius of 100 m (**Figure 3.59**)?

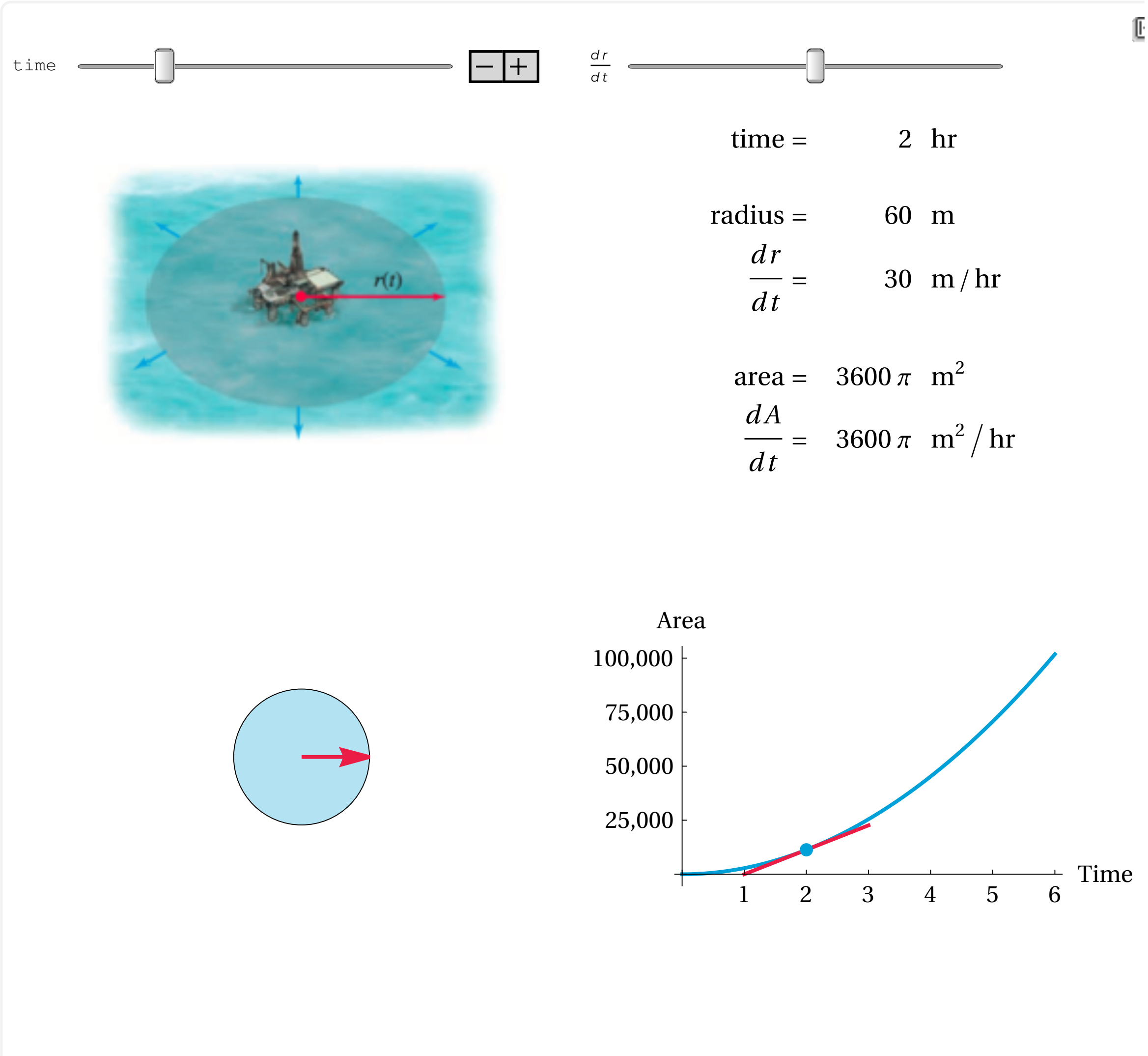


Figure 3.59

SOLUTION »

Two variables change simultaneously: the radius of the circle and its area. The key relationship between the radius and area is $A = \pi r^2$. It helps to rewrite the basic relationship showing explicitly which quantities vary in time. In this case, we rewrite A and r as $A(t)$ and $r(t)$ to emphasize that they change with respect to t (time). The general expression relating the radius and area at any time t is $A(t) = \pi r(t)^2$.

The goal is to find the rate of change of the area of the circle, which is $A'(t)$. In order to introduce derivatives into the problem, we differentiate the area relation $A(t) = \pi r(t)^2$ with respect to t:

$$\begin{aligned} A'(t) &= \frac{d}{dt}\left(\pi r(t)^2\right) \\ &= \pi \frac{d}{dt}\left(r(t)^2\right) \\ &= \pi (2 r(t)) r'(t) && \text{Chain Rule} \\ &= 2\pi r(t) r'(t). && \text{Simplify.} \end{aligned}$$

Substituting the given values $r(t) = 100$ m and $r'(t) = 30$ m/hr, we have (including units)

$$\begin{aligned} A'(t) &= 2\pi\, r(t)\, r'(t) \\ &= 2\pi\,(100\text{ m})\left(30\,\frac{\text{m}}{\text{hr}}\right) \\ &= 6000\,\pi\,\frac{\text{m}^2}{\text{hr}}. \end{aligned}$$

Note »

> It is important to remember that substitution of specific values of the variables occurs *after* differentiating.

We see that the area of the oil spill increases at a rate of $6000\pi \approx 18{,}850\ \text{m}^2/\text{hr}$. Including units is a simple way to check your work. In this case, we expect an answer with units of area per unit time, so m^2/hr makes sense.

Notice that the rate of change of the area depends on the radius of the spill. As the radius increases, the rate of change of the area also increases.

Related Exercises 5, 15 ◆

Quick Check 1 In Example 1, what is the rate of change of the area when the radius is 200 m? 300 m? ◆

Answer »

> $12{,}000\,\pi\ \text{m}^2/\text{hr}$, $18{,}000\,\pi\ \text{m}^2/\text{hr}$

Using Example 1 as a template, we offer a set of guidelines for solving related-rate problems. There are always variations that arise for individual problems, but here is a general procedure.

PROCEDURE **Steps for Related-Rate Problems**

1. Read the problem carefully, making a sketch to organize the given information. Identify the rates that are given and the rate that is to be determined.

2. Write one or more equations that express the basic relationships among the variables.

3. Introduce rates of change by differentiating the appropriate equation(s) with respect to time t.

4. Substitute known values and solve for the desired quantity.

5. Check that units are consistent and the answer is reasonable. (For example, does it have the correct sign?)

EXAMPLE 2 Converging airplanes

Two small planes approach an airport, one flying due west at 120 mi/hr and the other flying due north at 150 mi/hr. Assuming they fly at the same constant elevation, how fast is the distance between the planes changing when the westbound plane is 180 mi from the airport and the northbound plane is 225 mi from the airport?

SOLUTION »

A sketch such as **Figure 3.60** helps us visualize the problem and organize the information. Let $x(t)$ and $y(t)$ denote the distance from the airport to the westbound and northbound planes, respectively. The paths of the two planes form the legs of a right triangle and the distance between them, denoted $z(t)$, is the hypotenuse. By the Pythagorean theorem, $z^2 = x^2 + y^2$.

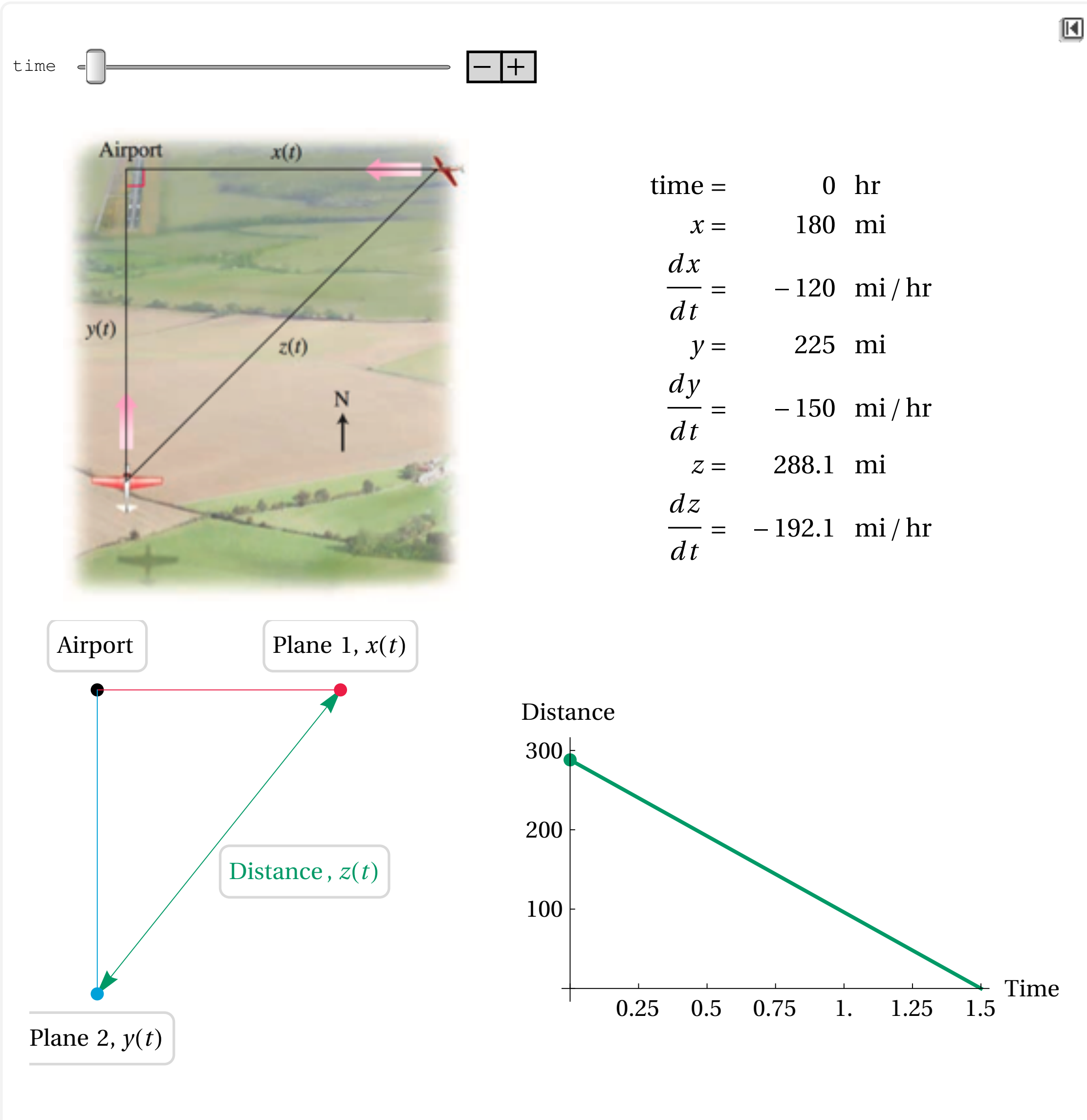


Figure 3.60

Our aim is to find $\dfrac{dz}{dt}$, the rate of change of the distance between the planes. We first differentiate both sides of $z^2 = x^2 + y^2$ with respect to t:

$$\frac{d}{dt}\left(z^2\right) = \frac{d}{dt}\left(x^2 + y^2\right) \Rightarrow 2\,z\,\frac{dz}{dt} = 2\,x\,\frac{dx}{dt} + 2\,y\,\frac{dy}{dt}.$$

Note »

In Example 1, we replaced A and r by $A(t)$ and $r(t)$, respectively, to remind us of the independent variable. After some practice, this replacement is not necessary.

Notice that the Chain Rule is needed because x, y, and z are functions of t. Solving for $\frac{dz}{dt}$ results in

$$\frac{dz}{dt} = \frac{2x\frac{dx}{dt} + 2y\frac{dy}{dt}}{2z} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{z}.$$

Note »

One could solve the equation $z^2 = x^2 + y^2$ for z, with the result $z = \sqrt{x^2 + y^2}$, and then differentiate. However, it is much easier to differentiate implicitly as shown in the example.

This equation relates the unknown rate $\frac{dz}{dt}$ to the known quantities x, y, z, $\frac{dx}{dt}$, and $\frac{dy}{dt}$. For the westbound plane $\frac{dx}{dt} = -120$ mi/hr (negative because the distance is decreasing), and for the northbound plane $\frac{dy}{dt} = -150$ mi/hr. At the moment of interest, when $x = 180$ mi and $y = 225$ mi, the distance between the planes is

$$z = \sqrt{x^2 + y^2} = \sqrt{180^2 + 225^2} \approx 288 \text{ mi}.$$

Substituting these values gives

$$\begin{aligned} \frac{dz}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{z} &\approx \frac{(180 \text{ mi})(-120 \text{ mi/hr}) + (225 \text{ mi})(-150 \text{ mi/hr})}{288 \text{ mi}} \\ &\approx -192 \text{ mi/hr}. \end{aligned}$$

Notice that $\frac{dz}{dt} < 0$, which means the distance between the planes is *decreasing* at a rate of about 192 mi/hr.

Related Exercises 22–23 ◆

Quick Check 2 Assuming the same plane speeds as in Example 2, how fast is the distance between the planes changing if $x = 60$ mi and $y = 75$ mi? ◆

Answer »

-192 mi/hr

EXAMPLE 3 Morning coffee

Coffee is draining out of a conical filter at a rate of 2.25 in^3/min. If the cone is 5 in tall and has a radius of 2 in, how fast is the coffee level dropping when the coffee is 3 in deep?

SOLUTION »

A sketch of the problem (**Figure 3.61a**) shows the three relevant variables: the volume V, the radius r, and the height h of the coffee in the filter. The aim is to find the rate of change of the height dh/dt at the instant that $h = 3$ in, given that $dV/dt = -2.25 \text{ in}^3/\text{min}$.

Note »

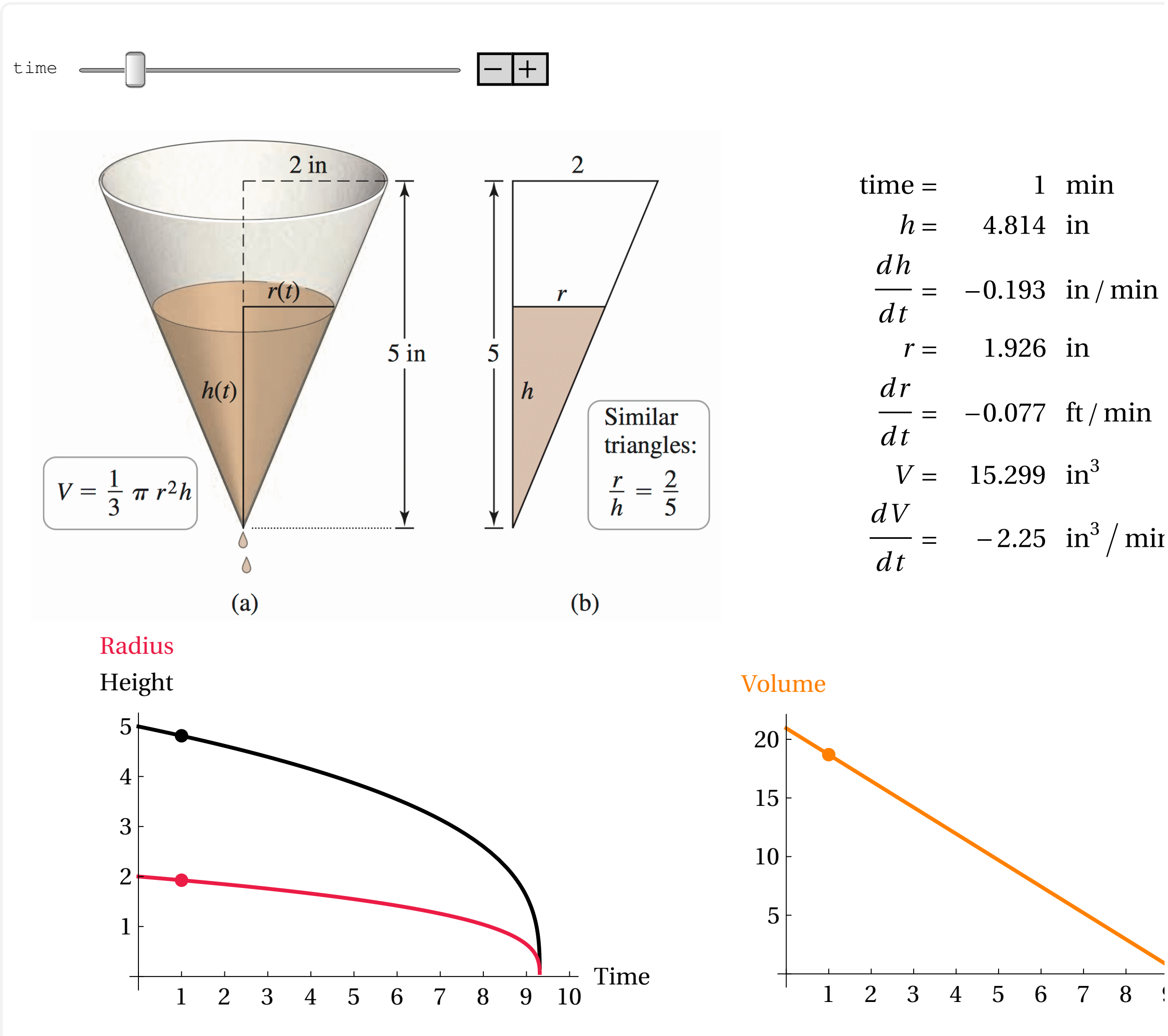


Figure 3.61

The volume formula for a cone, $V = \frac{1}{3}\pi r^2 h$, expresses the basic relationship among the relevant variables. Using similar triangles (**Figure 3.61b**), we see that the ratio of the radius of the coffee in the cone to the height of the coffee is $2/5$ at all times; that is,

$$\frac{r}{h} = \frac{2}{5} \quad \text{or} \quad r = \frac{2}{5} h.$$

Substituting $r = \frac{2}{5} h$ into the volume formula gives V in terms of h:

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\left(\frac{2}{5} h\right)^2 h = \frac{4\pi}{75} h^3.$$

Rates of change are introduced by differentiating both sides of $V = \frac{4\pi}{75} h^3$ with respect to t:

$$\frac{dV}{dt} = \frac{4\pi}{25} h^2 \frac{dh}{dt}.$$

Now we find dh/dt at the instant when $h = 3$, given that $dV/dt = -2.25\ \text{in}^3/\text{min}$. Solving for dh/dt and substituting these values, we have

$$\frac{dh}{dt} = \frac{25}{4\pi h^2}\frac{dV}{dt} \qquad \text{Solve for } \frac{dh}{dt}.$$

$$= \frac{25}{4\pi(3\ \text{in})^2}\left(-2.25\ \frac{\text{in}^3}{\text{min}}\right) \approx -0.497\ \frac{\text{in}}{\text{min}}. \quad \text{Substitute for } \frac{dV}{dt} \text{ and } h.$$

At the instant that the coffee is 3 inches deep, the height decreases at a rate of 0.497 in/min (almost half an inch per minute). Notice that the units work out consistently.

Related Exercises 36–37 ◆

Quick Check 3 In Example 3, what is the rate of change of the height when $h = 2$ in? ◆

Answer »

-1.12 in/min

EXAMPLE 4 Observing a launch

An observer stands 200 m from the launch site of a hot-air balloon at an elevation equal to the elevation of the launch site. The balloon rises vertically at a constant rate of 4 m/s. How fast is the angle of elevation of the balloon increasing 30 s after the launch? (The angle of elevation is the angle between the ground and the observer's line of sight to the balloon.)

SOLUTION »

Figure 3.62 shows the geometry of the launch. As the balloon rises, its distance from the ground y and its angle of elevation θ change simultaneously. An equation expressing the relationship between these variables is $\tan\theta = \frac{y}{200}$.

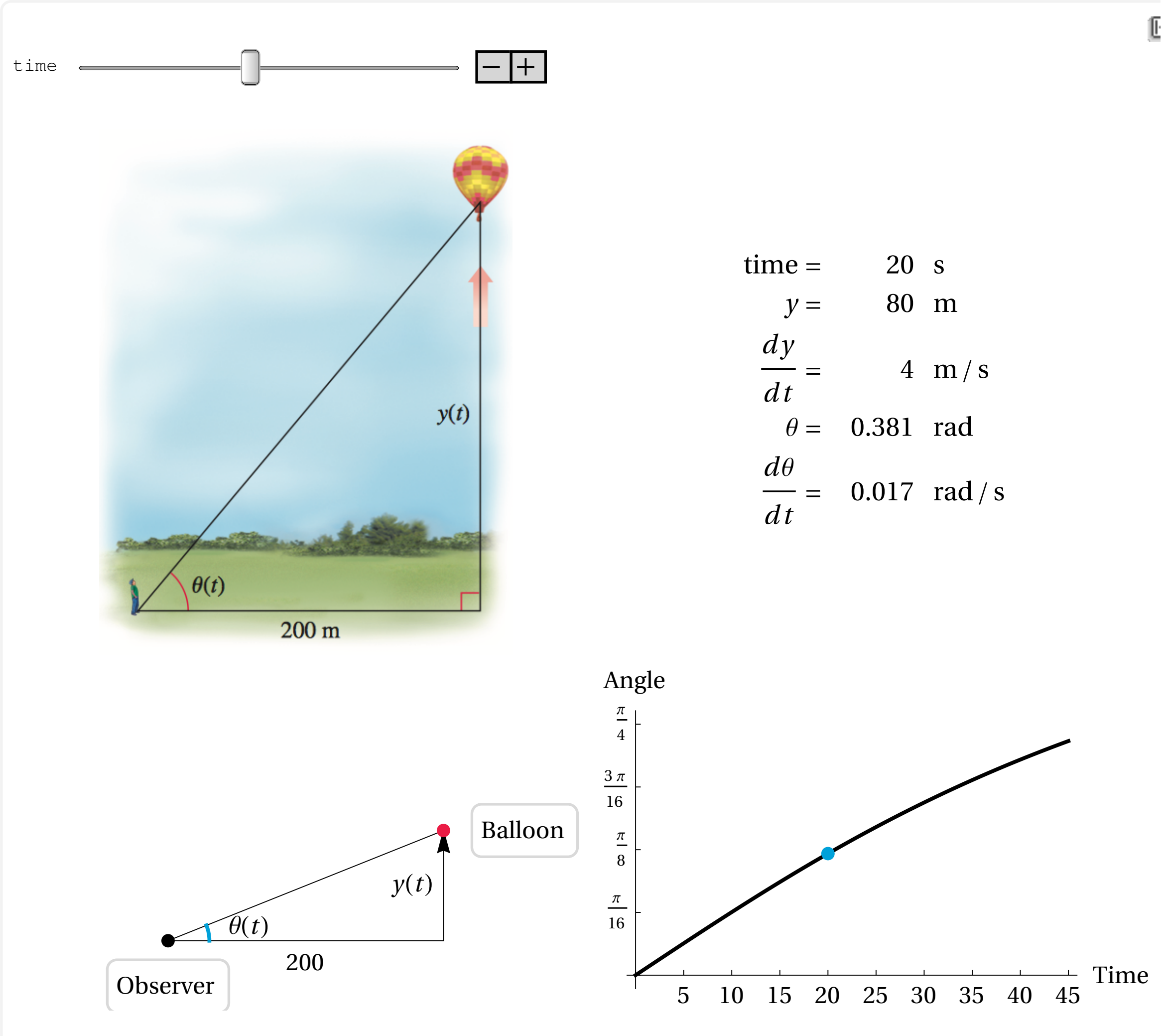


Figure 3.62

To find $\frac{d\theta}{dt}$, we differentiate both sides of this relationship using the Chain Rule:

$$\sec^2 \theta \frac{d\theta}{dt} = \frac{1}{200} \frac{dy}{dt}.$$

Next we solve for $\frac{d\theta}{dt}$:

$$\frac{d\theta}{dt} = \frac{\frac{dy}{dt}}{200 \sec^2 \theta} = \frac{\left(\frac{dy}{dt}\right) \cdot \cos^2 \theta}{200}.$$

The rate of change of the angle of elevation depends on the angle of elevation and the speed of the balloon. Thirty seconds after the launch, the balloon has risen $y = (4 \text{ m/s})(30 \text{ s}) = 120$ m. To complete the problem, we need the value of $\cos \theta$. Note that when $y = 120$ m, the distance between the observer and the balloon is

$$d = \sqrt{120^2 + 200^2} \approx 233.24 \text{ m}.$$

Therefore, $\cos\theta \approx \dfrac{200}{233.24} \approx 0.86$ (**Figure 3.63**), and the rate of change of the angle of elevation is

$$\frac{d\theta}{dt} = \frac{(dy/dt)\cdot\cos^2\theta}{200} \approx \frac{(4 \text{ m/s})\left(0.86^2\right)}{200 \text{ m}} = 0.015 \text{ rad/s}.$$

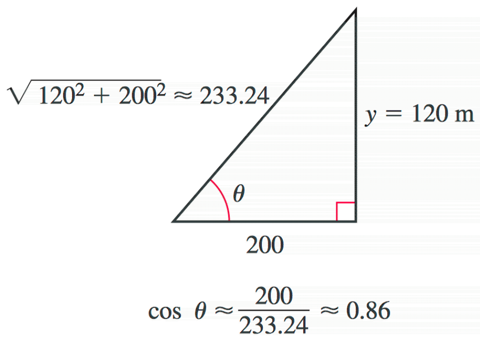


Figure 3.63

Note »

The solution to Example 4 is reported in units of rad/s. Where did radians come from? Because a radian has no physical dimensions (it is the ratio of an arc length and a radius), no unit appears. We write rad/s for clarity because $\dfrac{d\theta}{dt}$ is the rate of change of an angle.

At this instant, the balloon is rising at an angular rate of 0.015 rad/s, or slightly less than $1°/s$, as seen by the observer.

Note »

Recall that to convert radians to degrees, we use

$$\text{degrees} = \frac{180}{\pi}\cdot\text{radians}.$$

Related Exercise 46 ◆

Quick Check 4 In Example 4, notice that as the balloon rises (as θ increases), the rate of change of the angle of elevation decreases to zero. When does the maximum value of $\theta'(t)$ occur and what is it? ◆

Answer »

$t = 0, \theta = 0, \theta'(0) = 0.02$ rad/s

Exercises »

Getting Started »

Practice Exercises »

11. Expanding square The sides of a square increase in length at a rate of 2 m/s.

a. At what rate is the area of the square changing when the sides are 10 m long?

b. At what rate is the area of the square changing when the sides are 20 m long?

12. **Shrinking square** The sides of a square decrease in length at a rate of 1 m/s.

a. At what rate is the area of the square changing when the sides are 5 m long?

b. At what rate are the lengths of the diagonals of the square changing?

13. **Expanding isosceles triangle** The legs of an isosceles right triangle increase in length at a rate of 2 m/s.

a. At what rate is the area of the triangle changing when the legs are 2 m long?

b. At what rate is the area of the triangle changing when the hypotenuse is 1 m long?

c. At what rate is the length of the hypotenuse changing?

14. **Shrinking isosceles triangle** The hypotenuse of an isosceles right triangle decreases in length at a rate of 4 m/s.

a. At what rate is the area of the triangle changing when the legs are 5 m long?

b. At what rate are the lengths of the legs of the triangle changing?

c. At what rate is the area of the triangle changing when the area is 4 m^2?

15. **Expanding circle** The area of a circle increases at a rate of 1 cm^2/s.

a. How fast is the radius changing when the radius is 2 cm?

b. How fast is the radius changing when the circumference is 2 cm?

16. **Expanding cube** The edges of a cube increase at a rate of 2 cm/s. How fast is the volume changing when the length of each edge is 50 cm?

17. **Shrinking circle** A circle has an initial radius of 50 ft when the radius begins decreasing at a rate of 2 ft/min. What is the rate of change of the area at the instant the radius is 10 ft?

18. **Shrinking cube** The volume of a cube decreases at a rate of 0.5 ft^3/min. What is the rate of change of the side length when the side lengths are 12 ft?

19. **Balloons** A spherical balloon is inflated and its volume increases at a rate of 15 in^3/min. What is the rate of change of its radius when the radius is 10 in?

20. **Expanding rectangle** A rectangle initially has dimensions 2 cm by 4 cm. All sides begin increasing in length at a rate of 1 cm/s. At what rate is the area of the rectangle increasing after 20 s?

21. **Melting snowball** A spherical snowball melts at a rate proportional to its surface area. Show that the rate of change of the radius is constant. (*Hint:* Surface area $= 4\pi r^2$.)

22. **Divergent paths** Two boats leave a port at the same time; one travels west at 20 mi/hr and the other travels south at 15 mi/hr.

a. After 30 minutes, how far is each boat from port?

b. At what rate is the distance between the boats changing 30 minutes after they leave the port?

23. **Time-lagged flights** An airliner passes over an airport at noon traveling 500 mi/hr due west. At 1:00 P.M., another airliner passes over the same airport at the same elevation traveling due north at 550 mi/hr. Assuming both airliners maintain their (equal) elevations, how fast is the distance between them changing at 2:30 P.M.?

24. Flying a kite Once Kate's kite reaches a height of 50 ft (above her hands), it rises no higher but drifts due east in a wind blowing 5 ft/s. How fast is the string running through Kate's hands at the moment that she has released 120 ft of string?

25. Rope on a boat A rope passing through a capstan on a dock is attached to a boat offshore. The rope is pulled in at a constant rate of 3 ft/s and the capstan is 5 ft vertically above the water. How fast is the boat traveling when it is 10 ft from the dock?

26. Bug on a parabola A bug is moving along the right side of the parabola $y = x^2$ at a rate such that its distance from the origin is increasing at 1 cm/min.

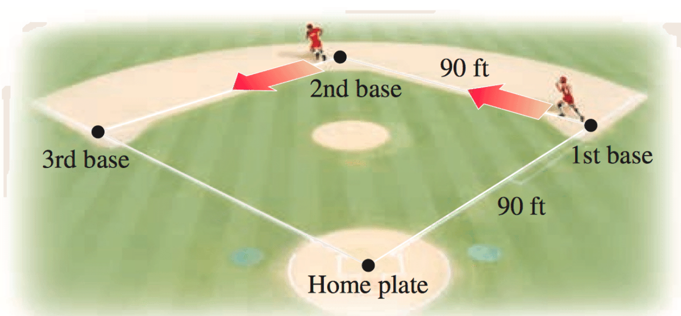
a. At what rate is the x-coordinate of the bug increasing at the point (2, 4)?

b. Use the equation $y = x^2$ to find an equation relating $\dfrac{dy}{dt}$ to $\dfrac{dx}{dt}$.

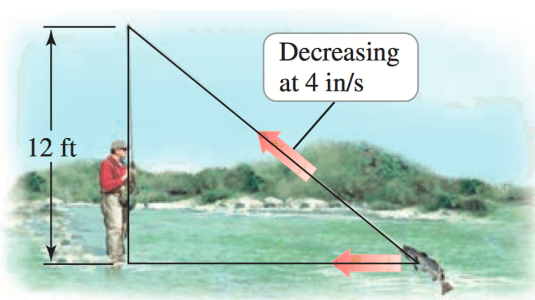
c. At what rate is the y-coordinate of the bug increasing at the point (2, 4)?

27. Balloons and motorcycles A hot-air balloon is 150 ft above the ground when a motorcycle (traveling in a straight line on a horizontal road) passes directly beneath it going 40 mi/hr (58.67 ft/s). If the balloon rises vertically at a rate of 10 ft/s, what is the rate of change of the distance between the motorcycle and the balloon 10 seconds later?

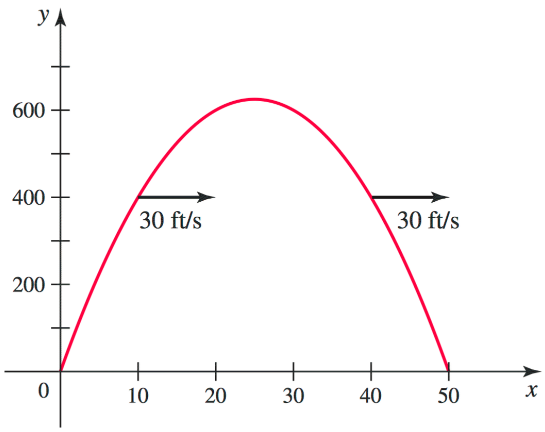
28. Baseball runners Runners stand at first and second base in a baseball game. At the moment a ball is hit, the runner at first base runs to second base at 18 ft/s; simultaneously the runner on second runs to third base at 20 ft/s. How fast is the distance between the runners changing 1 s after the ball is hit (see figure)? (*Hint:* The distance between consecutive bases is 90 ft and the bases lie at the corners of a square.)



29. Fishing story An angler hooks a trout and reels in his line at 4 in/s. Assume the tip of the fishing rod is 12 ft above the water and directly above the angler, and the fish is pulled horizontally directly towards the angler (see figure). Find the horizontal speed of the fish when it is 20 ft from the angler.



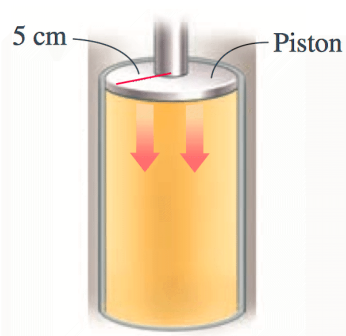
30. Parabolic motion An arrow is shot into the air and moves along the parabolic path $y = x(50 - x)$ (see figure). The horizontal component of velocity is always 30 ft/s. What is the vertical component of velocity when (a) $x = 10$ and (b) $x = 40$?



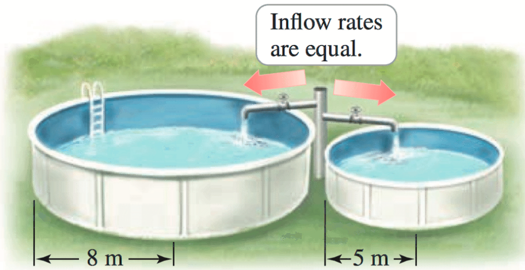
31. Draining a water heater A water heater that has the shape of a right cylindrical tank with a radius of 1 ft and a height of 4 ft is being drained. How fast is water draining out of the tank (in ft^3/min) if the water level is dropping at 6 in/min?

32. Drinking a soda At what rate (in in^3/s) is soda being sucked out of a cylindrical glass that is 6 in tall and has a radius of 2 in? The depth of the soda decreases at a constant rate of 0.25 in/s.

33. Piston compression A piston is seated at the top of a cylindrical chamber with radius 5 cm when it starts moving into the chamber at a constant speed of 3 cm/s (see figure). What is the rate of change of the volume of the cylinder when the piston is 2 cm from the base of the chamber?

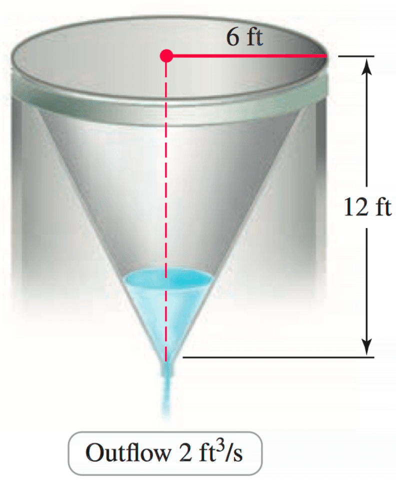


34. Filling two pools Two cylindrical swimming pools are being filled simultaneously at the same rate (in m^3/min; see figure). The smaller pool has a radius of 5 m, and the water level rises at a rate of 0.5 m/min. The larger pool has a radius of 8 m. How fast is the water level rising in the larger pool?



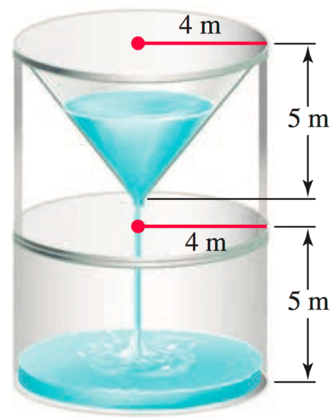
35. **Growing sandpile** Sand falls from an overhead bin and accumulates in a conical pile with a radius that is always three times its height. Suppose the height of the pile increases at a rate of 2 cm/s when the pile is 12 cm high. At what rate is the sand leaving the bin at that instant?

36. **Draining a tank** An inverted conical water tank with a height of 12 ft and a radius of 6 ft is drained through a hole in the vertex at a rate of $2\ \text{ft}^3/\text{s}$ (see figure). What is the rate of change of the water depth when the water depth is 3 ft? (*Hint:* Use similar triangles.)

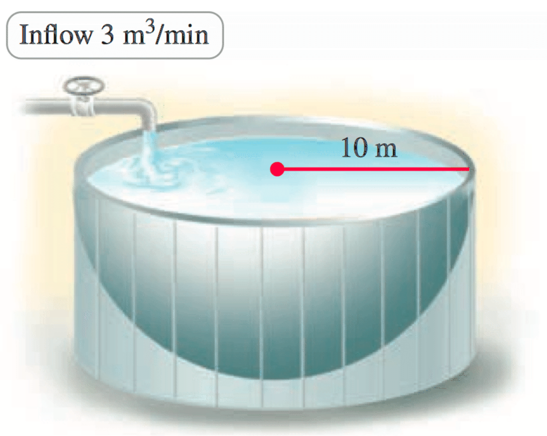


37. **Draining a cone** Water is drained out of an inverted cone, having the same dimensions as the cone depicted in Exercise 36. If the water level drops at 1 ft/min, at what rate is water (in ft^3/min) draining from the tank when the water depth is 6 ft?

38. **Two tanks** A conical tank with an upper radius of 4 m and a height of 5 m drains into a cylindrical tank with a radius of 4 m and a height of 5 m (see figure). If the water level in the conical tank drops at a rate of 0.5 m/min, at what rate does the water level in the cylindrical tank rise when the water level in the conical tank is 3 m? 1 m?

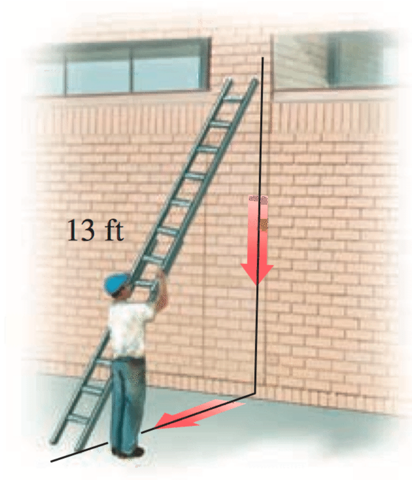


39. Filling a hemispherical tank A hemispherical tank with a radius of 10 m is filled from an inflow pipe at a rate of $3\ \text{m}^3/\text{min}$ (see figure). How fast is the water level rising when the water level is 5 m from the bottom of the tank? (*Hint:* The volume of a cap of thickness h sliced from a sphere of radius r is $\pi h^2(3r - h)/3$.)



40. Surface area of hemispherical tank For the situation described in Exercise 39, what is the rate of change of the area of the exposed surface of the water when the water is 5 m deep?

41. Ladder against the wall A 13-ft ladder is leaning against a vertical wall (see figure) when Jack begins pulling the foot of the ladder away from the wall at a rate of 0.5 ft/s. How fast is the top of the ladder sliding down the wall when the foot of the ladder is 5 ft from the wall?

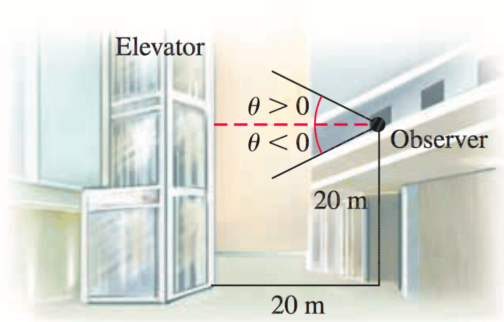


42. Ladder against the wall again A 12-ft ladder is leaning against a vertical wall when Jack begins pulling the foot of the ladder away from the wall at a rate of 0.2 ft/s. What is the configuration of the ladder at the instant that the vertical speed of the top of the ladder equals the horizontal speed of the foot of the ladder?

43. Moving shadow A 5-ft-tall woman walks at 8 ft/s toward a streetlight that is 20 ft above the ground. What is the rate of change of the length of her shadow when she is 15 ft from the streetlight? At what rate is the tip of her shadow moving?

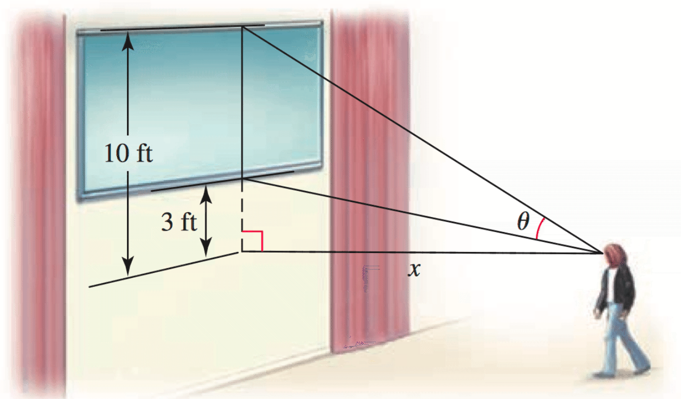
44. Another moving shadow A landscape light on level ground lights up the side of a tall building that is 15 feet from the light. A 6-ft-tall man starts walking from the light directly towards the building. How fast is he walking when he is 9 feet from the light if his shadow on the building is shrinking at 2 ft/s at that instant?

45. Watching an elevator An observer is 20 m above the ground floor of a large hotel atrium looking at a glass-enclosed elevator shaft that is 20 m horizontally from the observer (see figure). The angle of elevation of the elevator is the angle that the observer's line of sight makes with the horizontal (it may be positive or negative). Assuming the elevator rises at a rate of 5 m/s, what is the rate of change of the angle of elevation when the elevator is 10 m above the ground? When the elevator is 40 m above the ground?



T **46. Observing a launch** An observer stands 300 ft from the launch site of a hot-air balloon at an elevation equal to the elevation of the launch site. The balloon is launched vertically and maintains a constant upward velocity of 20 ft/s. What is the rate of change of the angle of elevation of the balloon when it is 400 ft from the ground? (*Hint*: The angle of elevation is the angle θ between the observer's line of sight to the balloon and the ground.)

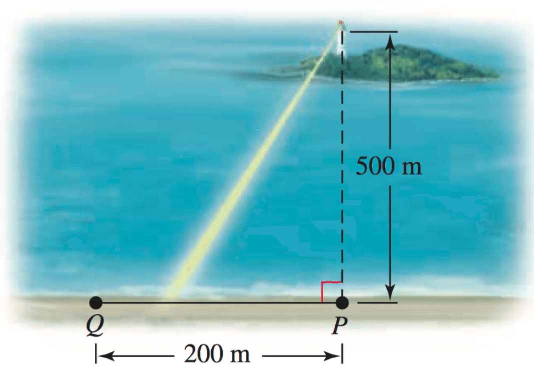
T **47. Viewing angle** The bottom of a large theater screen is 3 ft above your eye level and the top of the screen is 10 ft above your eye level. Assume you walk away from the screen (perpendicular to the screen) at a rate of 3 ft/s while looking at the screen. What is the rate of change of the viewing angle θ when you are 30 ft from the wall on which the screen hangs, assuming the floor is horizontal (see figure)?



48. Altitude of a jet A jet ascends at a 10° angle from the horizontal with an airspeed of 550 mi/hr (its speed along its line of flight is 550 mi/hr). How fast is the altitude of the jet increasing? If the sun is directly overhead, how fast is the shadow of the jet moving on the ground?

49. Rate of dive of a submarine A surface ship is moving (horizontally) in a straight line at 10 km/hr. At the same time, an enemy submarine maintains a position directly below the ship while diving at an angle that is 20° below the horizontal. How fast is the submarine's altitude decreasing?

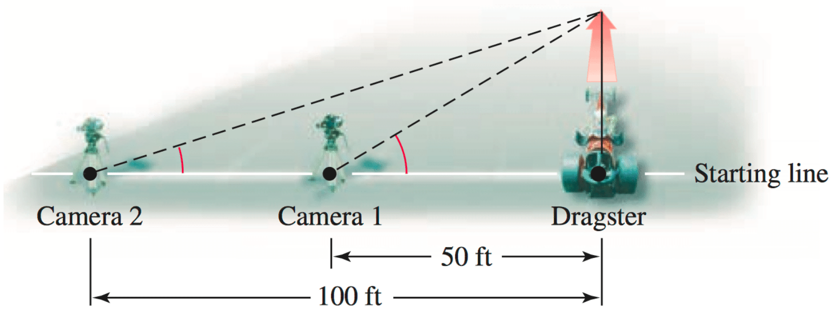
50. Revolving light beam A lighthouse stands 500 m off of a straight shore, and the focused beam of its light revolves (at a constant rate) four times each minute. As shown in the figure, P is the point on shore closest to the lighthouse and Q is a point on the shore 200 m from P. What is the speed of the beam along the shore when it strikes the point Q? Describe how the speed of the beam along the shore varies with the distance between P and Q. Neglect the height of the lighthouse.



51. **Filming a race** A camera is set up at the starting line of a drag race 50 ft from a dragster at the starting line (camera 1 in the figure). Two seconds after the start of the race, the dragster has traveled 100 ft and the camera is turning at 0.75 rad/s while filming the dragster.

a. What is the speed of the dragster at this point?

b. A second camera (camera 2 in the figure) filming the dragster is located on the starting line 100 ft away from the dragster at the start of the race. How fast is this camera turning 2 s after the start of the race?



52. **Fishing reel** An angler hooks a trout and begins turning her circular reel at 1.5 rev/s. Assume the radius of the reel (and the fishing line on it) is 2 inches.

a. Let R equal the number of revolutions the angler has turned her reel and suppose L is the amount of line that she has reeled in. Find an equation for L as a function of R.

b. How fast is she reeling in her fishing line?

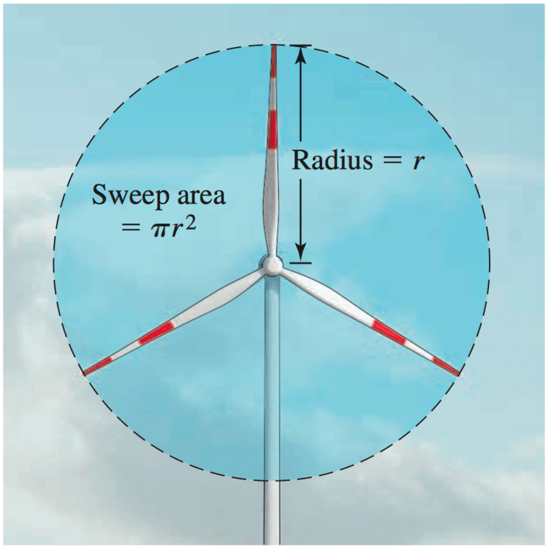
53. **Wind energy** The kinetic energy E (in joules) of a mass in motion satisfies the equation $E = \frac{1}{2} m v^2$, where mass m is measured in kg and velocity v is measured in m/s.

a. Power P is defined to be $\frac{dE}{dt}$, the rate of change in energy with respect to time. Power is measured in units of watts (W), where 1 W = 1 joule/s. If the velocity v is constant, use implicit differentiation to find an equation for power P in terms of the derivative $\frac{dm}{dt}$.

b. Wind turbines use kinetic energy in the wind to create electrical power. In this case, the derivative $\frac{dm}{dt}$ is called the **mass flow rate** and it satisfies the equation $\frac{dm}{dt} = \rho A v$, where ρ is the density of the air in kg/m^3, A is the sweep area in m^2 of the wind turbine (see figure), and v is the velocity of the wind in m/s. Show that $P = \frac{1}{2} \rho A v^3$.

c. Suppose a blade on a small wind turbine has a length of 3 m. Find the available power P if the wind is blowing at 10 m/s. (*Hint*: Use $\rho = 1.23\ \text{kg}/\text{m}^3$ for the density of air. The density of air varies, but this is a reasonable average value.)

d. Wind turbines convert only a small percentage of the available wind power into electricity. Assume the wind turbine described in this exercise converts only 25% of the available wind power into electricity. How much electrical power is produced?



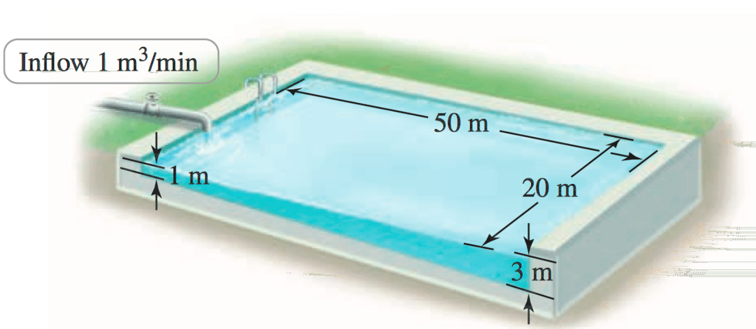
54. **Boyle's law** Robert Boyle (1627-1691) found that for a given quantity of gas at a constant temperature, the pressure P (in kPa) and volume V of the gas (in m^3) are accurately approximated by the equation $V = k/P$, where $k > 0$ is constant. Suppose the volume of an expanding gas is increasing at a rate of 0.15 m^3/min when the volume $V = 0.5\ m^3$ and the pressure is $P = 50$ kPa. At what rate is pressure changing at this moment?

Explorations and Challenges »

55. **Clock hands** The hands of the clock in the tower of the Houses of Parliament in London are approximately 3 m and 2.5 m in length. How fast is the distance between the tips of the hands changing at 9:00? (*Hint:* Use the Law of Cosines.)

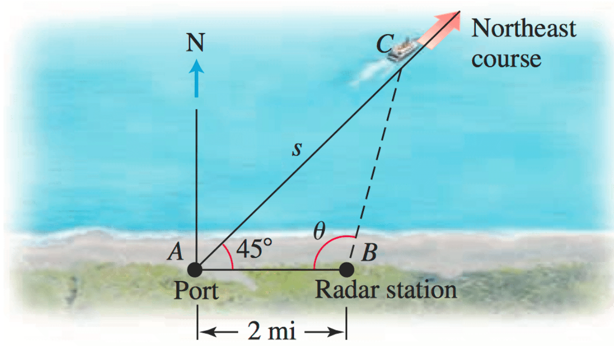
56. **Divergent paths** Two boats leave a port at the same time, one traveling west at 20 mi/hr and the other traveling southwest (45° south of west) at 15 mi/hr. After 30 minutes, how far apart are the boats and at what rate is the distance between them changing? (*Hint*: Use the Law of Cosines.)

57. **Filling a pool** A swimming pool is 50 m long and 20 m wide. Its depth decreases linearly along the length from 3 m to 1 m (see figure). It is initially empty and is filled at a rate of 1 m^3/min. How fast is the water level rising 250 min after the filling begins? How long will it take to fill the pool?

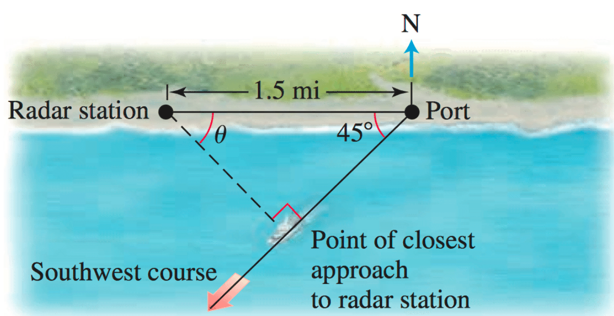


58. Disappearing triangle An equilateral triangle initially has sides of length 20 ft when each vertex moves toward the midpoint of the opposite side at a rate of 1.5 ft/min. Assuming the triangle remains equilateral, what is the rate of change of the area of the triangle at the instant the triangle disappears?

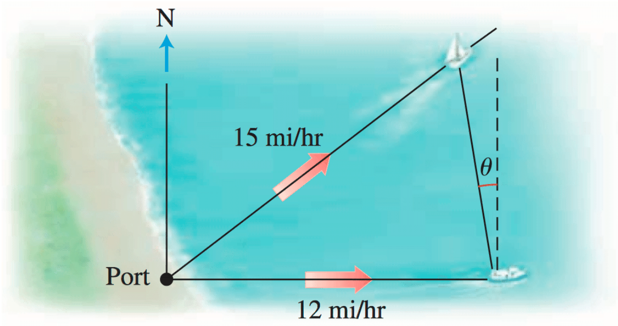
T 59. Oblique tracking A port and a radar station are 2 mi apart on a straight shore running east and west (see figure). A ship leaves the port at noon traveling northeast at a rate of 15 mi/hr. If the ship maintains its speed and course, what is the rate of change of the tracking angle θ between the shore and the line between the radar station and the ship at 12:30 P.M.? (*Hint:* Use the Law of Sines.)



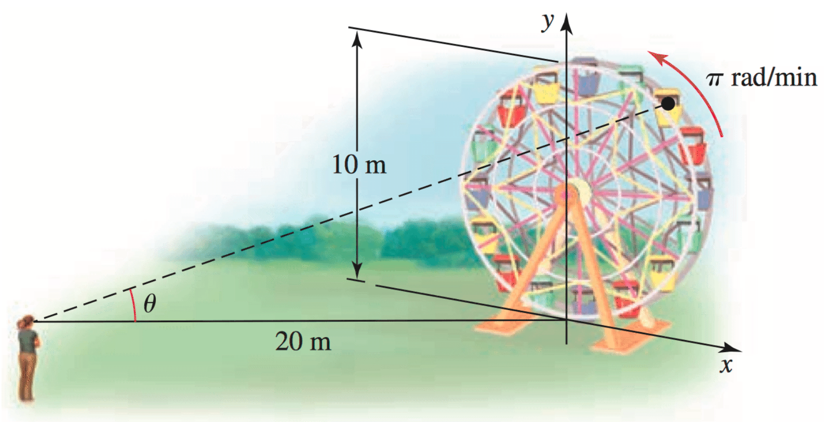
T 60. Oblique tracking A ship leaves port traveling southwest at a rate of 12 mi/hr. At noon, the ship reaches its closest approach to a radar station, which is on the shore 1.5 mi from the port. If the ship maintains its speed and course, what is the rate of change of the tracking angle θ between the radar station and the ship at 1:30 P.M. (see figure)? (*Hint:* Use the Law of Sines.)



61. Navigation A boat leaves a port traveling due east at 12 mi/hr. At the same time, another boat leaves the same port traveling northeast at 15 mi/hr. The angle θ of the line between the boats is measured relative to due north (see figure). What is the rate of change of this angle 30 min after the boats leave the port? 2 hr after the boats leave the port?



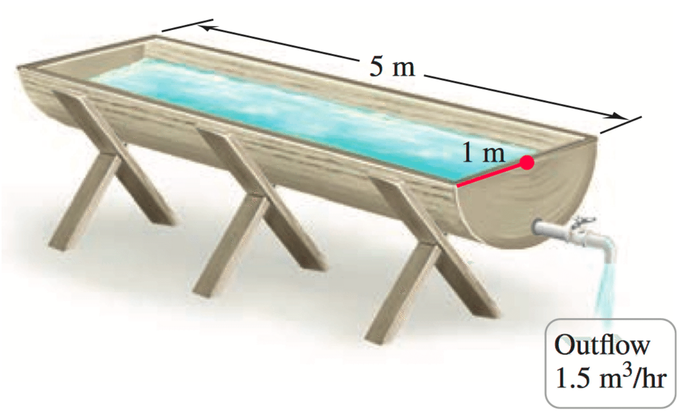
62. Watching a Ferris wheel An observer stands 20 m from the bottom of a 10-m-tall Ferris wheel on a line that is perpendicular to the face of the Ferris wheel. The wheel revolves at a rate of π rad/min and the observer's line of sight with a specific seat on the wheel makes an angle θ with the ground (see figure). Forty seconds after that seat leaves the lowest point on the wheel, what is the rate of change of θ? Assume the observer's eyes are level with the bottom of the wheel.



63. Draining a trough A trough in the shape of a half cylinder has length 5 m and radius 1 m. The trough is full of water when a valve is opened, and water flows out of the bottom of the trough at a rate of 1.5 m^3/hr (see figure). (*Hint:* The area of a sector of a circle of radius r subtended by an angle θ is $r^2\theta/2$.)

a. How fast is the water level changing when the water level is 0.5 m from the bottom of the trough?

b. What is the rate of change of the surface area of the water when the water is 0.5 m deep?



64. Searchlight—wide beam A revolving searchlight, which is 100 m from the nearest point on a straight highway, casts a horizontal beam along a highway (see figure). The beam leaves the spotlight at an angle of $\pi/16$ rad and revolves at a rate $\pi/6$ rad/s. Let w be the width of the beam as it sweeps along the highway and θ be the angle that the center of the beam makes with the perpendicular to the highway. What is the rate of change of w when $\theta = \pi/3$? Neglect the height of the searchlight.