3.8 Implicit Differentiation

Implicit Differentiation »

This chapter has been devoted to calculating derivatives of functions of the form $y = f(x)$, where y is defined *explicitly* as a function of x. However, relationships between variables are often expressed *implicitly*. For example, the equation of the unit circle $x^2 + y^2 = 1$ does not specify y directly, but rather defines y implicitly. This equation does not represent a single function because its graph fails the vertical line test (**Figure 3.54a**). If, however, the equation $x^2 + y^2 = 1$ is solved for y, then *two* functions emerge: $y = -\sqrt{1-x^2}$ and $y = \sqrt{1-x^2}$ (**Figure 3.54b**). Having identified two explicit functions that form the circle, their derivatives are found using the Chain Rule:

$$\text{If } y = \sqrt{1-x^2}, \text{ then } \frac{dy}{dx} = -\frac{x}{\sqrt{1-x^2}}. \qquad (1)$$

$$\text{If } y = -\sqrt{1-x^2}, \text{ then } \frac{dy}{dx} = \frac{x}{\sqrt{1-x^2}}. \qquad (2)$$

We use equation (1) to find the slope of the curve at any point on the upper half of the unit circle and equation (2) to find the slope of the curve at any point on the lower half of the circle.

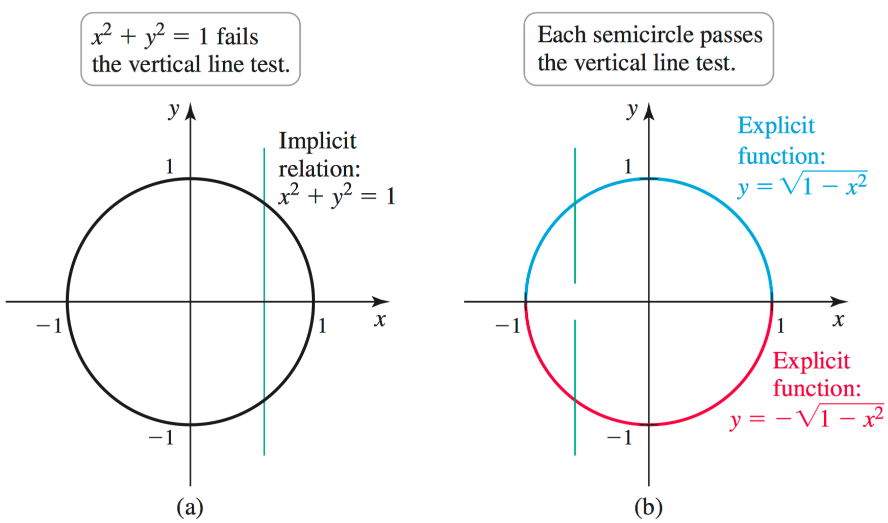


Figure 3.54

Quick Check 1 The equation $x - y^2 = 0$ implicitly defines what two functions?

Answer »

$y = \sqrt{x}$ and $y = -\sqrt{x}$

Although it is straightforward to solve some implicit equations for y (such as $x^2 + y^2 = 1$ or $x - y^2 = 0$), it is difficult or impossible to solve other equations for y. For example, the graph of $x + y^3 - xy = 1$ (**Figure 3.55a**) represents three functions: the upper half of a parabola $y = f_1(x)$, the lower half of a parabola $y = f_2(x)$, and the horizontal line $y = f_3(x)$ (**Figure 3.55b**). Solving for y to obtain these three functions is challenging (Exercise 69), and even after solving for y, derivatives for each of the three functions must be calculated separately. The goal of this section is to find a *single* expression for the derivative *directly* from an equation $F(x, y) = 0$ without first solving for y. This technique, called **implicit differentiation**, is demonstrated through examples.

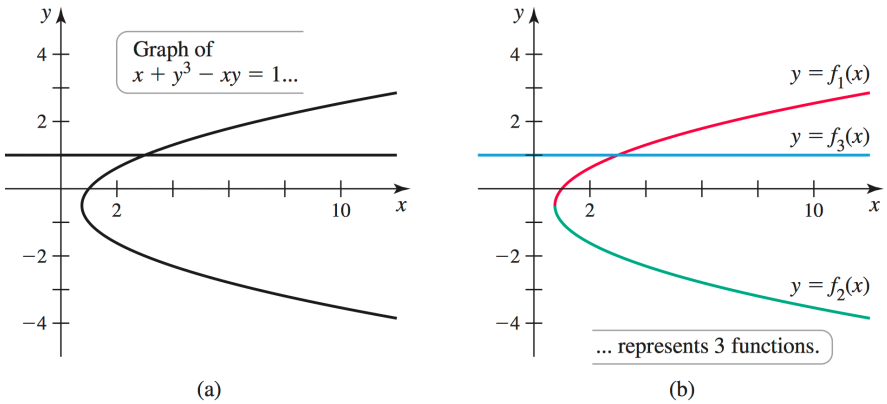


Figure 3.55

EXAMPLE 1 Implicit differentiation

a. Calculate $\frac{dy}{dx}$ directly from the equation for the unit circle $x^2 + y^2 = 1$.

b. Find the slope of the unit circle at $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.

SOLUTION »

a. To indicate the choice of x as the independent variable, it is helpful to replace the variable y with $y(x)$:

$$x^2 + (y(x))^2 = 1. \quad \text{Replace } y \text{ with } y(x).$$

We now take the derivative of each term in the equation *with respect to* x:

$$\underbrace{\frac{d}{dx}\left(x^2\right)}_{2x} + \underbrace{\frac{d}{dx}(y(x))^2}_{\text{Use the Chain Rule}} = \underbrace{\frac{d}{dx}(1)}_{0}.$$

By the Chain Rule, $\frac{d}{dx}(y(x))^2 = 2\,y(x)\,y'(x)$, or $\frac{d}{dx}\left(y^2\right) = 2\,y\,\frac{dy}{dx}$. Substituting this result, we have

$$2x + 2y\frac{dy}{dx} = 0.$$

The last step is to solve for $\frac{dy}{dx}$:

$$2y\frac{dy}{dx} = -2x \quad \text{Subtract } 2x \text{ from both sides.}$$

$$\frac{dy}{dx} = -\frac{x}{y}. \quad \text{Divide by } 2y \text{ and simplify.}$$

This result holds provided $y \neq 0$. At the points $(1, 0)$ and $(-1, 0)$, the circle has vertical tangent lines.

b. Notice that the derivative $\frac{dy}{dx} = -\frac{x}{y}$ depends on *both* x and y. Therefore, to find the slope of the circle at $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, we substitute both $x = \frac{1}{2}$ and $y = \frac{\sqrt{3}}{2}$ into the derivative formula. The result is

$$\left.\frac{dy}{dx}\right|_{\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)} = -\frac{1/2}{\sqrt{3}/2} = -\frac{1}{\sqrt{3}}.$$

Note »

Implicit differentiation usually produces an expression for $\frac{dy}{dx}$ in terms of both x and y. The notation $\left.\frac{dy}{dx}\right|_{(a,b)}$ tells us to replace x with a and y with b in the expression for $\frac{dy}{dx}$.

The slope of the curve at $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ is

$$\left.\frac{dy}{dx}\right|_{\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)} = -\frac{1/2}{-\sqrt{3}/2} = \frac{1}{\sqrt{3}}.$$

The curve and tangent lines are shown in **Figure 3.56**.

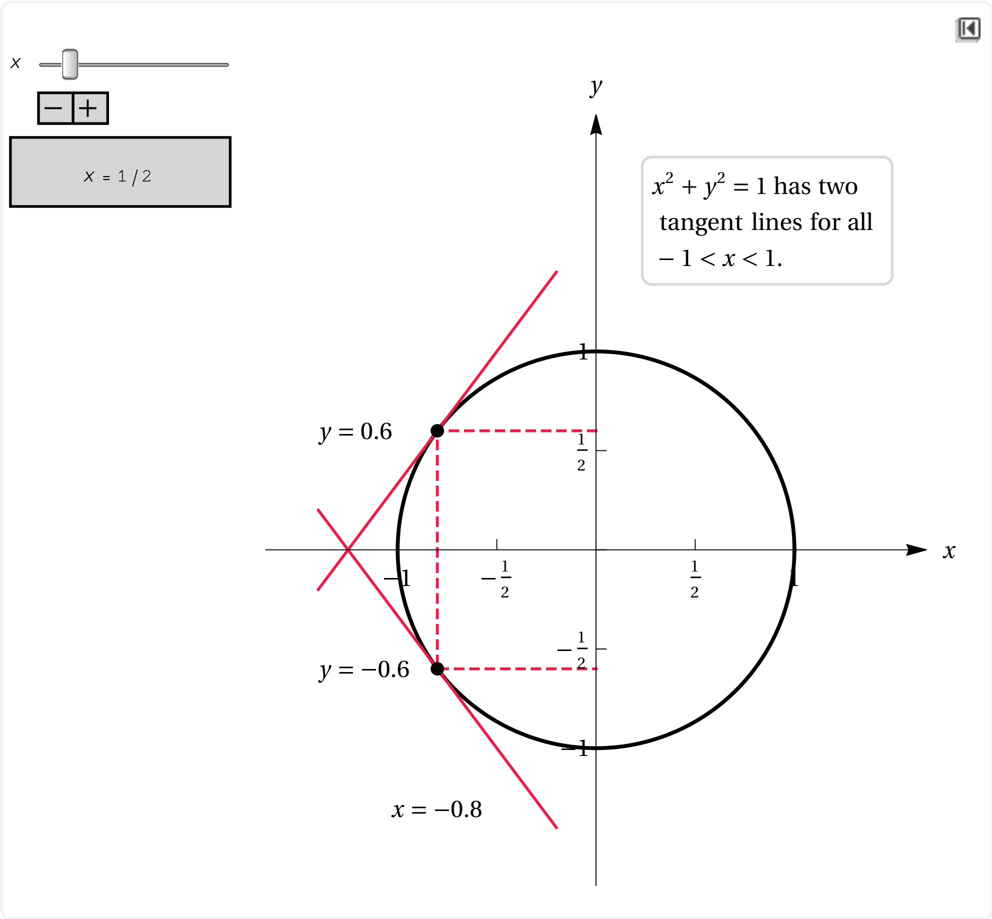


Figure 3.56

Related Exercises 13, 15 ◆

Example 1 illustrates the technique of implicit differentiation. It is done without solving for y, and it produces $\frac{dy}{dx}$ in terms of *both* x and y. The derivative obtained in Example 1 is consistent with the derivatives calculated explicitly in equations (1) and (2). For the upper half of the circle, substituting $y = \sqrt{1 - x^2}$ into the implicit derivative $\frac{dy}{dx} = -\frac{x}{y}$ gives

$$\frac{dy}{dx} = -\frac{x}{y} = -\frac{x}{\sqrt{1 - x^2}},$$

which agrees with equation (1). For the lower half of the circle, substituting $y = -\sqrt{1 - x^2}$ into $\frac{dy}{dx} = -\frac{x}{y}$ gives

$$\frac{dy}{dx} = -\frac{x}{y} = \frac{x}{\sqrt{1 - x^2}},$$

which is consistent with equation (2). Therefore, implicit differentiation gives a single unified derivative

$$\frac{dy}{dx} = -\frac{x}{y}.$$

EXAMPLE 2 **Implicit differentiation**

Find $y'(x)$ when $\sin xy = x^2 + y$.

SOLUTION »

It is impossible to solve this equation for y in terms of x, so we differentiate implicitly. Differentiating both sides of the equation with respect to x, using the Chain Rule and the Product Rule on the left side, gives

$$\cos xy\,(y + x\,y') = 2\,x + y'.$$

We now solve for y':

$$\begin{aligned} x\,y' \cos xy - y' &= 2\,x - y \cos xy && \text{Rearrange terms.} \\ y'(x \cos xy - 1) &= 2\,x - y \cos xy && \text{Factor on left side.} \\ y' &= \frac{2\,x - y \cos xy}{x \cos xy - 1}. && \text{Solve for } y'. \end{aligned}$$

Notice that the final result gives y' in terms of both x and y.

Related Exercises 31, 33 ◆

Quick Check 2 Use implicit differentiation to find $\frac{dy}{dx}$ for $x - y^2 = 3$.◀

Answer »

> $$\frac{dy}{dx} = \frac{1}{2\,y}$$

Slopes of Tangent Lines »

Derivatives obtained by implicit differentiation typically depend on x *and* y. Therefore, the slope of a curve at a particular point (x, y) requires both the x- and y-coordinates of the point. These coordinates are also needed to find an equation of the tangent line at that point.

Quick Check 3 If a function is defined explicitly in the form $y = f(x)$, which coordinates are needed to find the slope of a tangent line—the x-coordinate, the y-coordinate, or both?◀

Answer »

> Only the x-coordinate is needed.

EXAMPLE 3 **Finding tangent lines with implicit functions**

Find an equation of the line tangent to the curve $x^2 + x\,y - y^3 = 7$ at $(3, 2)$.

SOLUTION »

We calculate the derivative of each term of the equation $x^2 + x\,y - y^3 = 7$ with respect to x:

$$\frac{d}{dx}\left(x^2\right) + \frac{d}{dx}(x\,y) - \frac{d}{dx}\left(y^3\right) = \frac{d}{dx}(7) \quad \text{Differentiate each term.}$$

$$2\,x + \underbrace{y + x\,y'}_{\text{Product Rule}} - \underbrace{3\,y^2\,y'}_{\text{Chain Rule}} = 0 \quad \text{Calculate the derivatives.}$$

$$3\,y^2\,y' - x\,y' = 2\,x + y \quad \text{Group the terms containing } y'.$$

$$y' = \frac{2\,x + y}{3\,y^2 - x}. \quad \text{Factor and solve for } y'.$$

Note »

Because y is a function of x, we have

$$\frac{d}{dx}(x) = 1 \quad \text{and}$$

$$\frac{d}{dx}(y) = y'.$$

To differentiate y^3 with respect to x, we need the Chain Rule.

To find the slope of the tangent line at (3, 2), we substitute $x = 3$ and $y = 2$ into the derivative formula:

$$\left.\frac{dy}{dx}\right|_{(3,2)} = \left.\frac{2\,x + y}{3\,y^2 - x}\right|_{(3,2)} = \frac{8}{9}.$$

An equation of the line passing through (3, 2) with slope $\frac{8}{9}$ is

$$y - 2 = \frac{8}{9}(x - 3) \quad \text{or} \quad y = \frac{8}{9}\,x - \frac{2}{3}.$$

Figure 3.57 shows the graphs of the curve $x^2 + x\,y - y^3 = 7$ and the tangent line.

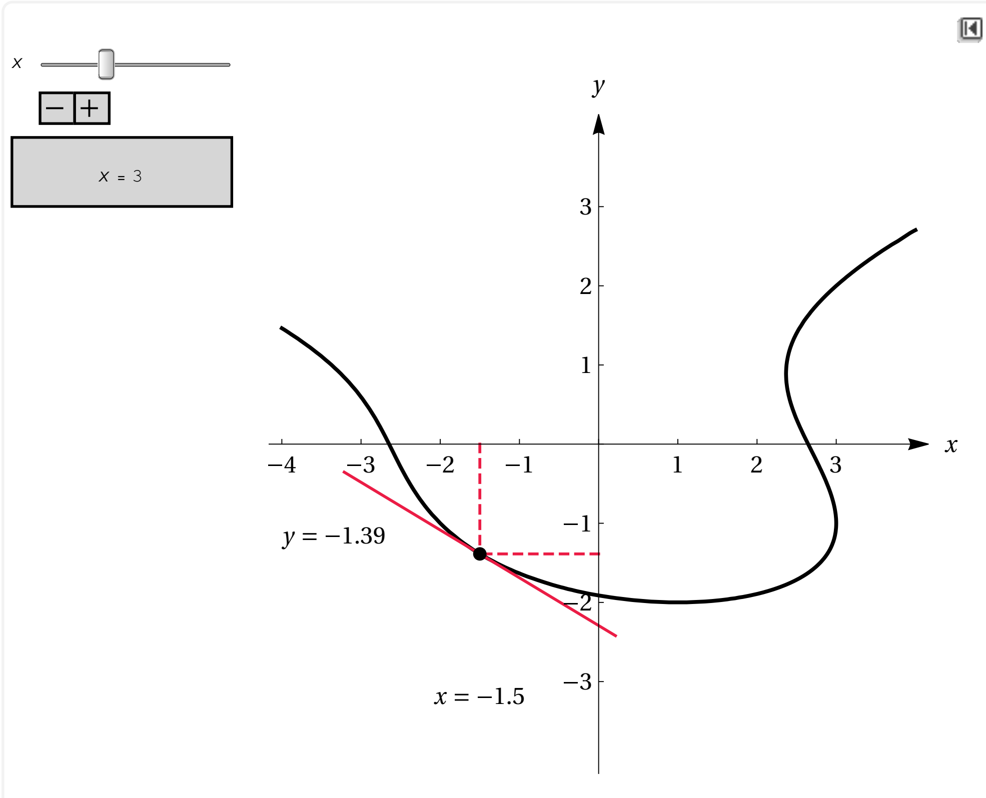


Figure 3.57

Related Exercises 47–48 ◆

EXAMPLE 4 Slope of a curve

Find the slope of the curve $2(x + y)^{1/3} = y$ at the point (4, 4).

SOLUTION »

We begin by differentiating both sides of the given equation:

$$\frac{2}{3}(x+y)^{-2/3}\left(1+\frac{dy}{dx}\right)=\frac{dy}{dx} \qquad \text{Implicit differentiation, Chain Rule, Power Rule}$$

$$\frac{2}{3}(x+y)^{-2/3}=\frac{dy}{dx}-\frac{2}{3}(x+y)^{-2/3}\frac{dy}{dx} \qquad \text{Expand and collect like terms.}$$

$$\frac{2}{3}(x+y)^{-2/3}=\frac{dy}{dx}\left(1-\frac{2}{3}(x+y)^{-2/3}\right). \qquad \text{Factor out } \frac{dy}{dx}.$$

Solving for $\frac{dy}{dx}$, we find that

$$\frac{dy}{dx} = \frac{\frac{2}{3}(x+y)^{-2/3}}{1-\frac{2}{3}(x+y)^{-2/3}} \quad \text{Divide by } 1-\frac{2}{3}(x+y)^{-2/3}.$$

$$= \frac{2}{3(x+y)^{2/3}-2}. \quad \text{Multiply by } 3(x+y)^{2/3} \text{ and simplify.}$$

At this stage, we could evaluate $\frac{dy}{dx}$ at the point (4, 4) to determine the slope of the curve. However, the derivative can be simplified by employing a subtle maneuver. The original equation $2(x+y)^{1/3} = y$ implies that $(x+y)^{2/3} = \frac{y^2}{4}$. Therefore,

$$\frac{dy}{dx} = \frac{2}{3\underbrace{(x+y)^{2/3}}_{y^2/4}-2} = \frac{8}{3y^2-8}.$$

With this simplified formula, finding the slope of the curve at the point (4, 4) (**Figure 3.58**) requires only the y-coordinate:

$$\left.\frac{dy}{dx}\right|_{(4,4)} = \left.\frac{8}{3y^2-8}\right|_{y=4} = \frac{8}{3(4)^2-8} = \frac{1}{5}.$$

Note »

> The procedure outlined in Example 4 for simplifying dy/dx is a useful trick. When computing derivatives using implicit differentiation, look back to the original equation to determine whether a substitution can be made to simplify the derivative. This technique is illustrated again in Example 5.

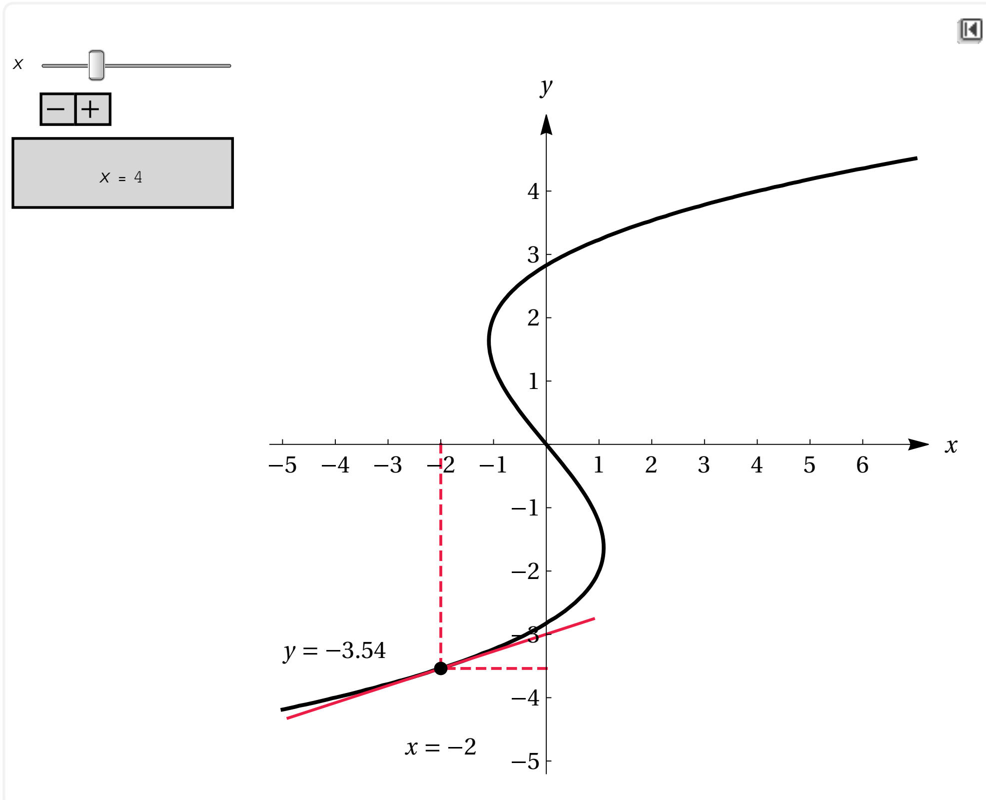


Figure 3.58

Related Exercises 25–26 ◆

Higher-Order Derivatives of Implicit Functions »

In previous sections of this chapter, we found higher-order derivatives $\frac{d^n y}{dx^n}$ by first calculating $\frac{dy}{dx}, \frac{d^2 y}{dx^2}, \ldots,$ and $\frac{d^{n-1} y}{dx^{n-1}}$. The same approach is used with implicit differentiation.

EXAMPLE 5 **A second derivative**

Find $\frac{d^2 y}{dx^2}$ if $x^2 + y^2 = 1$.

SOLUTION »

The first derivative $\frac{dy}{dx} = -\frac{x}{y}$ was computed in Example 1.

We now calculate the derivative of each side of this equation and simplify the right side:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(-\frac{x}{y}\right) \quad \text{Take derivatives with respect to } x.$$

$$\frac{d^2y}{dx^2} = -\frac{y \cdot 1 - x\,\frac{dy}{dx}}{y^2} \quad \text{Quotient Rule}$$

$$= -\frac{y - x\left(-\frac{x}{y}\right)}{y^2} \quad \text{Substitute for } \frac{dy}{dx}.$$

$$= -\frac{x^2 + y^2}{y^3} \quad \text{Simplify.}$$

$$= -\frac{1}{y^3}. \quad x^2 + y^2 = 1$$

In the last step of this calculation we used the original equation $x^2 + y^2 = 1$ to simplify the formula for $\frac{d^2y}{dx^2}$.

Related Exercises 51–52 ◆

Exercises »

Getting Started »

Practice Exercises »

13–26. Implicit differentiation *Carry out the following steps.*

a. *Use implicit differentiation to find* $\frac{dy}{dx}$.

b. *Find the slope of the curve at the given point.*

13. $x^4 + y^4 = 2;\ (1, -1)$

14. $y^2 + 1 = 2x;\ (1, 1)$

15. $y^2 = 4x;\ (1, 2)$

16. $y^2 + 3x = 8;\ \left(1, \sqrt{5}\right)$

17. $\sin y = 5x^4 - 5;\ (1, \pi)$

18. $\sqrt{x} - 2\sqrt{y} = 0;\ (4, 1)$

19. $\cos y = x;\ \left(0, \frac{\pi}{2}\right)$

20. $\tan xy = x + y;\ (0, 0)$

21. $xy = 7;\ (1, 7)$

22. $\dfrac{x}{y^2+1} = 1$; $(10, 3)$

23. $\sqrt[3]{x} + \sqrt[3]{y^4} = 2$; $(1, 1)$

24. $x^{2/3} + y^{2/3} = 2$; $(1, 1)$

25. $x\sqrt[3]{y} + y = 10$; $(1, 8)$

26. $(x+y)^{2/3} = y$; $(4, 4)$

27–40. Implicit differentiation *Use implicit differentiation to find* $\dfrac{dy}{dx}$.

27. $\sin x + \sin y = y$

28. $y = x \sin y$

29. $x + y = \cos y$

30. $x + 2y = \sqrt{y}$

31. $\sin xy = x + y$

32. $\tan(x+y) = 2y$

33. $\cos y^2 + x = y^2$

34. $y = \dfrac{x+1}{y-1}$

35. $x^3 = \dfrac{x+y}{x-y}$

36. $(xy+1)^3 = x - y^2 + 8$

37. $6x^3 + 7y^3 = 13xy$

38. $\sin x \cos y = \sin x + \cos y$

39. $\sqrt{x^4 + y^2} = 5x + 2y^3$

40. $\sqrt{x + y^2} = \sin y$

41. **Cobb-Douglas production function** The output of an economic system Q, subject to two inputs, such as labor L and capital K, is often modeled by the Cobb-Douglas production function $Q = cL^a K^b$. When $a + b = 1$, the case is called *constant returns to scale*. Suppose $Q = 1280$, $a = \dfrac{1}{3}$, $b = \dfrac{2}{3}$, and $c = 40$.

a. Find the rate of change of capital with respect to labor, dK/dL.

b. Evaluate the derivative in part (a) with $L = 8$ and $K = 64$.

42. **Surface area of a cone** The lateral surface area of a cone of radius r and height h (the surface area excluding the base) is $A = \pi r\sqrt{r^2 + h^2}$.

a. Find dr/dh for a cone with a lateral surface area of $A = 1500\pi$.

b. Evaluate this derivative when $r = 30$ and $h = 40$.

43. **Volume of a spherical cap** Imagine slicing through a sphere with a plane (sheet of paper). The smaller piece produced is called a spherical cap. Its volume is $V = \pi h^2(3r - h)/3$, where r is the radius of the sphere and h is the thickness of the cap.

a. Find dr/dh for a spherical cap with a volume of $5\pi/3$.

b. Evaluate this derivative when $r = 2$ and $h = 1$.

44. **Volume of a torus** The volume of a torus (doughnut or bagel) with an inner radius of a and an outer radius of b is $V = \pi^2(b + a)(b - a)^2/4$.

a. Find db/da for a torus with a volume of $64\pi^2$.

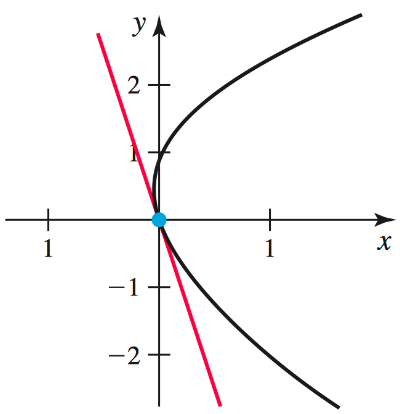
b. Evaluate this derivative when $a = 6$ and $b = 10$.

45–50. Tangent lines *Carry out the following steps.*

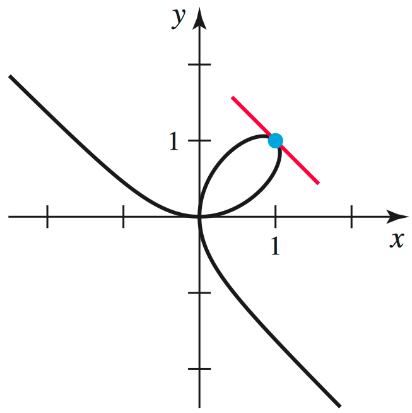
a. *Verify that the given point lies on the curve.*

b. *Determine an equation of the line tangent to the curve at the given point.*

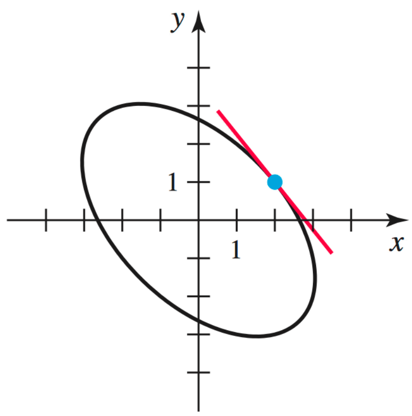
45. $\sin y + 5x = y^2$; $(0, 0)$



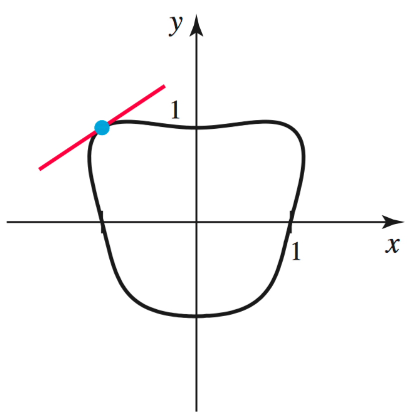
46. $x^3 + y^3 = 2xy$; $(1, 1)$



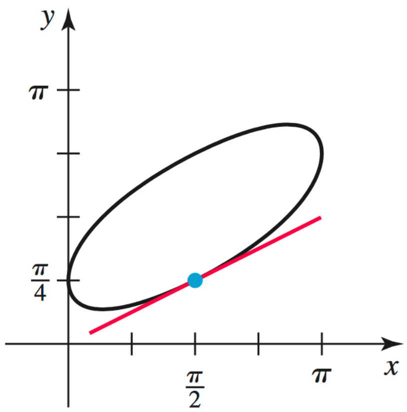
47. $x^2 + x\,y + y^2 = 7; (2,\ 1)$



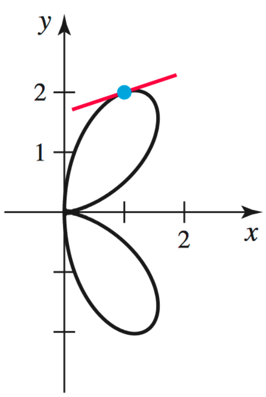
48. $x^4 - x^2\,y + y^4 = 1; (-1,\ 1)$



49. $\cos\,(x - y) + \sin\,y = \sqrt{2}\,; \left(\frac{\pi}{2},\ \frac{\pi}{4}\right)$



50. $\left(x^2 + y^2\right)^2 = \dfrac{25}{4} x y^2$; (1, 2)



51–56. Second derivatives *Find* $\dfrac{d^2 y}{dx^2}$.

51. $x + y^2 = 1$

52. $2x^2 + y^2 = 4$

53. $x + y = \sin y$

54. $x^4 + y^4 = 64$

55. $\cos y + x = y$

56. $\sin x + x^2 y = 10$

57. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. For any equation containing the variables x and y, the derivative dy/dx can be found by first using algebra to rewrite the equation in the form $y = f(x)$.

b. For the equation of a circle of radius r, $x^2+y^2=r^2$, we have $\dfrac{dy}{dx}=-\dfrac{x}{y}$, for $y\neq 0$ and any real number $r>0$.

c. If $x=1$, then by implicit differentiation, $1=0$.

d. If $xy=1$, then $y'=1/x$.

58–59. *Carry out the following steps.*

a. *Use implicit differentiation to find* $\dfrac{dy}{dx}$.

b. *Find the slope of the curve at the given point.*

58. $xy^{5/2}+x^{3/2}y=12;\ (4,\ 1)$

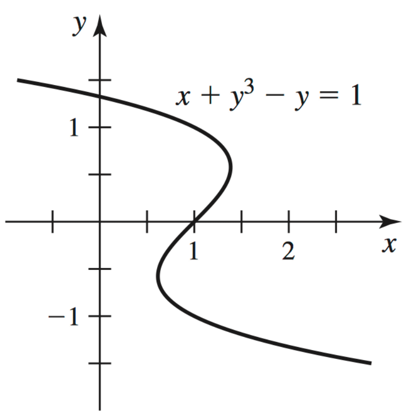
59. $xy+x^{3/2}y^{-1/2}=2;\ (1,\ 1)$

T **60–62. Multiple tangent lines** *Complete the following steps.*

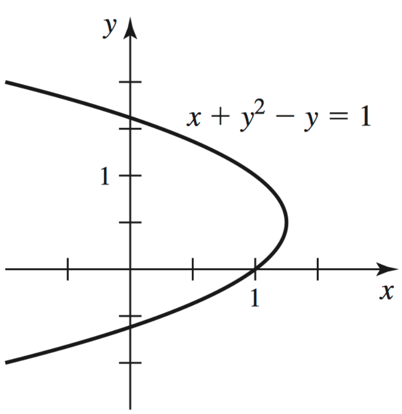
a. *Find equations of all lines tangent to the curve at the given value of* x.

b. *Graph the tangent lines on the given graph.*

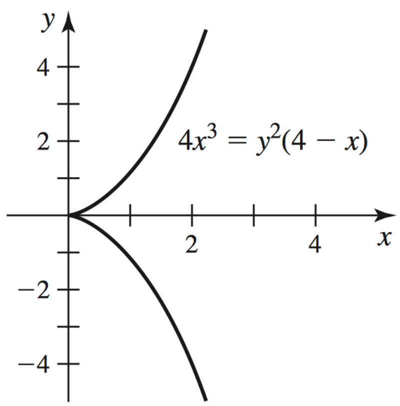
60. $x+y^3-y=1;\ x=1$



61. $x+y^2-y=1;\ x=1$



62. $4x^3=y^2(4-x);\ x=2$ (cissoid of Diocles)



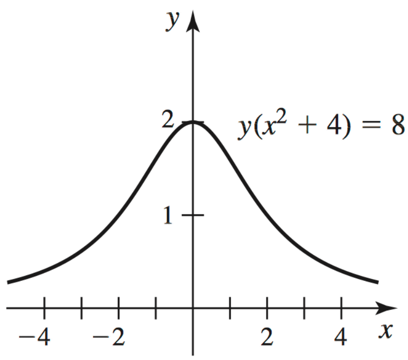
63. Witch of Agnesi Let $y\left(x^2+4\right)=8$ (see figure).

a. Use implicit differentiation to find $\frac{dy}{dx}$.

b. Find equations of all lines tangent to the curve $y\left(x^2+4\right)=8$ when $y=1$.

c. Solve the equation $y\left(x^2+4\right)=8$ for y to find an explicit expression for y and then calculate $\frac{dy}{dx}$.

d. Verify that the results of parts (a) and (c) are consistent.



64. Vertical tangent lines

a. Determine the points where the curve $x+y^3-y=1$ has a vertical tangent line (see Exercise 60).

b. Does the curve have any horizontal tangent lines? Explain.

65. Vertical tangent lines

a. Determine the points where the curve $x+y^2-y=1$ has a vertical tangent line (Exercise 61).

b. Does the curve have any horizontal tangent lines? Explain.

T **66–67. Tangent lines for ellipses** *Find the equations of the vertical and horizontal tangent lines of the following ellipses.*

66. $x^2+4y^2+2xy=12$

67. $9x^2+y^2-36x+6y+36=0$

Explorations and Challenges »

T **68–72. Identifying functions from an equation** *The following equations implicitly define one or more functions.*

- ***a.*** *Find* $\frac{dy}{dx}$ *using implicit differentiation.*
- ***b.*** *Solve the given equation for y to identify the implicitly defined functions* $y = f_1(x),\ y = f_2(x), \ldots$.
- ***c.*** *Use the functions found in part (b) to graph the given equation.*

68. $y^3 = a x^2$ (Neile's semicubical parabola)

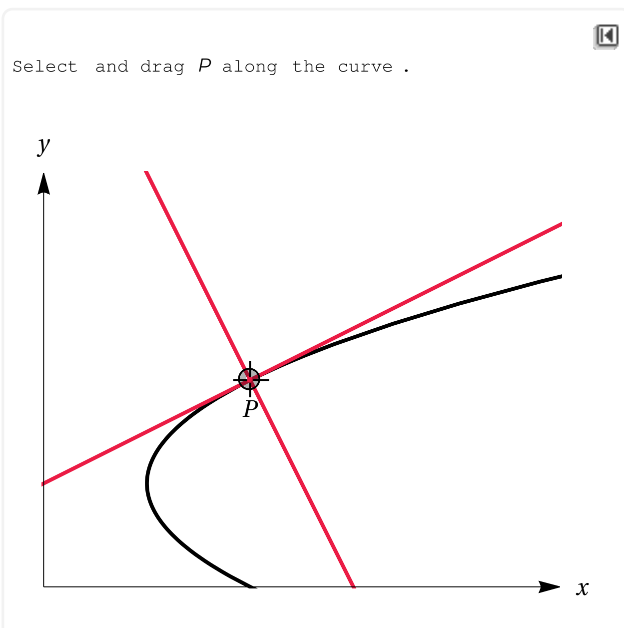
69. $x + y^3 - xy = 1$ (*Hint*: Rewrite as $y^3 - 1 = xy - x$ and then factor both sides.)

70. $y^2 = \frac{x^2(4-x)}{4+x}$ (right strophoid)

71. $x^4 = 2\left(x^2 - y^2\right)$ (eight curve)

72. $y^2(x+2) = x^2(6-x)$ (trisectrix)

T **73–78. Normal lines** *A* **normal line** *at a point P on a curve passes through P and is perpendicular to the line tangent to the curve at P (see figure). Use the following equations and graphs to determine an equation of the normal line at the given point. Illustrate your work by graphing the curve with the normal line.*



73. Exercise 45

74. Exercise 46

75. Exercise 47

76. Exercise 48

77. Exercise 49

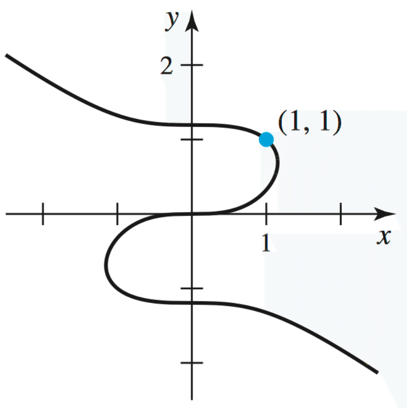
78. Exercise 50

T 79–82. Visualizing tangent and normal lines

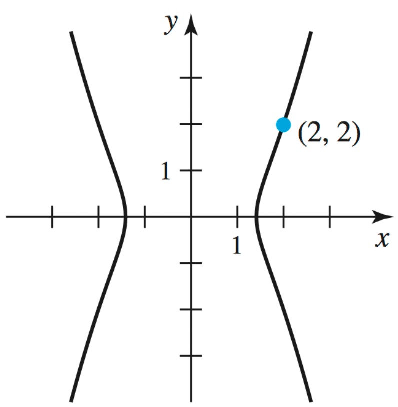
a. *Determine an equation of the tangent line and normal line at the given point* (x_0, y_0) *on the following curves. (See instructions for Exercises 73–78.)*

b. *Graph the tangent and normal lines on the given graph.*

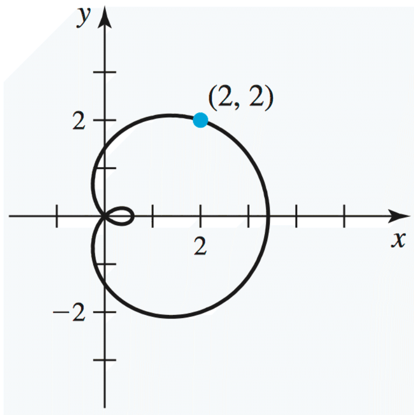
79. $3x^3 + 7y^3 = 10y$; $(x_0, y_0) = (1, 1)$



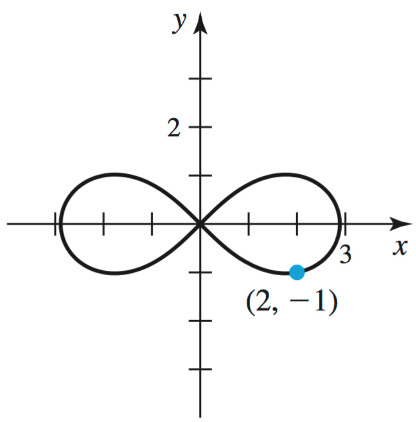
80. $x^4 = 2x^2 + 2y^2$; $(x_0\ y_0) = (2, 2)$ (kampyle of Eudoxus)



81. $\left(x^2 + y^2 - 2x\right)^2 = 2\left(x^2 + y^2\right)$; $(x_0, y_0) = (2, 2)$ (limaçon of Pascal)



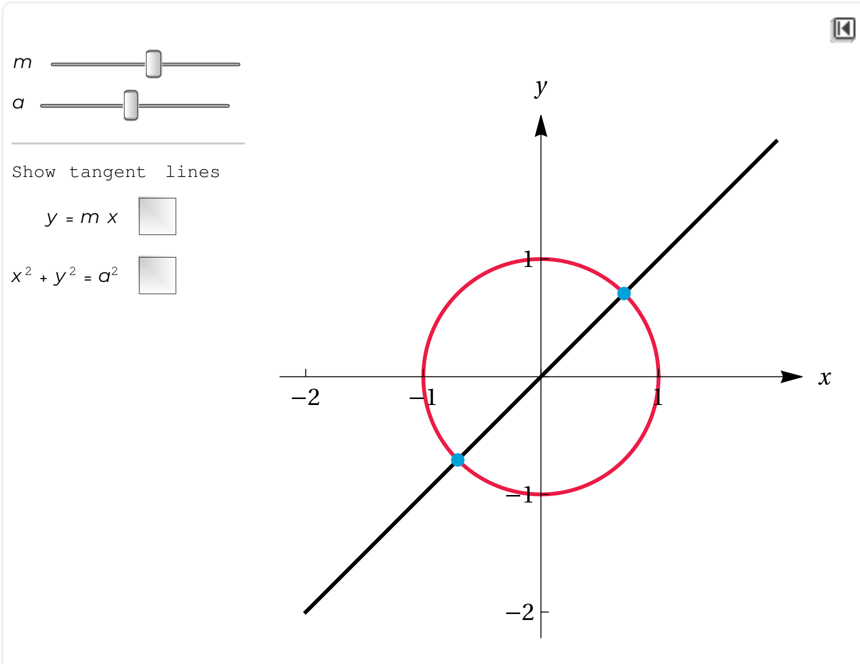
82. $\left(x^2 + y^2\right)^2 = \frac{25}{3}\left(x^2 - y^2\right)$; $(x_0, y_0) = (2, -1)$ (lemniscate of Bernoulli)



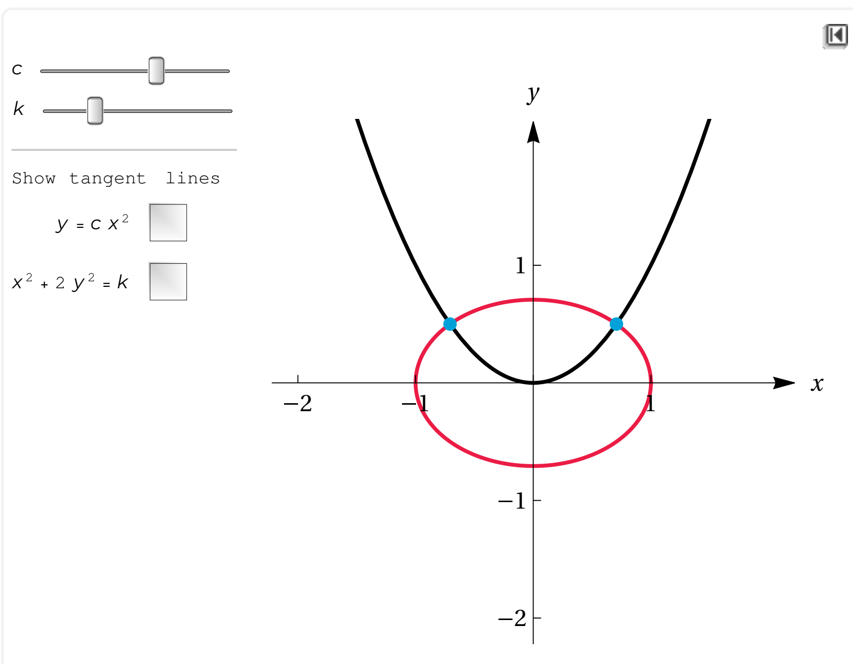
83–85. Orthogonal trajectories *Two curves are* **orthogonal** *to each other if their tangent lines are perpendicular at each point of intersection (recall that two lines are perpendicular to each other if their slopes are negative reciprocals). A family of curves forms* **orthogonal trajectories** *with another family of curves if each curve in one family is orthogonal to each curve in the other family. For example, the parabolas $y = c\,x^2$ form orthogonal trajectories with the family of ellipses $x^2 + 2\,y^2 = k$, where c and k are constants (see figure).*

Find dy/dx for each equation of the following pairs. Use the derivatives to explain why the families of curves form orthogonal trajectories.

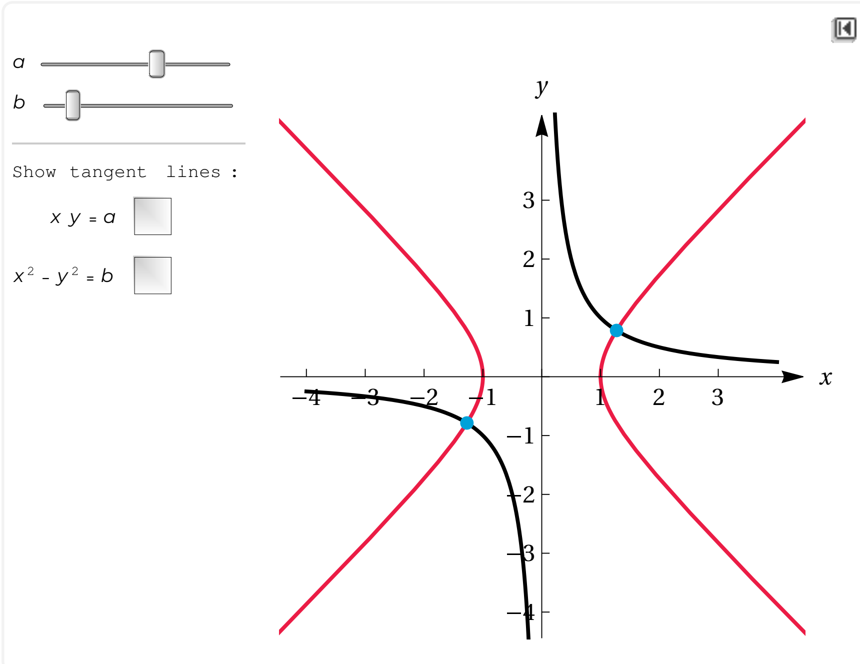
83. $y = m\,x$; $x^2 + y^2 = a^2$, where m and a are constants



84. $y = c\,x^2;\ x^2 + 2\,y^2 = k$, where c and k are constants



85. $x\,y = a;\ x^2 - y^2 = b$, where a and b are constants



86. Finding slope Find the slope of the curve $5\sqrt{x} - 10\sqrt{y} = \sin x$ at the point $(4\pi, \pi)$.

87. A challenging derivative Find $\dfrac{dy}{dx}$, where $\left(x^2 + y^2\right)\left(x^2 + y^2 + x\right) = 8xy^2$.

88. A challenging derivative Find $\dfrac{dy}{dx}$, where $\sqrt{3x^7 + y^2} = \sin^2 y + 100xy$.

89. A challenging second derivative Find $\dfrac{d^2y}{dx^2}$, where $\sqrt{y} + xy = 1$.

T 90–93. Work carefully *Proceed with caution when using implicit differentiation to find points at which a curve has a specified slope. For the following curves, find the points on the curve (if they exist) at which the tangent line is horizontal or vertical. Once you have found possible points, make sure that they actually lie on the curve. Confirm your results with a graph.*

90. $y^2 - 3xy = 2$

91. $x^2\left(3y^2 - 2y^3\right) = 4$

92. $x\left(1 - y^2\right) + y^3 = 0$