3.7 The Chain Rule

The differentiation rules presented so far allow us to find derivatives of many functions. However, these rules are inadequate for finding the derivatives of most *composite functions*. Here is a typical situation. If $f(x) = x^3$ and $g(x) = 5x + 4$, then their composition is $f(g(x)) = (5x + 4)^3$. One way to find the derivative is by expanding $(5x + 4)^3$ and differentiating the resulting polynomial. Unfortunately, this strategy becomes prohibitive for functions such as $(5x + 4)^{100}$. We need a better approach.

Quick Check 1 Explain why it is not practical to calculate $\frac{d}{dx}(5x + 4)^{100}$ by first expanding $(5x + 4)^{100}$.

◆

Answer »

The expansion of $(5x + 4)^{100}$ contains 101 terms. It would take too much time to calculate both the expansion and the derivative.

Chain Rule Formulas »

An efficient method for differentiating composite functions, called the *Chain Rule*, is motivated by the following example. Suppose Yancey, Uri, and Xan pick apples. Let y, u, and x represent the number of apples picked in some period of time by Yancey, Uri, and Xan, respectively. Yancey picks apples three times faster than Uri, which means the rate at which Yancey picks apples with respect to Uri is $\frac{dy}{du} = 3$. Uri picks apples twice as fast as Xan, so $\frac{du}{dx} = 2$. Therefore, Yancey picks apples at a rate that is $3 \cdot 2 = 6$ times greater than Xan's rate, which means that $\frac{dy}{dx} = 6$ (**Figure 3.53**). Observe that

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 3 \cdot 2 = 6.$$

The equation $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ is one form of the Chain Rule.

Note »

Expressions such as $\frac{dy}{dx}$ should not be treated as fractions. Nevertheless, you can check symbolically that you have written the Chain Rule correctly by noting that du appears in the "numerator" and "denominator." If it were "canceled," the Chain Rule would have $\frac{dy}{dx}$ on both sides.

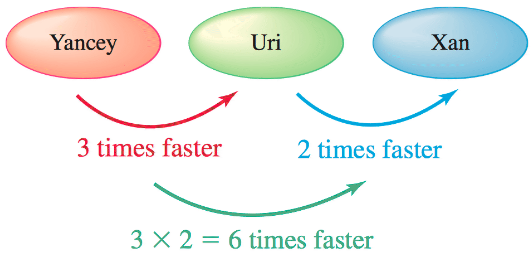

Figure 3.53

Alternatively, the Chain Rule may be expressed in terms of composite functions. Let $y = f(u)$ and $u = g(x)$, which means y is related to x through the composite function $y = f(u) = f(g(x))$. The derivative $\frac{dy}{dx}$ is now expressed as the product

$$\underbrace{\frac{d}{dx}(f(g(x)))}_{\frac{dy}{dx}} = \underbrace{f'(u)}_{\frac{dy}{du}} \cdot \underbrace{g'(x)}_{\frac{du}{dx}}.$$

Replacing u with $g(x)$ results in

$$\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x).$$

THEOREM 3.12 **The Chain Rule**

Suppose $y = f(u)$ is differentiable at $u = g(x)$ and $u = g(x)$ is differentiable at x. The composite function $y = f(g(x))$ is differentiable at x, and its derivative can be expressed in two equivalent ways.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \qquad (1)$$

$$\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x) \qquad (2)$$

Note »

The two equations in Theorem 3.13 differ only in notation (Leibniz notation for the derivative versus function notation). Mathematically, they are identical. The second equation states that the derivative of $y = f(g(x))$ is the derivative of f evaluated at $g(x)$ multiplied by the derivative of g evaluated at x.

A proof of the Chain Rule is given at the end of this section. For now, it's important to learn how to use it. With the composite function $f(g(x))$, we refer to g as the *inner function* and f as the *outer function*. The key to using the Chain Rule is identifying the inner and outer functions. The following four steps outline the differentiation process, although you will soon find that the procedure can be streamlined.

Note »

There may be several ways to choose an inner function $u = g(x)$ and an outer function $y = f(u)$. Nevertheless, we refer to *the* inner and *the* outer function for the most obvious choices.

PROCEDURE **Using the Chain Rule**

Assume the differentiable function $y = f(g(x))$ is given.

1. Identify the outer function f, the inner function g, and let $u = g(x)$.

2. Replace $g(x)$ by u to express y in terms of u:

$$y = f(\underbrace{g(x)}_{u}) \Longrightarrow y = f(u).$$

3. Calculate the product $\dfrac{dy}{du} \cdot \dfrac{du}{dx}$.

4. Replace u by $g(x)$ in $\dfrac{dy}{du}$ to obtain $\dfrac{dy}{dx}$.

Quick Check 2 Identify an inner function (call it g) of $y = (5x + 4)^3$. Let $u = g(x)$ and express the outer function f in terms of u. ◆

Answer »

The inner function is $u = 5x + 4$, and the outer function is $y = u^3$.

EXAMPLE 1 The Chain Rule

For each of the following composite functions, find the inner function $u = g(x)$ and the outer function $y = f(u)$. Use equation (1) of the Chain Rule to find $\dfrac{dy}{dx}$.

a. $y = (5x + 4)^{3/2}$

b. $y = \sin^3 x$

c. $y = \sin x^3$

SOLUTION »

a. The inner function of $y = (5x + 4)^{3/2}$ is $u = 5x + 4$, and the outer function is $y = u^{3/2}$. Using equation (1) of the Chain Rule, we have

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\
&= \frac{3}{2} u^{1/2} \cdot 5 && y = u^{3/2} \Longrightarrow \frac{dy}{du} = \frac{3}{2} u^{1/2};\ u = 5x + 4 \Longrightarrow \frac{du}{dx} = 5 \\
&= \frac{3}{2} (5x + 4)^{1/2} \cdot 5 && \text{Replace } u \text{ by } 5x + 4. \\
&= \frac{15}{2} \sqrt{5x + 4}.
\end{aligned}$$

b. Replacing the shorthand form $y = \sin^3 x$ with $y = (\sin x)^3$, we identify the inner function as $u = \sin x$. Letting $y = u^3$, we have

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 3\,u^2 \cdot \cos x = \underbrace{3\sin^2 x}_{3\,u^2} \cos x.$$

Note »

With trigonometric functions, expressions such as $\sin^n x$ always mean $(\sin x)^n$, except when $n = -1$. In Example 1, $\sin^3 x = (\sin x)^3$.

c. Although $y = \sin x^3$ appears to be similar to the function $y = \sin^3 x$ in part (b), the inner function in this case is $u = x^3$ and the outer function is $y = \sin u$. Therefore,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (\cos u) \cdot 3\,x^2 = 3\,x^2 \cos x^3.$$

Related Exercises 13–15 ◆

Equation (2) of the Chain Rule, $\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$, is equivalent to equation (1); it just uses different derivative notation. With equation (2), we identify an outer function $y = f(u)$ and an inner function $u = g(x)$. Then, $\frac{d}{dx}(f(g(x)))$ is the product of $f'(u)$ evaluated at $u = g(x)$ and $g'(x)$.

EXAMPLE 2 The Chain Rule

Use equation (2) of the Chain Rule to calculate the derivatives of the following functions.

a. $\left(6\,x^3 + 3\,x + 1\right)^{10}$

b. $\sqrt{5\,x^2 + 1}$

c. $\left(\frac{5\,t^2}{3\,t^2 + 2}\right)^3$

SOLUTION »

a. The inner function of $\left(6\,x^3 + 3\,x + 1\right)^{10}$ is $g(x) = 6\,x^3 + 3\,x + 1$, and the outer function is $f(u) = u^{10}$. The derivative of the outer function is $f'(u) = 10\,u^9$, which when evaluated at $g(x)$ is $10\left(6\,x^3 + 3\,x + 1\right)^9$. The derivative of the inner function is $g'(x) = 18\,x^2 + 3$. Multiplying the derivatives of the outer and inner functions, we have

$$\frac{d}{dx}\left(\left(6\,x^3 + 3\,x + 1\right)^{10}\right) = \underbrace{10\left(6\,x^3 + 3\,x + 1\right)^9}_{f'(u)\text{ evaluated at }g(x)} \cdot \underbrace{\left(18\,x^2 + 3\right)}_{g'(x)}$$

$$= 30\left(6\,x^2 + 1\right)\left(6\,x^3 + 3\,x + 1\right)^9. \quad \text{Factor and simplify.}$$

b. The inner function of $\sqrt{5\,x^2 + 1}$ is $g(x) = 5\,x^2 + 1$, and the outer function is $f(u) = \sqrt{u}$. The derivatives of these functions are $f'(u) = \frac{1}{2\sqrt{u}}$ and $g'(x) = 10\,x$. Therefore,

$$\frac{d}{dx}\sqrt{5x^2+1} = \underbrace{\frac{1}{2\sqrt{5x^2+1}}}_{\substack{f'(u)\text{ evaluated}\\ \text{at } g(x)}} \cdot \underbrace{10x}_{g'(x)} = \frac{5x}{\sqrt{5x^2+1}}.$$

c. The inner function of $\left(\frac{5t^2}{3t^2+2}\right)^3$ is $g(t) = \frac{5t^2}{3t^2+2}$. The outer function is $f(u) = u^3$, whose derivative is $f'(u) = 3u^2$. The derivative of the inner function requires the Quotient Rule. Applying the Chain Rule, we have

$$\frac{d}{dt}\left(\frac{5t^2}{3t^2+2}\right)^3 = \underbrace{3\left(\frac{5t^2}{3t^2+2}\right)^2}_{\substack{f'(u)\text{ evaluated}\\ \text{at } g(t)}} \cdot \underbrace{\frac{(3t^2+2)\,10t - 5t^2(6t)}{(3t^2+2)^2}}_{g'(t)\text{ by the Quotient Rule}} = \frac{1500t^5}{(3t^2+2)^4}.$$

Related Exercises 26, 27, 31 ◆

The Chain Rule is also used to calculate the derivative of a composite function for a specific value of the variable. If $h(x) = f(g(x))$ and a is a real number, the $h'(a) = f'(g(a))\,g'(a)$, provided the necessary derivatives exist. Therefore, $h'(a)$ is the derivative of f evaluated at $g(a)$ multiplied by the derivative of g evaluated at a.

EXAMPLE 3 Calculating derivatives at a point

Let $h(x) = f(g(x))$. Use the values in Table 3.2 to calculate $h'(1)$ and $h'(2)$.

Table 3.2

x	$f'(x)$	$g(x)$	$g'(x)$
1	5	2	3
2	7	1	4

SOLUTION »

We use $h'(a) = f'(g(a))\,g'(a)$ with $a = 1$:

$$h'(1) = f'(g(1))\,g'(1) = f'(2)\,g'(1) = 7 \cdot 3 = 21.$$

With $a = 2$, we have

$$h'(2) = f'(g(2))\,g'(2) = f'(1)\,g'(2) = 5 \cdot 4 = 20.$$

Related Exercises 23–24 ◆

EXAMPLE 4 Applying the Chain Rule

A trail runner programs her GPS unit to record her altitude a (in feet) every 10 minutes during a training run in the mountains; the resulting data are shown in Table 3.3. Meanwhile, at a nearby weather station, a weather probe records the atmospheric pressure p (in hectopascals, or hPa) at various altitudes, shown in Table 3.4.

Table 3.3

t **(minutes)**	0	10	20	30	40	50	60	70	80
***a*(*t*)** **(altitude)**	10,000	10,220	10,510	10,980	11,660	12,330	12,710	13,330	13,440

Table 3.4

a **(altitude)**	5485	7795	10,260	11,330	12,330	13,330	14,330	15,830	16,230
***p*(*a*)** **(pressure)**	1000	925	840	821	793	765	738	700	690

Use the Chain Rule to estimate the rate of change in pressure per unit time experienced by the trail runner when she is 50 minutes into her run.

SOLUTION »

We seek the rate of change in the pressure $\frac{dp}{dt}$, which is given by the Chain Rule:

$$\frac{dp}{dt} = \frac{dp}{da}\frac{da}{dt}.$$

The runner is at an altitude of 12,330 feet 50 minutes into her run, so we must compute $\frac{dp}{da}$ when $a = 12{,}330$ and $\frac{da}{dt}$ when $t = 50$. These derivatives can be approximated using the following forward difference quotients.

$$\left.\frac{dp}{da}\right|_{a=12{,}330} \approx \frac{p(12{,}330+1000)-p(12{,}330)}{1000} = \frac{765-793}{1000} = -0.028\,\frac{\text{hPa}}{\text{ft}}$$

$$\left.\frac{dp}{dt}\right|_{t=50} \approx \frac{a(50+10)-a(50)}{10} = \frac{12{,}710-12{,}330}{10} = 38.0\,\frac{\text{ft}}{\text{min}}$$

Note »

The difference quotient $\frac{p(a+h)-p(a)}{h}$ is a *forward difference quotient* when $h>0$ (see Exercises 62–65 in Section 3.1).

We now compute the rate of change of the pressure with respect to time:

$$\frac{dp}{dt} = \frac{dp}{da}\frac{da}{dt} \approx -0.028\,\frac{\text{hPa}}{\text{ft}}\cdot 38.0\,\frac{\text{ft}}{\text{min}} = -1.06\,\frac{\text{hPa}}{\text{min}}.$$

As expected, $\frac{dp}{dt}$ is negative because the pressure decreases with increasing altitude (Table 3.4) as the runner ascends the trail. Note also that the units are consistent.

Related Exercises 72–73 ◆

Chain Rule for Powers »

The Chain Rule leads to a general derivative rule for powers of differentiable functions. In fact, we have already used it in several examples. Consider the function $f(x)=(g(x))^p$, where p is a real number. Letting $f(u)=u^n$ be the outer function and $u=g(x)$ be the inner function, we obtain the Chain Rule for powers of functions.

THEOREM 3.13 **Chain Rule for Powers**

If g is differentiable for all x in its domain and p is a real number, then

$$\frac{d}{dx}\left((g(x))^p\right) = p\,(g(x))^{p-1}\,g'(x).$$

EXAMPLE 5 **Chain Rule for powers**

Find $\frac{d}{dx}(\tan x+10)^{21}$.

SOLUTION »

With $g(x)=\tan x+10$, the Chain Rule gives

$$\frac{d}{dx}(\tan x+10)^{21} = 21\,(\tan x+10)^{20}\,\frac{d}{dx}(\tan x+10) = 21\,(\tan x+10)^{20}\sec^2 x.$$

Related Exercises 43–44 ◆

The Composition of Three or More Functions »

We can differentiate the composition of three or more functions by applying the Chain Rule repeatedly, as shown in the following example.

EXAMPLE 6 Composition of three functions

Calculate the derivative of $\sin\left(\cos x^2\right)$.

SOLUTION

The inner function of $\sin\left(\cos x^2\right)$ is $\cos x^2$. Because $\cos x^2$ is also a composition of two functions, the Chain Rule is used again to calculate $\frac{d}{dx}\left(\cos x^2\right)$, where x^2 is the inner function:

$$\begin{aligned}
\frac{d}{dx}\Big(\underbrace{\sin}_{\text{outer}}\underbrace{\left(\cos x^2\right)}_{\text{inner}}\Big) &= \cos\left(\cos x^2\right)\frac{d}{dx}\left(\cos x^2\right) && \text{Chain Rule}\\
&= \cos\left(\cos x^2\right)\underbrace{\left(-\sin x^2\right)\cdot\frac{d}{dx}\left(x^2\right)}_{\frac{d}{dx}\left(\cos x^2\right)} && \text{Chain Rule}\\
&= \cos\left(\cos x^2\right)\cdot\left(-\sin x^2\right)\cdot 2x && \text{Differentiate } x^2.\\
&= -2x\cos\left(\cos x^2\right)\cdot\sin x^2. && \text{Simplify.}
\end{aligned}$$

Related Exercises 49–50 ◆

Quick Check 3 Let $y = \tan^{10}\left(x^5\right)$. Find f, g, and h such that $y = f(u)$, where $u = g(v)$ and $v = h(x)$. ◆

Answer »

> $f(u) = u^{10}$; $u = g(v) = \tan v$; $v = h(x) = x^5$

The Chain Rule is often used in combination with the other derivative rules you have learned. Example 7 illustrates how several differentiation rules are combined.

EXAMPLE 7 Combining rules

Find $\frac{d}{dx}\left(x^2\sqrt{x^2+1}\right)$.

SOLUTION »

The given function is the produce of x^2 and $\sqrt{x^2+1}$, and $\sqrt{x^2+1}$ is a composite function. We apply the Product Rule and then the Chain Rule:

$$\frac{d}{dx}\left(x^2\sqrt{x^2+1}\right) = \underbrace{\frac{d}{dx}\left(x^2\right)}_{2x}\cdot\sqrt{x^2+1} + x^2\cdot\underbrace{\frac{d}{dx}\left(\sqrt{x^2+1}\right)}_{\text{Use Chain Rule}} \quad \text{Product Rule}$$

$$= 2x\sqrt{x^2+1} + x^2\cdot\frac{1}{2\sqrt{x^2+1}}\cdot 2x \quad \text{Chain Rule}$$

$$= 2x\sqrt{x^2+1} + \frac{x^3}{\sqrt{x^2+1}} \quad \text{Simplify.}$$

$$= \frac{3x^3+2x}{\sqrt{x^2+1}}. \quad \text{Simplify.}$$

Related Exercises 62–63 ◆

Proof of the Chain Rule »

Suppose f is differentiable at $u = g(a)$, g is differentiable at a, and $h(x) = f(g(x))$. By the definition of the derivative of h,

$$h'(a) = \lim_{x\to a}\frac{h(x)-h(a)}{x-a} = \lim_{x\to a}\frac{f(g(x))-f(g(a))}{x-a}. \tag{3}$$

We assume that $g(a) \neq g(x)$ for values of x near a but not equal to a. This assumption holds for most, but not all, functions encountered in this text. For a proof of the Chain Rule without this assumption, see Exercise 101.

We multiply the right side of equation (3) by $\frac{g(x)-g(a)}{g(x)-g(a)}$, which equals 1, and let $v = g(x)$ and $u = g(a)$. The result is

$$\begin{aligned} h'(a) &= \lim_{x\to a}\frac{f(g(x))-f(g(a))}{g(x)-g(a)}\cdot\frac{g(x)-g(a)}{x-a} \\ &= \lim_{x\to a}\frac{f(v)-f(u)}{v-u}\cdot\frac{g(x)-g(a)}{x-a}. \end{aligned}$$

By assumption, g is a differentiable at a; therefore, it is continuous at a. This means that $\lim_{x\to a} g(x) = g(a)$, so $v \to u$ as $x \to a$. Consequently,

$$h'(a) = \underbrace{\lim_{v\to u}\frac{f(v)-f(u)}{v-u}}_{f'(u)}\cdot\underbrace{\lim_{x\to a}\frac{g(x)-g(a)}{x-a}}_{g'(a)} = f'(u)\,g'(a).$$

Because f and g are differentiable at u and a, respectively, the two limits in this expression exist; therefore $h'(a)$ exists. Noting that $u = g(a)$, we have $h'(a) = f'(g(a))\,g'(a)$. Replacing a with the variable x gives the Chain Rule: $h'(x) = f'(g(x))\,g'(x)$. ◆

Exercises »

Getting Started »

Practice Exercises »

13–22. *For each of the following composite functions, find an inner function $u = g(x)$ and an outer function $y = f(u)$ such that $y = f(g(x))$. Then calculate $\frac{dy}{dx}$.*

13. $y = (3x + 7)^{10}$

14. $y = \left(5x^2 + 11x\right)^{4/3}$

15. $y = \sin^5 x$

16. $y = \sin x^5$

17. $y = \sqrt{x^2 + 1}$

18. $y = \sqrt{7x - 1}$

19. $y = \sqrt{2 + \sin x}$

20. $y = (\cos x + \sin x)^2$

21. $y = \tan 5x^2$

22. $y = \sin \frac{x}{4}$

23. **Derivatives using tables** Let $h(x) = f(g(x))$ and $p(x) = g(f(x))$. Use the table to compute the following derivatives.

a. $h'(3)$
b. $h'(2)$
c. $p'(4)$
d. $p'(2)$
e. $h'(5)$

x	1	2	3	4	5
$f(x)$	0	3	5	1	0
$f'(x)$	5	2	−5	−8	−10
$g(x)$	4	5	1	3	2
$g'(x)$	2	10	20	15	20

24. **Derivatives using tables** Let $h(x) = f(g(x))$ and $k(x) = g(g(x))$. Use the table to compute the following derivatives.

a. $h'(1)$
b. $h'(2)$
c. $h'(3)$
d. $k'(3)$

e. $k'(1)$

f. $k'(5)$

x	1	2	3	4	5
$f'(x)$	−6	−3	8	7	2
$g(x)$	4	1	5	2	3
$g'(x)$	9	7	3	−1	−5

25–70. *Calculate the derivative of the following functions.*

25. $y = \left(3x^2 + 7x\right)^{10}$

26. $y = \left(x^2 + 2x + 7\right)^8$

27. $y = \sqrt{10x + 1}$

28. $y = \sqrt[3]{x^2 + 9}$

29. $y = 5\left(7x^3 + 1\right)^{-3}$

30. $y = \cos 5t$

31. $y = \sec(3x + 1)$

32. $y = \csc \sqrt{x}$

33. $y = \tan \sqrt{w}$

34. $y = \sin(\cos x)$

35. $y = \sin\left(4x^3 + 3x + 1\right)$

36. $y = \csc\left(t^2 + t\right)$

37. $y = (5x + 1)^{2/3}$

38. $y = x(x + 1)^{1/3}$

39. $y = \sqrt[4]{\dfrac{2x}{4x - 3}}$

40. $y = \cos^4 \theta + \sin^4 \theta$

41. $y = (\sec x + \tan x)^5$

42. $y = \sin(4 \cos z)$

43. $y = \left(2x^6 - 3x^3 + 3\right)^{25}$

44. $y = (\cos x + 2 \sin x)^8$

45. $y = (1 + 2\tan u)^{4.5}$

46. $y = \left(1 - \sqrt{x}\right)^4$

47. $y = \sqrt{1 + \cot^2 x}$

48. $y = \sqrt{(3x - 4)^2 + 3x}$

49. $y = \left(\sin^2 x + 1\right)^4$

50. $y = \left(\sin x^2 + 1\right)^4$

51. $y = \sin^5(\cos 3x)$

52. $y = \cos^{7/4}\left(4x^3\right)$

53. $y = \sqrt{x + \sqrt{x}}$

54. $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$

55. $y = f\left(g\left(x^2\right)\right)$, where f and g are differentiable for all real numbers

56. $y = (f(g(x^m)))^n$, where f and g are differentiable for all real numbers and m and n are constants

57. $y = \left(\frac{x}{x + 1}\right)^5$

58. $y = \left(\frac{x - 1}{x + 1}\right)^8$

59. $y = x\left(x^2 + 1\right)^3$

60. $y = \frac{x}{\left(x^2 + 1\right)^2}$

61. $y = \theta^2 \sec 5\theta$

62. $y = \left(\frac{3x}{4x + 2}\right)^5$

63. $y = \left((x + 2)\left(x^2 + 1\right)\right)^4$

64. $y = \left(\frac{\sin x}{\sin x + 1}\right)^2$

65. $y = \sqrt[5]{x^4 + \cos 2x}$

66. $y = \frac{x\cos x}{\sin x + 1}$

67. $y = (p+3)^2 \sin p^2$

68. $y = (2z+5)^{1.75} \tan z$

69. $y = \sqrt{f(x)}$, where f is differentiable at x

70. $y = \sqrt[5]{f(x)\,g(x)}$, where f and g are differentiable at x

71. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. The function $f(x) = x \sin x$ can be differentiated without using the Chain Rule.

b. The function $f(x) = \left(x^2+1\right)^{-12}$ should be differentiated using the Chain Rule.

c. The derivative of a product is *not* the product of the derivatives, but the derivative of a composition is a product of derivatives.

d. $\dfrac{d}{dx} P(Q(x)) = P'(x)\,Q'(x)$

72. Changing temperature The *lapse rate* is the rate at which the temperature in Earth's atmosphere decreases with altitude. For example, a lapse rate of 6.5° Celsius/km means the temperature *decreases* at a rate of 6.5°C per kilometer of altitude. The lapse rate varies with location and with other variables such as humidity. However, at a given time and location, the lapse rate is often nearly constant in the first 10 kilometers of the atmosphere. A radiosonde (weather balloon) is released from Earth's surface, and its altitude (measured in km above sea level) at various times (measured in hours) is given in the table below.

Time (hr)	0	0.5	1	1.5	2	2.5
Altitude (km)	0.5	1.2	1.7	2.1	2.5	2.9

a. Assuming a lapse rate of 6.5°C/km, what is the approximate rate of change of the temperature with respect to time as the balloon rises 1.5 hours into the flight? Specify the units of your result and use a forward difference quotient when estimating the required derivative.

b. How does an increase in lapse rate change your answer in part (a)?

c. Is it necessary to know the actual temperature to carry out the calculation in part (a)? Explain.

73. Applying the Chain Rule Use the data in Tables 3.3 and 3.4 of Example 4 to estimate the rate of change in pressure with respect to time experienced by the runner when she is at an altitude of 13,330 ft. Make use of a forward difference quotient when estimating the required derivatives.

74–77. Second derivatives *Find* $\dfrac{d^2y}{dx^2}$ *for the following functions.*

74. $y = x \cos x^2$

75. $y = \sin x^2$

76. $y = \sqrt{x^2+2}$

77. $y = \left(x^2+1\right)^{-2}$

78. Derivatives by different methods

a. Calculate $\frac{d}{dx}\left(x^2+x\right)^2$ using the Chain Rule. Simplify your answer.

b. Expand $\left(x^2+x\right)^2$ first and then calculate the derivative. Verify that your answer agrees with part (a).

79. Tangent lines Determine an equation of the line tangent to the graph of $y=\frac{\left(x^2-1\right)^2}{x^3-6x-1}$ at the point $(0, -1)$.

T **80. Tangent lines** Determine equations of the lines tangent to the graph of $y = x\sqrt{5-x^2}$ at the points $(1, 2)$ and $(-2, -2)$. Graph the function and the tangent lines.

81. Tangent lines Assume f and g are differentiable on their domains with $h(x) = f(g(x))$. Suppose the equation of the line tangent to the graph of g at the point $(4, 7)$ is $y = 3x - 5$ and the equation of the line tangent to the graph of f at $(7, 9)$ is $y = -2x + 23$.

a. Calculate $h(4)$ and $h'(4)$.

b. Determine an equation of the line tangent to the graph of h at $(4, h(4))$.

82. Tangent lines Assume f is a differentiable function whose graph passes through the point $(1, 4)$. Suppose $g(x) = f\left(x^2\right)$ and the line tangent to the graph of f at $(1, 4)$ is $y = 3x + 1$. Find each of the following.

a. $g(1)$

b. $g'(x)$

c. $g'(1)$

d. An equation of the line tangent to the graph of g when $x = 1$

83. Tangent lines Find the equation of the line tangent to $y = \sec 2x$ at $x = \frac{\pi}{6}$. Graph the function and the tangent line.

84. Composition containing sin *x* Suppose f is differentiable on $[-2, 2]$ with $f'(0) = 3$ and $f'(1) = 5$. Let $g(x) = f(\sin x)$. Evaluate the following expressions.

a. $g'(0)$

b. $g'\left(\frac{\pi}{2}\right)$

c. $g'(\pi)$

85. Composition containing sin *x* Suppose f is differentiable for all real numbers with $f(0) = -3$, $f(1) = 3$, $f'(0) = 3$, and $f'(1) = 5$. Let $g(x) = \sin(\pi f(x))$. Evaluate the following expressions.

a. $g'(0)$

b. $g'(1)$

86–88. Vibrations of a spring *Suppose an object of mass m is attached to the end of a spring hanging from the ceiling. The mass is at its equilibrium position $y = 0$ when the mass hangs at rest. Suppose you push the mass to a position y_0 units above its equilibrium position and release it. As the mass oscillates up and down (neglecting any friction in the system), the position y of the mass after t seconds is*

$$y = y_0 \cos\left(t\sqrt{\frac{k}{m}}\right), \tag{4}$$

where $k > 0$ is a constant measuring the stiffness of the spring (the larger the value of k, the stiffer the spring) and y is positive in the upward direction.

86. Use equation (4) to answer the following questions.

a. Find $\frac{dy}{dt}$, the velocity of the mass. Assume k and m are constant.

b. How would the velocity be affected if the experiment were repeated with four times the mass on the end of the spring?

c. How would the velocity be affected if the experiment were repeated with a spring having four times the stiffness (k is increased by a factor of 4)?

d. Assume y has units of meters, t has units of seconds, m has units of kg and k has units of kg/s^2. Show that the units of the velocity in part (a) are consistent.

87. Use equation (4) to answer the following questions.

a. Find the second derivative $\frac{d^2y}{dt^2}$.

b. Verify that $\frac{d^2y}{dt^2} = -\frac{k}{m}y$.

88. Use equation (4) to answer the following questions.

a. The *period* T is the time required by the mass to complete one oscillation. Show that $T = 2\pi\sqrt{\frac{m}{k}}$.

b. Assume k is constant and calculate $\frac{dT}{dm}$.

c. Give a physical explanation of why $\frac{dT}{dm}$ is positive.

T **89. Hours of daylight** The number of hours of daylight at any point on Earth fluctuates throughout the year. In the northern hemisphere, the shortest day is on the winter solstice and the longest day is on the summer solstice. At 40° north latitude, the length of a day is approximated by

$$D(t) = 12 - 3\cos\left(\frac{2\pi(t+10)}{365}\right),$$

where D is measured in hours and $0 \le t \le 365$ is measured in days, with $t = 0$ corresponding to January 1.

a. Approximately how much daylight is there on March 1 ($t = 59$)?

b. Find the rate at which the daylight function changes.

c. Find the rate at which the daylight function changes on March 1. Convert your answer to units of min/day and explain what this result means.

d. Graph the function $y = D'(t)$ using a graphing utility.

e. At what times of the year is the length of day changing most rapidly? Least rapidly?

Explorations and Challenges »

T **90.** **A mixing tank** A 500-liter (L) tank is filled with pure water. At time $t = 0$, a salt solution begins flowing into the tank at a rate of 5 L/min. At the same time, the (fully mixed) solution flows out of the tank at a rate of 5.5 L/min. The mass of salt in grams in the tank at any time $t \geq 0$ is given by

$$M(t) = 250\,(1000 - t)\left(1 - 10^{-30}\,(1000 - t)^{10}\right)$$

and the volume of solution in the tank is given by

$$V(t) = 500 - 0.5\,t.$$

a. Graph the mass function and verify that $M(0) = 0$.

b. Graph the volume function and verify that the tank is empty when $t = 1000$ min.

c. The concentration of the salt solution in the tank (in g/L) is given by $C(t) = M(t)/V(t)$. Graph the concentration function and comment on its properties. Specifically, what are $C(0)$ and $\lim\limits_{t \to 1000^-} C(t)$?

d. Find the rate of change of the mass $M'(t)$, for $0 \leq t \leq 1000$.

e. Find the rate of change of the concentration $C'(t)$, for $0 \leq t \leq 1000$.

f. For what times is the concentration of the solution increasing? Decreasing?

T **91.** **Power and energy** The total energy in megawatt-hr (MWh) used by a town is given by

$$E(t) = 400\,t + \frac{2400}{\pi}\sin\frac{\pi\,t}{12},$$

where $t \geq 0$ is measured in hours, with $t = 0$ corresponding to noon.

a. Find the power, or rate of energy consumption, $P(t) = E'(t)$ in units of megawatts (MW).

b. At what time of day is the rate of energy consumption a maximum? What is the power at that time of day?

c. At what time of day is the rate of energy consumption a minimum? What is the power at that time of day?

d. Sketch a graph of the power function reflecting the times when energy use is a minimum or a maximum.

92. **Deriving trigonometric identities**

a. Differentiate both sides of the identity $\cos 2t = \cos^2 t - \sin^2 t$ to prove that $\sin 2t = 2\sin t\cos t$.

b. Verify that you obtain the same identity for $\sin 2t$ as in part (a) if you differentiate the identity $\cos 2t = 2\cos^2 t - 1$.

c. Differentiate both sides of the identity $\sin 2t = 2\sin t\cos t$ to prove that $\cos 2t = \cos^2 t - \sin^2 t$.

93. **Quotient Rule derivation** Suppose you forgot the Quotient Rule for calculating $\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right)$. Use the Chain Rule and Product Rule with the identity $\dfrac{f(x)}{g(x)} = f(x)\,(g(x))^{-1}$ to derive the Quotient Rule.

94. **The Chain Rule for second derivatives**

a. Derive a formula for the second derivative, $\dfrac{d^2}{dx^2}(f(g(x)))$.

b. Use the formula in part (a) to calculate $\dfrac{d^2}{dx^2}\left(\sin\left(3x^4 + 5x^2 + 2\right)\right)$.

T 95–98. Calculating limits *The following limits are the derivatives of a composite function g at a point a.*

a. *Find a possible function g and number a.*

b. *Use the Chain Rule to find each limit. Verify your answer by using a calculator.*

95. $\displaystyle\lim_{x\to 2} \frac{\left(x^2 - 3\right)^5 - 1}{x - 2}$

96. $\displaystyle\lim_{x\to 0} \frac{\sqrt{4 + \sin x} - 2}{x}$

97. $\displaystyle\lim_{h\to 0} \frac{\sin(\pi/2 + h)^2 - \sin\left(\pi^2/4\right)}{h}$

98. $\displaystyle\lim_{h\to 0} \frac{\frac{1}{3\left((1+h)^5+7\right)^{10}} - \frac{1}{3(8)^{10}}}{h}$

99. Limit of a difference quotient Assuming f is differentiable for all x, simplify $\displaystyle\lim_{x\to 5} \frac{f\left(x^2\right) - f(25)}{x - 5}$.

100. Derivatives of even and odd functions Recall that f is even if $f(-x) = f(x)$, for all x in the domain of f, and f is odd if $f(-x) = -f(x)$, for all x in the domain of f.

a. If f is a differentiable, even function on its domain, determine whether f' is even, odd, or neither.

b. If f is a differentiable, odd function on its domain, determine whether f' is even, odd, or neither.

101. A general proof of the Chain Rule Let f and g be differentiable functions with $h(x) = f(g(x))$. For a given constant a, let $u = g(a)$ and $v = g(x)$, and define

$$H(v) = \begin{cases} \dfrac{f(v) - f(u)}{v - u} - f'(u) & \text{if } v \neq u \\ 0 & \text{if } v = u. \end{cases}$$

a. Show that $\displaystyle\lim_{v\to u} H(v) = 0$.

b. For any value of u, show that $f(v) - f(u) = (H(v) + f'(u))(v - u)$.

c. Show that $h'(a) = \displaystyle\lim_{x\to a}\left((H(g(x)) + f'(g(a)))\cdot\frac{g(x) - g(a)}{x - a}\right)$.

d. Show that $h'(a) = f'(g(a))\, g'(a)$.