3.6 Derivatives as Rates of Change

The theme of this section is the *derivative as a rate of change*. Observing the world around us, we see that almost everything is in a state of change: The size of the Internet is increasing; your blood pressure fluctuates; as supply increases, prices decrease; and the universe is expanding. This section explores a few of the many applications of this idea and demonstrates why calculus is called the mathematics of change.

One-Dimensional Motion »

Describing the motion of objects such as projectiles and planets was one of the challenges that led to the development of calculus in the 17th century. We begin by considering the motion of an object confined to one dimension; that is, the object moves along a line. This motion could be horizontal (for example, a car moving along a straight highway) or it could be vertical (such as a projectile launched vertically into the air).

Note »

When describing the motion of objects, it is customary to use t as the independent variable to represent time. Generally, motion is assumed to begin at $t = 0$.

Position and Velocity

Suppose an object moves along a straight line and its location at time t is given by the **position function** $s = f(t)$. All positions are measured relative to a reference point, which is often the origin at $s = 0$. The **displacement** of the object between $t = a$ and $t = a + \Delta t$ is $\Delta s = f(a + \Delta t) - f(a)$, where the elapsed time is Δt units (**Figure 3.36**).

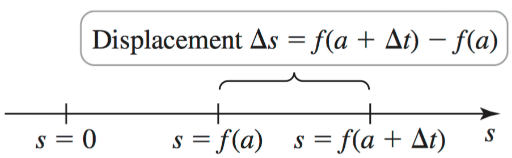


Figure 3.36

Recall from Section 2.1 that the *average velocity* of the object over the interval $[a, a + \Delta t]$ is the displacement Δs of the object divided by the elapsed time Δt:

$$v_{\text{av}} = \frac{\Delta s}{\Delta t} = \frac{f(a + \Delta t) - f(a)}{\Delta t}.$$

The average velocity is the slope of the secant line passing through the points $P(a, f(a))$ and $Q(a + \Delta t, f(a + \Delta t))$ (**Figure 3.37**).

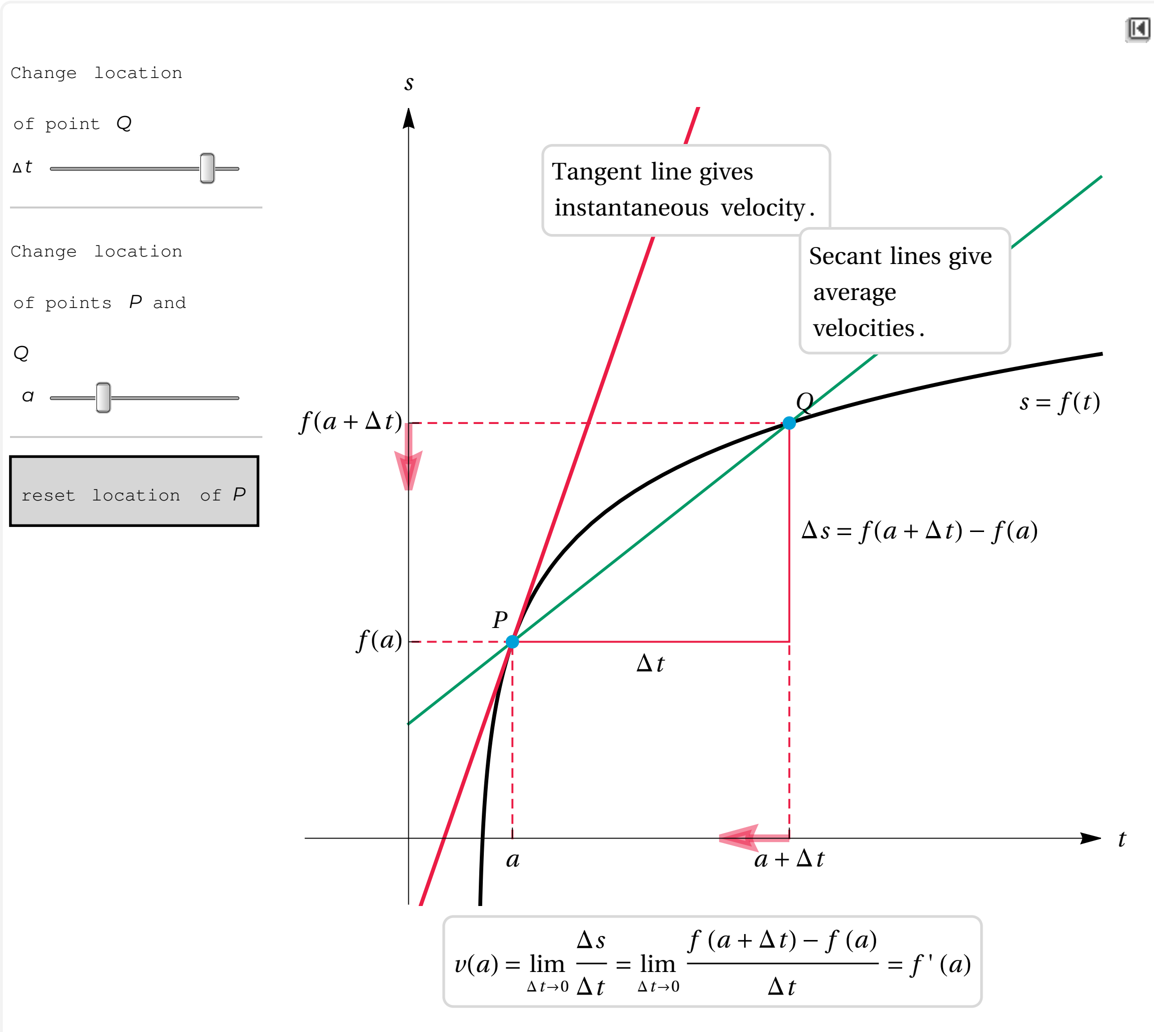


Figure 3.37

As Δt approaches 0, the average velocity is calculated over smaller and smaller time intervals, and the limiting value of these average velocities, when it exists, is the *instantaneous velocity* at a. This is the same argument used to arrive at the derivative. The conclusion is that the instantaneous velocity at time a, denoted $v(a)$, is the derivative of the position function evaluated at a:

$$v(a) = \lim_{\Delta t \to 0} \frac{f(a + \Delta t) - f(a)}{\Delta t} = f'(a).$$

Equivalently, the instantaneous velocity at a is the rate of change in the position function at a; it also equals the slope of the line tangent to the curve $s = f(t)$ at $P(a, f(a))$.

DEFINITION **Average and Instantaneous Velocity**

Let $s = f(t)$ be the position function of an object moving along a line. The **average velocity** of the object over the time interval $[a, a + \Delta t]$ is the slope of the secant line between $(a, f(a))$ and $(a + \Delta t, f(a + \Delta t))$:

$$v_{\text{av}} = \frac{f(a + \Delta t) - f(a)}{\Delta t}.$$

The **instantaneous velocity** at a is the slope of the line tangent to the position curve, which is the derivative of the position function:

$$v(a) = \lim_{\Delta t \to 0} \frac{f(a + \Delta t) - f(a)}{\Delta t} = f'(a).$$

Note »

Using the various derivative notations, the velocity is also written $v(t) = s'(t) = \frac{ds}{dt}$. If *average* or *instantaneous* is not specified, *velocity* is understood to mean instantaneous velocity.

Quick Check 1 Does the speedometer in your car measure average or instantaneous velocity? ◆

Answer »

Instantaneous velocity.

EXAMPLE 1 **Position and velocity of a patrol car**

Assume a police station is located along a straight east-west freeway. At noon $(t = 0)$, a patrol car leaves the station heading east. The position function of the car $s = f(t)$ gives the location of the car in miles east $(s > 0)$ or west $(s < 0)$ of the station t hours after noon (**Figure 3.38**).

a. Describe the location of the patrol car during the first 3.5 hr of the trip.

b. Calculate the displacement and average velocity of the car between 2:00 P.M. and 3:30 P.M. $(2 \le t \le 3.5)$.

c. At what time(s) is the instantaneous velocity greatest *as the car travels east*?

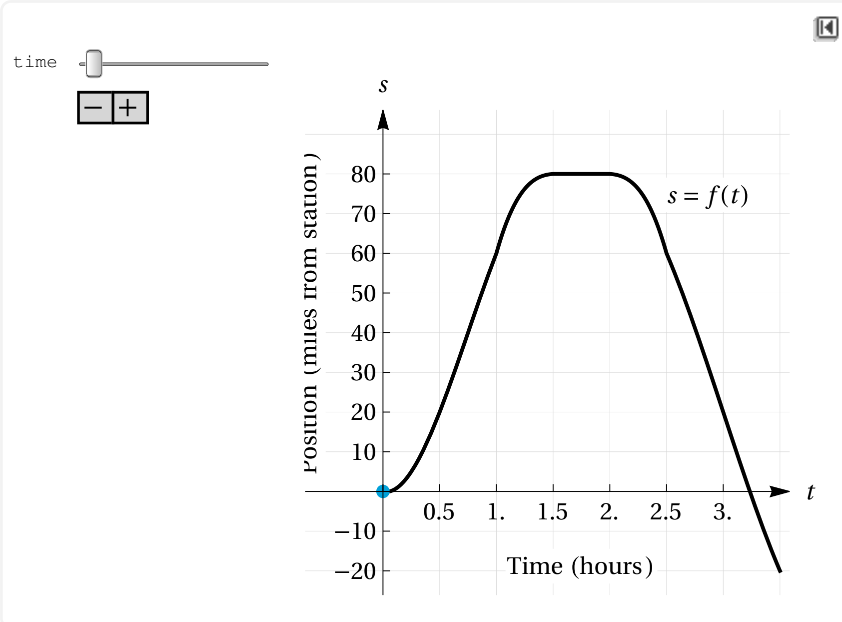


Figure 3.38

SOLUTION »

a. The graph of the position function indicates the car travels 80 mi east between $t = 0$ (noon) and $t = 1.5$ (1:30 P.M.). The car is at rest and its position does not change from $t = 1.5$ to $t = 2$ (that is, from 1:30 P.M. to 2:00 P.M.). Starting at $t = 2$, the car's distance from the station decreases, which means the car travels west, eventually ending up 20 miles west of the station at $t = 3.5$ (3:30 P.M.) (**Figure 3.39**).

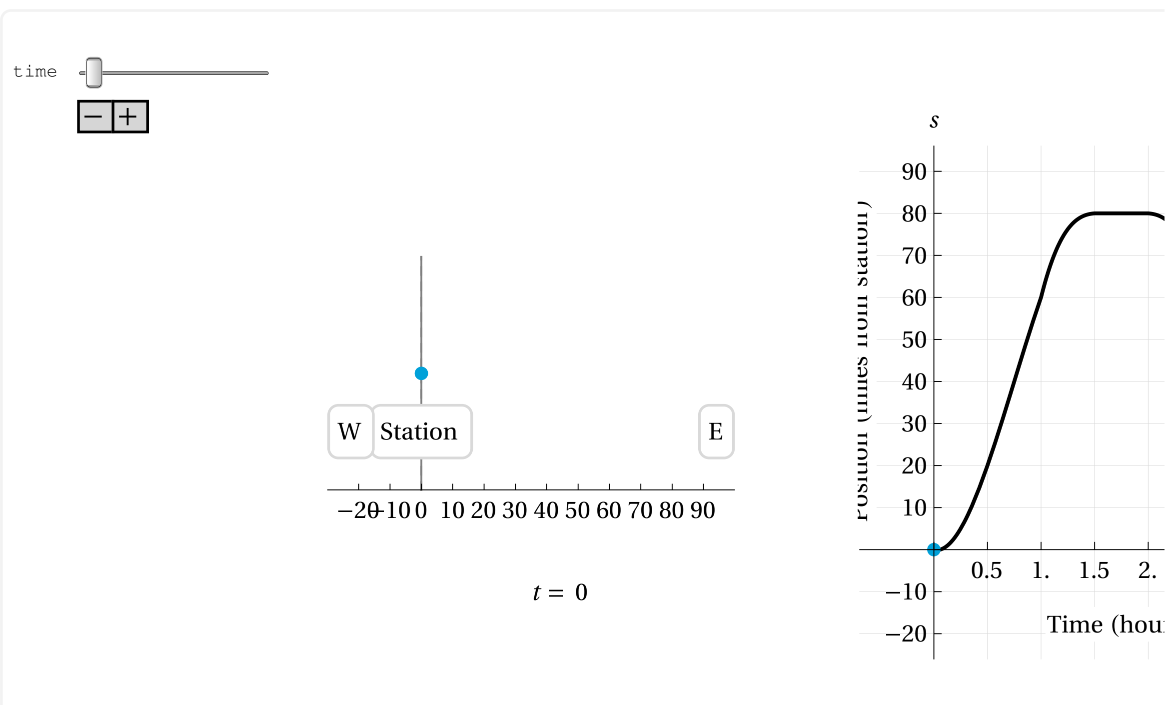


Figure 3.39

b. The position of the car at 3:30 P.M. is $f(3.5) = -20$ (the negative sign indicates the car is 20 miles *west* of the station), and the position of the car at 2:00 P.M. is $f(2) = 80$. Therefore, the displacement is

$$\Delta s = f(3.5) - f(2) = -20 \text{ mi} - 80 \text{ mi} = -100 \text{ mi}$$

during an elapsed time of $\Delta t = 3.5 - 2 = 1.5$ hr (the *negative* displacement indicates that the car moved 100 miles *west*). The average velocity is

$$v_{\text{av}} = \frac{\Delta s}{\Delta t} = \frac{-100 \text{ mi}}{1.5 \text{ hr}} \approx -66.7 \text{ mi/hr}.$$

c. The greatest eastward instantaneous velocity corresponds to points at which the graph of the position function has the greatest positive slope. The greatest slope appears to occur between $t = 0.5$ and $t = 1$. During this time interval, the car also has a nearly constant velocity because the curve is approximately linear. We conclude that the eastward velocity is largest from 12:30 P.M. to 1:00 P.M.

Related Exercises 11–12 ◆

Speed and Acceleration

When only the magnitude of the velocity is of interest, we use *speed*, which is the absolute value of the velocity:

$$\text{speed} = |v|.$$

For example, a car with an instantaneous velocity of −30 mi/hr has a speed of 30 mi/hr.

A more complete description of an object moving along a line includes its *acceleration*, which is the rate of change of the velocity; that is, acceleration is the derivative of the velocity function with respect to time t. If the acceleration is positive, the object's velocity increases; if it is negative, the object's velocity decreases. Because velocity is the derivative of the position function, acceleration is the second derivative of the position. Therefore,

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

Note »

> Newton's first law of motion says that in the absence of external forces, a moving object has no acceleration, which means the magnitude and direction of the velocity are constant.

DEFINITION **Velocity, Speed, and Acceleration**

Suppose an object moves along a line with position $s = f(t)$. Then,

the **velocity** at time t is $v = \frac{ds}{dt} = f'(t)$,

the **speed** at time t is $|v| = |f'(t)|$, and

the **acceleration** at time t is $a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = f''(t)$.

Quick Check 2 For an object moving along a line, is it possible for its velocity to increase while its speed decreases? Is it possible for its velocity to decrease while its speed increases? Give an example to support your answers. ◆

Answer »

Yes; yes

Note »

The units of derivatives are consistent with the notation. If s is measured in meters and t is measured in seconds, the units of the velocity $\frac{ds}{dt}$ are m/s. The units of the acceleration $\frac{d^2s}{dt^2}$ are $\mathrm{m/s^2}$.

EXAMPLE 2 Velocity and acceleration

Suppose the position (in feet) of an object moving horizontally at time t (in seconds) is $s = t^2 - 5t$ for $0 \le t \le 5$ (**Figure 3.40**). Assume that positive values of s correspond to positions to the right of $s = 0$.

a. Graph the velocity function on the interval $0 \le t \le 5$, and determine when the object is stationary, moving to the left, and moving to the right.

b. Graph the acceleration function on the interval $0 \le t \le 5$, and determine the acceleration of the object when its velocity is zero.

c. Describe the motion of the object.

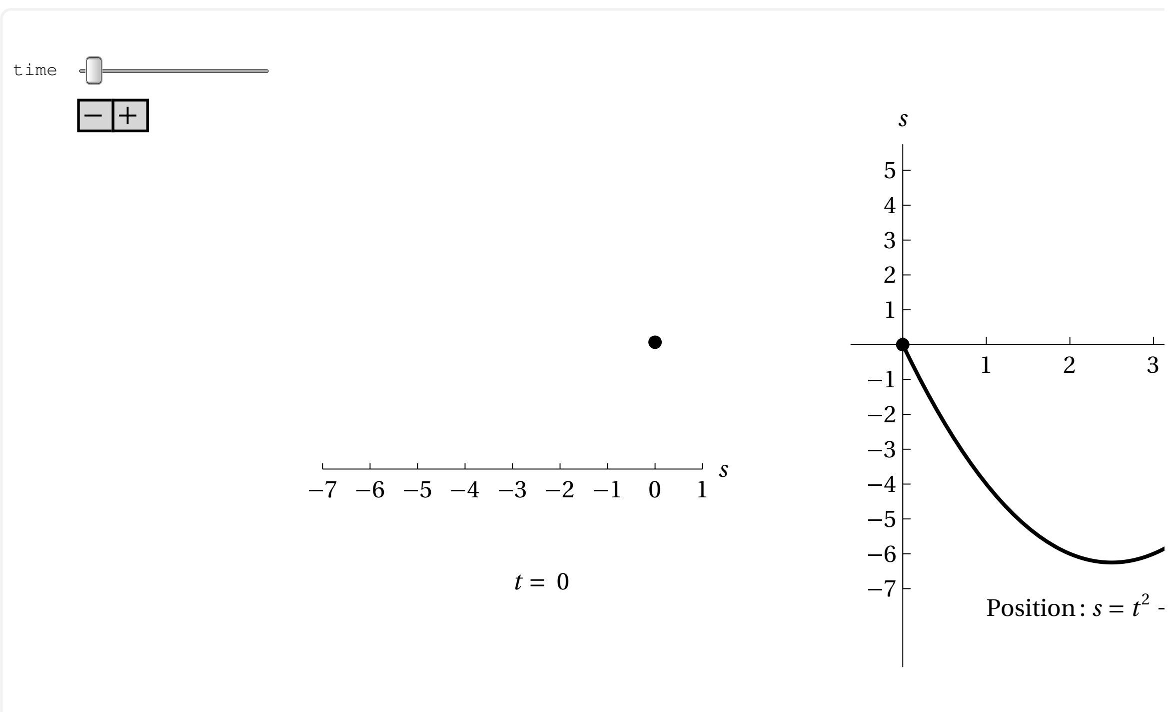


Figure 3.40

Note »

> The right side of Figure 3.40 gives the graph of the position function, not the path of the object. The motion is along a horizontal line as shown on the left side of the figure.

SOLUTION »

a. The velocity is $v = s'(t) = 2t - 5$. The object is stationary when $v = 2t - 5 = 0$, or at $t = 2.5$ s. Solving $v = 2t - 5 > 0$, the velocity is positive (motion to the right) for $\frac{5}{2} < t \leq 5$. Similarly, the velocity is negative (motion to the left) for $0 \leq t < \frac{5}{2}$. Though the velocity of the object is increasing at all times, its speed is decreasing for $0 \leq t < \frac{5}{2}$ and then increasing for $\frac{5}{2} < t \leq 5$. The graph of the velocity function (**Figure 3.41**) confirms these observations.

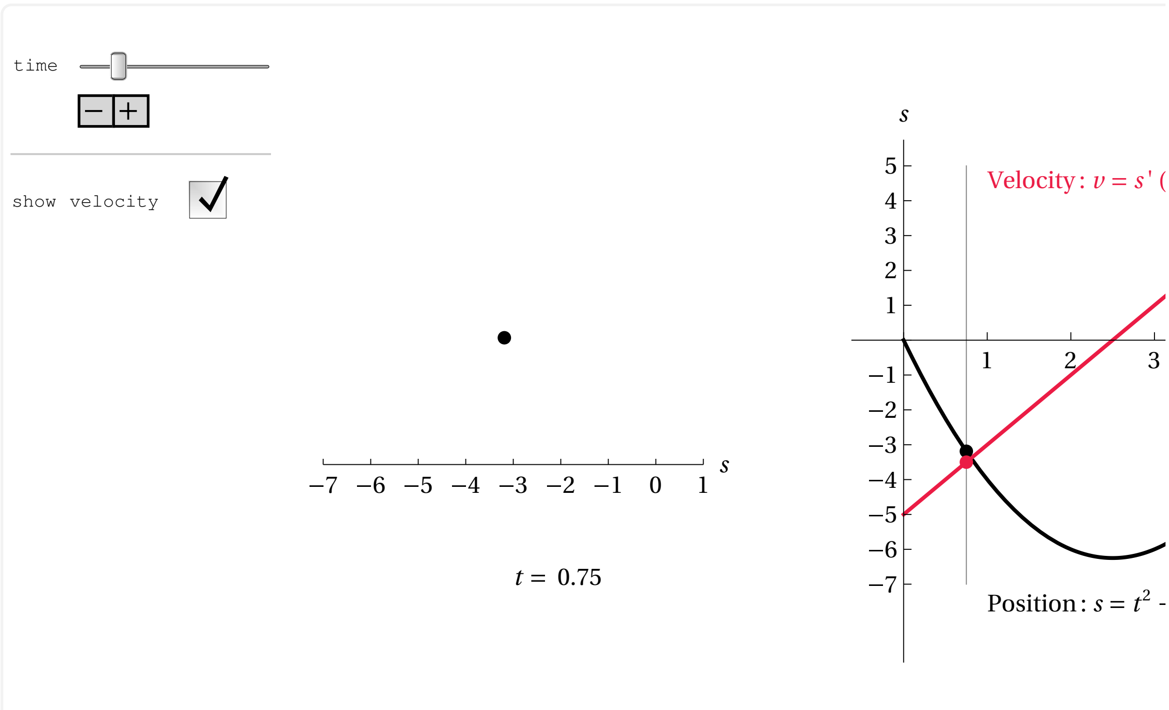


Figure 3.41

b. The acceleration is the derivative of the velocity or $a = v'(t) = s''(t) = 2$. This means that the acceleration is $2 \text{ ft}/\text{s}^2$, for $0 \leq t \leq 5$ (**Figure 3.42**).

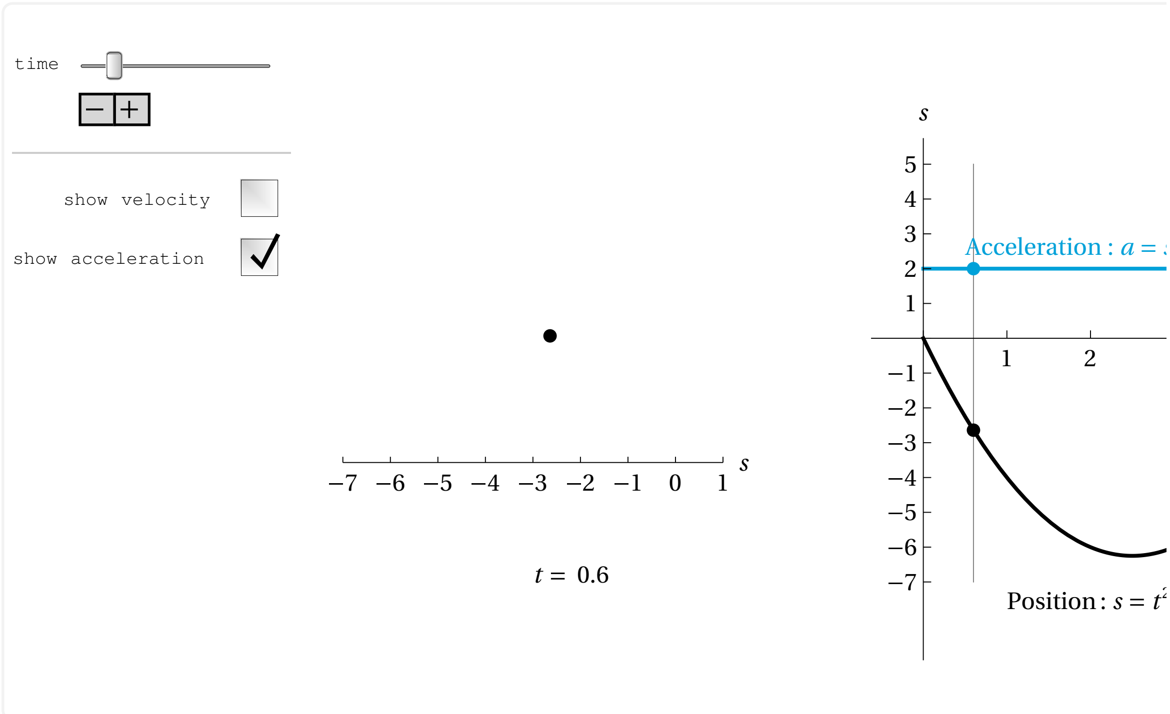


Figure 3.42

c. Starting at an initial position of $s(0) = 0$, the object moves in the negative direction (to the left) with decreasing speed until it comes to rest momentarily at $s\left(\frac{5}{2}\right) = -\frac{25}{4}$. The object then moves in the positive direction (to the right) with increasing speed, reaching its initial position at $t = 5$. During this time interval, the acceleration is constant.

Related Exercises 15–16 ◆

Quick Check 3 Describe in words the velocity of an object that has a positive constant acceleration. Could an object have a positive acceleration and a decreasing speed? ◆

Answer »

If an object has positive acceleration, then its velocity is increasing. If the velocity is negative but increasing, then the acceleration is positive and the speed is decreasing. For example, the velocity may increase from -2 m/s to -1 m/s to 0 m/s.

Free Fall

We now consider problems in which an object moves vertically in Earth's gravitational field, assuming that no other forces (such as air resistance) are at work.

Note »

The acceleration due to Earth's gravitational field is denoted g. In metric units $g \approx 9.8 \text{ m}/\text{s}^2$ on the surface of Earth; in the U.S. Customary System (USCS), $g \approx 32 \text{ ft}/\text{s}^2$.

EXAMPLE 3 Motion in a gravitational field

Suppose a stone is thrown vertically upward with an initial velocity of 64 ft/s from a bridge 96 ft above a river. By Newton's laws of motion, the position of the stone (measured as the height above the river) after t seconds is

$$s(t) = -16\,t^2 + 64\,t + 96,$$

where $s = 0$ is the level of the river (**Figure 3.43a**).

a. Find the velocity and acceleration functions.

b. What is the highest point above the river reached by the stone?

c. With what velocity will the stone strike the river?

Note »

The position function in Example 3 is derived in Section 6.1. Once again we mention that the graph of the position function is not the path of the stone.

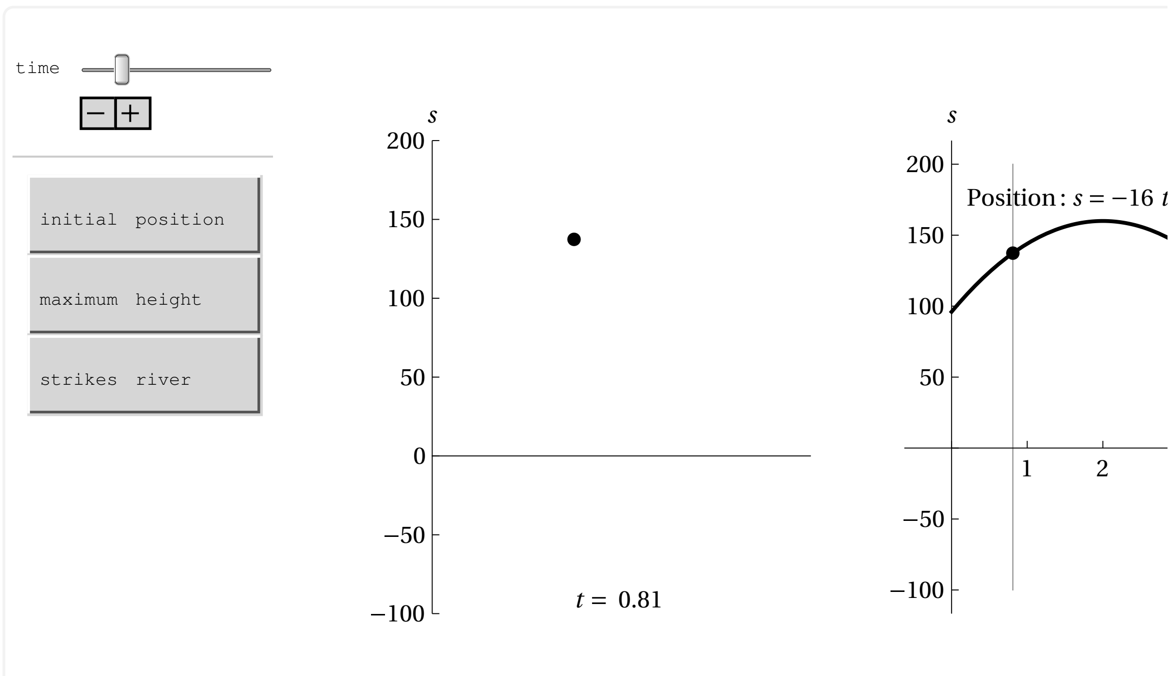


Figure 3.43 a

SOLUTION »

a. The velocity of the stone is the derivative of the position function, and its acceleration is the derivative of the velocity function. Therefore,

$$v = \frac{ds}{dt} = -32\,t + 64 \quad \text{and} \quad a = \frac{dv}{dt} = -32.$$

b. When the stone reaches its high point, its velocity is zero (**Figure 3.43b**). Solving $v(t) = -32\,t + 64 = 0$ yields $t = 2$; therefore, the stone reaches its maximum height 2 seconds after it is thrown. Its height at that instant is

$$s(2) = -16\,(2)^2 + 64\,(2) + 96 = 160.$$

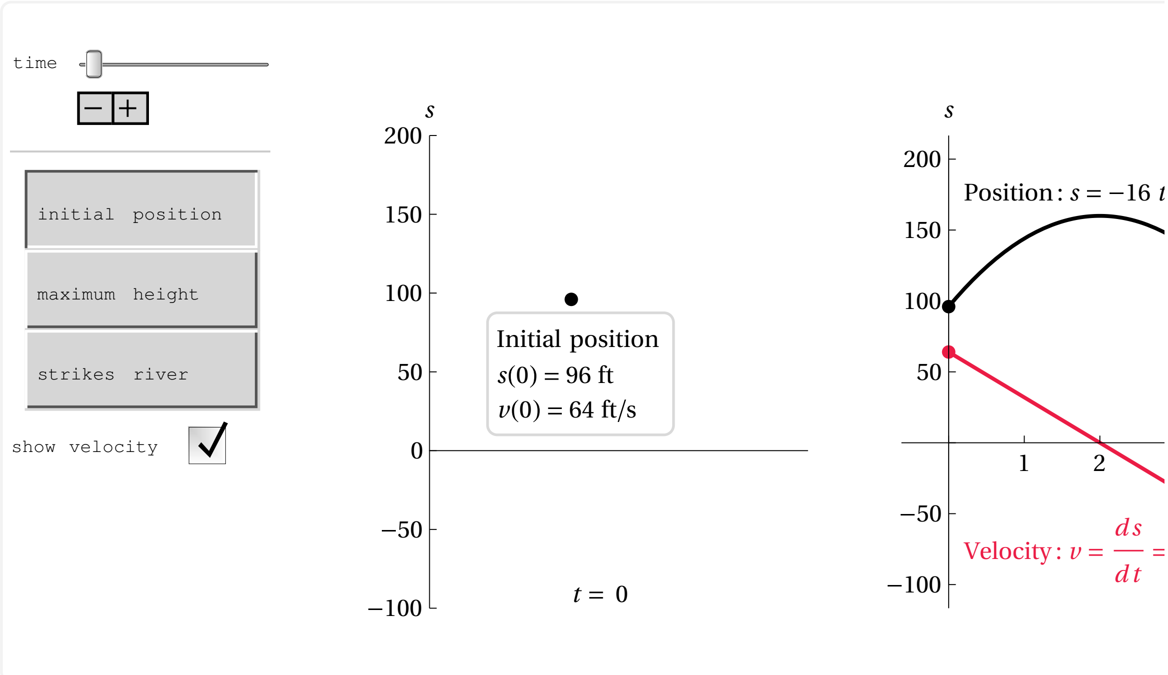


Figure 3.43 b

c. To determine the velocity at which the stone strikes the river, we first determine *when* it reaches the river. The stone strikes the river when $s(t) = -16\,t^2 + 64\,t + 96 = 0$. Dividing both sides of the equation by -16, we obtain $t^2 - 4\,t - 6 = 0$. Using the quadratic formula, the solutions are $t \approx 5.16$ or $t \approx -1.16$. Because the stone is thrown at $t = 0$, only positive values of t are of interest; therefore, the relevant root is $t \approx 5.16$. The velocity of the stone (in ft/s) when it strikes the river is approximately

$$v(5.16) = -32\,(5.16) + 64 = -101.12.$$

Related Exercises 24–25 ◆

Quick Check 4 In Example 3, does the rock have the greater speed at $t = 1$ or $t = 3$? ◆

Answer »

Growth Models »

Much of the change in the world around us can be classified as *growth*: Populations, prices, and computer networks all tend to increase in size. Modeling growth is important because it often leads to an understanding of underlying processes and allows for predictions.

We let $p = f(t)$ be the measure of a quantity of interest (for example, the population of a species or the consumer price index), where $t \geq 0$ represents time. The average growth rate of p between time $t = a$ and a later time $t = a + \Delta t$ is the change Δp divided by elapsed time Δt. Therefore, the **average growth rate** of p on the interval $[a, a + \Delta t]$ is

$$\frac{\Delta p}{\Delta t} = \frac{f(a + \Delta t) - f(a)}{\Delta t}.$$

If we now let $\Delta t \to 0$, then $\frac{\Delta p}{\Delta t}$ approaches the derivative $\frac{dp}{dt}$, which is the **instantaneous growth rate** (or simply **growth rate**) of p with respect to time:

$$\frac{dp}{dt} = \lim_{\Delta t \to 0} \frac{\Delta p}{\Delta t}.$$

Once again, we see the derivative appearing as an instantaneous rate of change. In the next example, a growth function and its derivative are approximated using real data.

EXAMPLE 4 Internet growth

The number of worldwide Internet users between 2000 and 2015 is shown in **Figure 3.44**. A reasonable fit to the data is given by the function $p(t) = 6t^2 + 98t + 431.2$, where t measures years after 2000.

a. Use the function p to approximate the average growth rate of Internet users from 2005 ($t = 5$) to 2010 ($t = 10$).

b. What was the instantaneous growth rate of the Internet in 2011?

c. Use a graphing utility to plot the growth rate dp/dt. What does the graph tell you about the growth rate between 2000 and 2015?

d. Assuming that the growth function can be extended beyond 2015, what is the predicted number of Internet users in 2020 ($t = 20$)?

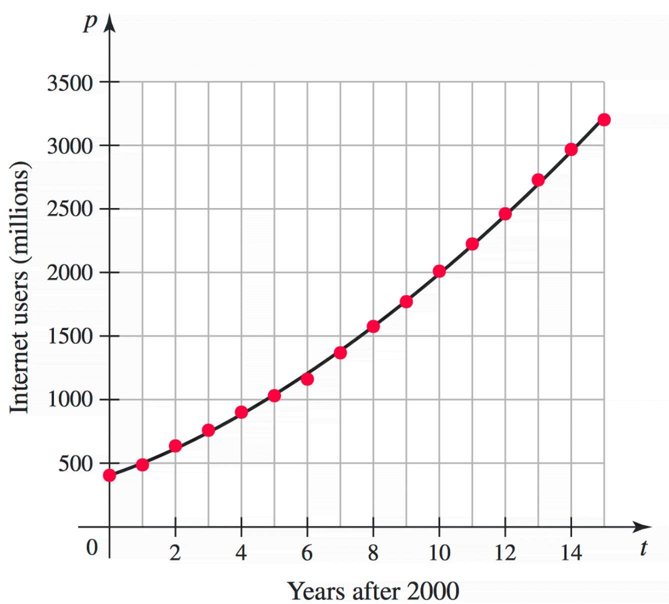


Figure 3.44

SOLUTION »

a. The average growth rate over the interval [5, 10] is

$$\frac{\Delta p}{\Delta t} = \frac{p(10) - p(5)}{10 - 5} \approx \frac{2011.2 - 1071.2}{5} \approx 188 \text{ million users/year}.$$

b. The growth rate at time t is $p'(t) = 12t + 98$. In 2011 ($t = 11$), the growth rate was $p'(11) \approx 230$ million users per year.

c. The graph of p', for $0 \le t \le 15$, is shown in **Figure 3.45**. We see that the growth rate is positive and increasing, for $t \ge 0$.

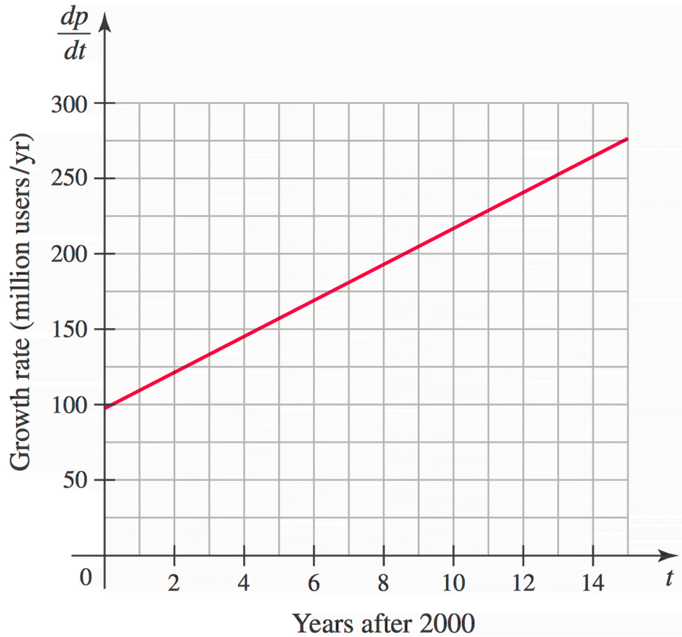


Figure 3.45

d. The number of Internet users in 2020 is predicted to be $p(20) \approx 4791$ million, or about 4.8 billion users, which is approximately 62% of the world's population, assuming a projected population of 7.7 billion people in 2020.

Related Exercise 28 ◆

Quick Check 5 Using the growth function in Example 4, compare the growth rates in 2001 and 2012. ◆

Answer »

> The growth rate in 2001 ($t = 1$) is 110 million users/year. It is less than half of the growth rate in 2012 ($t = 12$), which is 242 million users/year.

Economics and Business »

Our final examples illustrate how derivatives arise in economics and business. As you will see, the mathematics of derivatives is the same as it is in other applications. However, the vocabulary and interpretation are quite different.

Average and Marginal Cost

Imagine a company that manufactures large quantities of a product such as mousetraps, DVD players, or snowboards. Associated with the manufacturing process is a *cost function* $C(x)$ that gives the cost of manufacturing x items of the product. A simple cost function might have the form $y = C(x) = 500 + 0.1x$, as shown in **Figure 3.46**. It includes a **fixed cost** of \$500 (setup costs and overhead) that is independent of the number of items produced. It also includes a **unit cost**, or **variable cost**, of \$0.10 per item produced. For example, the cost of producing 1000 items is $C(1000) = \$600$.

Note »

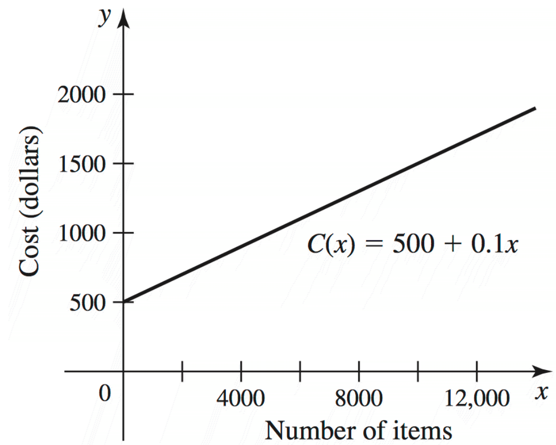


Figure 3.46

If the company produces x items at a cost of $C(x)$, then the *average cost* is $\frac{C(x)}{x}$ per item. For the cost function $C(x) = 500 + 0.1\,x$, the average cost is

$$\frac{C(x)}{x} = \frac{500 + 0.1\,x}{x} = \frac{500}{x} + 0.1.$$

For example, the average cost of manufacturing 1000 items is

$$\frac{C(1000)}{1000} = \frac{\$600}{1000} = \$0.60\,/\,\text{unit}.$$

Plotting $\frac{C(x)}{x}$, we see that the average cost decreases as the number of items produced increases (**Figure 3.47**).

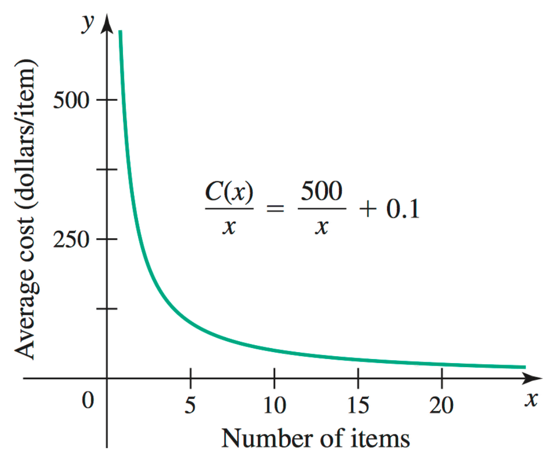


Figure 3.47

The average cost gives the cost of items already produced. But what about the cost of producing additional items? Having produced x items, the cost of producing another Δx items is $C(x + \Delta x) - C(x)$. Therefore, the average cost per item of producing those Δx additional items is

$$\frac{C(x+\Delta x)-C(x)}{\Delta x}=\frac{\Delta C}{\Delta x}.$$

If we let $\Delta x \to 0$, we see that

$$\lim_{\Delta x\to 0}\frac{\Delta C}{\Delta x}=C'(x),$$

which is called the *marginal cost*. In reality, we cannot let $\Delta x \to 0$ because Δx represents whole numbers of items. However, there is is a useful interpretation of the marginal cost. Suppose $\Delta x = 1$. Then, $\Delta C = C(x+1)-C(x)$ is the cost to produce *one* additional item. In this case we write

$$\frac{\Delta C}{\Delta x}=\frac{C(x+1)-C(x)}{1}.$$

If the *slope* of the cost curve does not vary significantly near the point x, then—as shown in **Figure 3.48**—we have

$$\frac{\Delta C}{\Delta x}\approx\lim_{\Delta x\to 0}\frac{\Delta C}{\Delta x}=C'(x).$$

Note »

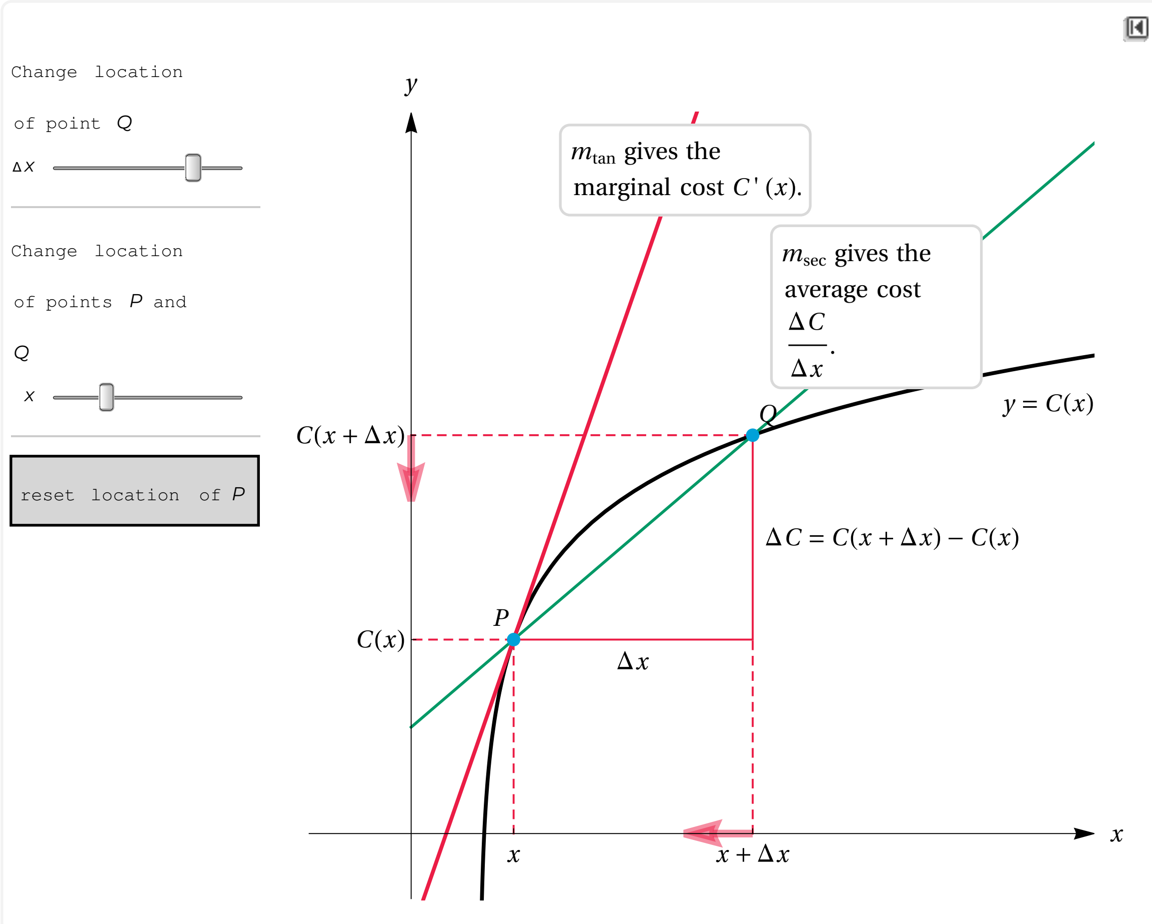


Figure 3.48

Therefore, the cost of producing one additional item, having already produced x items, is approximated by the marginal cost $C'(x)$. In the preceding example, we have $C'(x) = 0.1$, so if $x = 1000$ items have been produced, then the cost of producing the 1001st item is approximately $C'(1000) = \$0.10$. With this simple linear cost function, the marginal cost tells us what we already know: The cost of producing one additional item is the variable cost of $0.10. With more realistic cost functions, the marginal cost may be variable.

Note »

DEFINITION **Average and Marginal Cost**

The **cost function** $C(x)$ gives the cost to produce the first x items in a manufacturing process. The **average cost** to produce x items is $\overline{C}(x) = \frac{C(x)}{x}$. The **marginal cost** $C'(x)$ is the approximate cost to produce one additional item after producing x items.

EXAMPLE 5 Average and marginal costs

Suppose the cost of producing x items is given by the function (**Figure 3.49**)

$$C(x) = -0.02\,x^2 + 50\,x + 100, \quad \text{for} \quad 0 \le x \le 1000.$$

a. Determine the average and marginal cost functions.

b. Determine the average and marginal cost for the first 100 items and interpret these values.

c. Determine the average and marginal cost for the first 900 items and interpret these values.

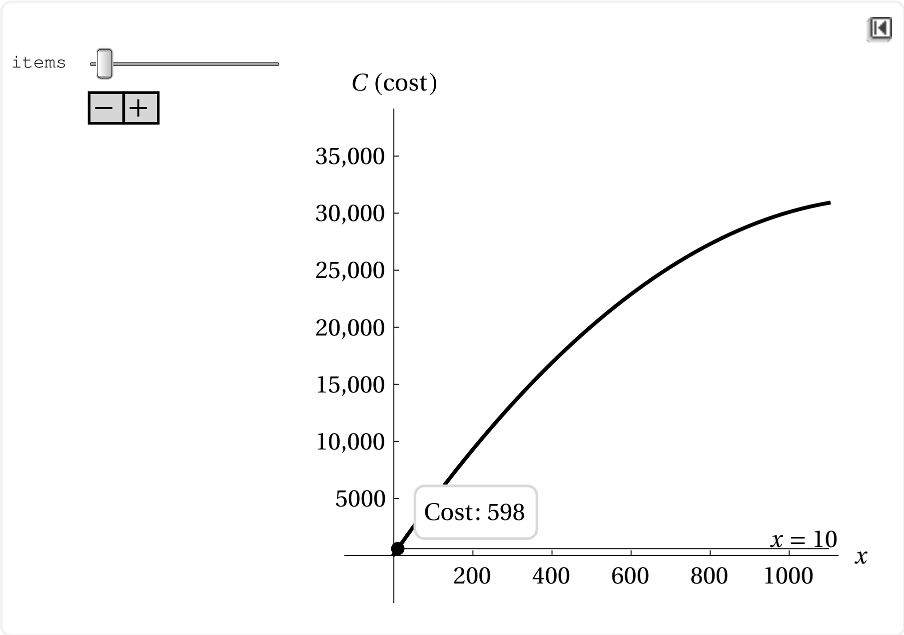


Figure 3.49

SOLUTION »

Quick Check 6 In Example 5, what happens to the average cost as the number of items produced increases from $x = 1$ to $x = 100$? ◆

Answer »

Elasticity in Economics

Economists apply the term *elasticity* to prices, income, capital, labor, and other variables in systems with input and output. Elasticity describes how changes in the input to a system are related to changes in the output. Because elasticity involves change, it also involves derivatives.

A general rule is that as the price p of an item increases, the number of sales of that item decreases. This relationship is expressed in a demand function. For example, suppose sales at a gas station have the linear demand function $D(p) = 1200 - 200\,p$ (**Figure 3.51**), where $D(p)$ is the number of gallons sold per day at a price p (measured in dollars). According to this function, if gas sells at \$3.60/gal, then the owner can expect to sell $D(3.6) = 480$ gallons. If the price is increased, sales decrease.

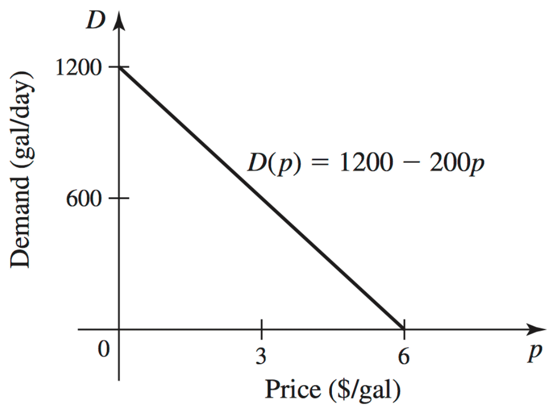


Figure 3.51

Suppose the price of a gallon of gasoline increases from \$3.60 to \$3.96 per gallon; call this change $\Delta p = \$0.36$. The resulting change in the number of gallons sold is $\Delta D = D(3.96) - D(3.60) = -0.72$. (The change is a decrease, so it is negative.) Comparisons of the variables are more meaningful if we work with percentages. Increasing the price from \$3.60 to \$3.96 per gallon is a percentage change of $\frac{\Delta p}{p} = \frac{\$0.36}{\$3.60} = 10\%$. Similarly, the corresponding percentage change in the gallons sold is $\frac{\Delta D}{D} = \frac{-72}{480} = -15\%$.

The *price elasticity of the demand* (or simply, *elasticity*) is the ratio of the percentage change in demand to the percentage change in price; that is, $E = \frac{\Delta D/D}{\Delta p/p}$. In the case of the gas demand function, the elasticity of this particular price change is $\frac{-15\%}{10\%} = -1.5$.

The elasticity is more useful when it is expressed as a function of the price. To do this, we consider small changes in p and assume the corresponding changes in D are also small. Using the definition of the derivative, the elasticity *function* is

$$E(p) = \lim_{\Delta p \to 0} \frac{\Delta D/D}{\Delta p/p} = \lim_{\Delta p \to 0} \frac{\Delta D}{\Delta p}\left(\frac{p}{D}\right) = \frac{dD}{dp}\,\frac{p}{D}.$$

Applying this definition to the gas demand function, we find that

$$\begin{aligned}
E(p) &= \frac{dD}{dp}\,\frac{p}{D} \\
&= \frac{d}{dp}\underbrace{(1200-200\,p)}_{D}\,\frac{p}{\underbrace{1200-200\,p}_{D}} && \text{Substitute } D = 1200 - 200\,p. \\
&= -200\left(\frac{p}{1200-200\,p}\right) && \text{Differentiate.} \\
&= \frac{p}{p-6}. && \text{Simplify.}
\end{aligned}$$

Given a particular price, the elasticity is interpreted as the percentage change in the demand that results for every 1% change in the price. For example, in the gas demand case, with $p = \$3.60$, the elasticity is

$E(3.6) = -1.5$; therefore, a 2% increase in the price results in a change of $-1.5 \cdot 2\% = -3\%$ (a decrease) in the number of gallons sold.

DEFINITION **Elasticity**

If the demand for a product varies with the price according to the function $D = f(p)$, then the **price elasticity of the demand** is $E(p) = \frac{dD}{dp} \frac{p}{D}$.

Note »

EXAMPLE 6 Elasticity in pork prices

The demand for processed pork in Canada is described by the function $D(p) = 286 - 20p$. (*Source:* J. Perloff, *Microeconomics*, Prentice Hall, 2012)

a. Compute and graph the price elasticity of the demand.

b. When $-\infty < E < -1$, the demand is said to be **elastic**. When $-1 < E < 0$, the demand is said to be **inelastic**. Interpret these terms.

c. For what prices is the demand for pork elastic? Inelastic?

SOLUTION »

a. Substituting the demand function into the definition of elasticity, we find that

$$\begin{aligned} E(p) &= \frac{dD}{dp} \frac{p}{D} \\ &= \frac{d}{dp} \underbrace{(286 - 20p)}_{D} \frac{p}{\underbrace{286 - 20p}_{D}} && \text{Substitute } D = 286 - 20p. \\ &= -20 \left(\frac{p}{286 - 20p} \right) && \text{Differentiate.} \\ &= -\frac{10p}{143 - 10p}. && \text{Simplify.} \end{aligned}$$

Notice that the elasticity is undefined at $p = 14.3$, which is the price at which the demand reaches zero. (According to the model, no pork can be sold at prices above $14.30.) Therefore, the domain of the elasticity function is $[0, 14.3)$, and on the interval $(0, 14.3)$, the elasticity is negative (**Figure 3.52**).

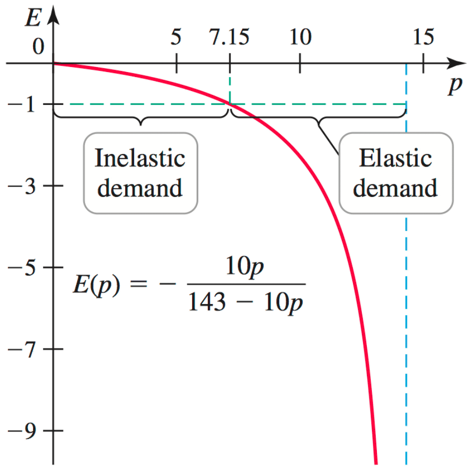


Figure 3.52

b. For prices with an elasticity in the interval $-\infty < E(p) < -1$, a $P\%$ increase in the price results in *more* than a $P\%$ decrease in the demand; this is the case of elastic (sensitive) demand. If a price has an elasticity in the interval $-1 < E(p) < 0$, a $P\%$ increase in the price results in *less* than a $P\%$ decrease in the demand; this is the case of inelastic (insensitive) demand.

c. Solving $E(p) = -\frac{10\,p}{143 - 10\,p} = -1$, we find that $E(p) < -1$, for $p > 7.15$. For prices in this interval, the demand is elastic (Figure 3.52). For prices with $0 < p < 7.15$, the demand is inelastic.

Note »

Related Exercises 33–34 ◆

Exercises »

Getting Started »

Practice Exercises »

15–20. Position, velocity, and acceleration *Suppose the position of an object moving horizontally along a line after t seconds is given by the following functions $s = f(t)$, where s is measured in feet, with $s > 0$ corresponding to positions right of the origin.*

- ***a.*** *Graph the position function.*
- ***b.*** *Find and graph the velocity function. When is the object stationary, moving to the right, and moving to the left?*
- ***c.*** *Determine the velocity and acceleration of the object at $t = 1$.*
- ***d.*** *Determine the acceleration of the object when its velocity is zero.*
- ***e.*** *On what intervals is the speed increasing?*

15. $f(t) = t^2 - 4\,t;\ 0 \le t \le 5$

16. $f(t) = -t^2 + 4\,t - 3;\ 0 \le t \le 5$

17. $f(t) = 2\,t^2 - 9\,t + 12;\ 0 \le t \le 3$

18. $f(t) = 18t - 3t^2; 0 \le t \le 8$

19. $f(t) = 2t^3 - 21t^2 + 60t; 0 \le t \le 6$

20. $f(t) = -6t^3 + 36t^2 - 54t; 0 \le t \le 4$

21. **A dropped stone on Earth** The height (in feet) of a stone dropped from a bridge 64 feet above a river at $t = 0$ seconds is given by $s(t) = -16t^2 + 64$. Find the velocity of the stone and its speed when it hits the water.

22. **A dropped stone on Mars** A stone is dropped off the edge of a 54-ft cliff on Mars, where the acceleration due to gravity is about $12 \text{ ft}/\text{s}^2$. The height (in feet) of the stone above the ground t seconds after it is dropped is $s(t) = -6t^2 + 54$. Find the velocity of the stone and its speed when it hits the ground.

23. **Throwing a stone** Suppose a stone is thrown vertically upward from the edge of a cliff on Earth with an initial velocity of 32 ft/s from a height of 48 ft above the ground. The height (in feet) of the stone above the ground t seconds after it is thrown is $s(t) = -16t^2 + 32t + 48$.

- **a.** Determine the velocity v of the stone after t seconds.
- **b.** When does the stone reach its highest point?
- **c.** What is the height of the stone at the highest point?
- **d.** When does the stone strike the ground?
- **e.** With what velocity does the stone strike the ground?
- **f.** On what intervals is the speed increasing?

24. Suppose a stone is thrown vertically upward from the edge of a cliff on Earth with an initial velocity of 19.6 m/s from a height of 24.5 m above the ground. The height (in meters) of the stone above the ground t seconds after it is thrown is $s(t) = -4.9t^2 + 19.6t + 24.5$.

- **a.** Determine the velocity v of the stone after t seconds.
- **b.** When does the stone reach its highest point?
- **c.** What is the height of the stone at the highest point?
- **d.** When does the stone strike the ground?
- **e.** With what velocity does the stone strike the ground?
- **f.** On what intervals is the speed increasing?

25. Suppose a stone is thrown vertically upward from the edge of a cliff on Earth with an initial velocity of 64 ft/s from a height of 32 ft above the ground. The height (in feet) of the stone above the ground t seconds after it is thrown is $s(t) = -16t^2 + 64t + 32$.

- **a.** Determine the velocity v of the stone after t seconds.
- **b.** When does the stone reach its highest point?
- **c.** What is the height of the stone at the highest point?
- **d.** When does the stone strike the ground?
- **e.** With what velocity does the stone strike the ground?
- **f.** On what intervals is the speed increasing?

26. Maximum height Suppose a baseball is thrown vertically upward from the ground with an initial velocity of v_0 ft/s. The approximate height of the ball (in feet) above the ground after t seconds is given by $s(t) = -16t^2 + v_0 t$.

a. What is the height of the ball at its highest point?

b. With what velocity does the ball strike the ground?

27. Initial velocity Suppose a baseball is thrown vertically upward from the ground with an initial velocity of v_0 ft/s. Its height above the ground after t seconds is given by $s(t) = -16t^2 + v_0 t$. Determine the initial velocity of the ball if it reaches a high point of 128 ft.

T **28. Population growth in Washington** The population of the state of Washington (in millions) from 2010 ($t = 0$) to 2016 ($t = 6$) is modeled by the polynomial $p(t) = 0.0078t^2 + 0.028t + 6.73$.

a. Determine the average growth rate from 2010 to 2016.

b. What was the growth rate for Washington State in 2011 ($t = 1$) and 2015 ($t = 5$)?

c. Use a graphing utility to graph p' for $0 \le t \le 6$. What does this graph tell you about population growth in Washington during the period of time from 2010 to 2016?

29–32. Average and marginal cost *Consider the following cost functions.*

a. *Find the average cost and marginal cost functions.*

b. *Determine the average and marginal cost when $x = a$.*

c. *Interpret the values obtained in part (b).*

29. $C(x) = 1000 + 0.1x$, $0 \le x \le 5000$, $a = 2000$

30. $C(x) = 500 + 0.02x$, $0 \le x \le 2000$, $a = 1000$

31. $C(x) = -0.01x^2 + 40x + 100$, $0 \le x \le 1500$, $a = 1000$

32. $C(x) = -0.04x^2 + 100x + 800$, $0 \le x \le 1000$, $a = 500$

33. Demand and elasticity Based on sales data over the past year, the owner of a DVD store devises the demand function $D(p) = 40 - 2p$, where $D(p)$ is the number of DVDs that can be sold in one day at a price of p dollars.

a. According to the model, how many DVDs can be sold in a day at a price of $10?

b. According to the model, what is the maximum price that can be charged (above which no DVDs can be sold)?

c. Find the elasticity function for this demand function.

d. For what prices is the demand elastic? Inelastic?

e. If the price of DVDs is raised from $10.00 to $10.25, what is the exact percentage decrease in demand (using the demand function)?

f. If the price of DVDs is raised from $10.00 to $10.25, what is the approximate percentage decrease in demand (using the elasticity function)?

34. Demand and elasticity The economic advisor of a large tire store proposes the demand function $D(p) = \dfrac{1800}{p - 40}$, where $D(p)$ is the number of tires of one brand and size that can be sold in one day at a price p.

a. Recalling that the demand must be positive, what is the domain of this function?

b. According to the model, how many tires can be sold in a day at a price of \$60 per tire?

c. Find the elasticity function on the domain of the demand function.

d. For what prices is the demand elastic? Inelastic?

e. If the price of tires is raised from \$60 to \$62, what is the approximate percentage decrease in demand (using the elasticity function)?

35. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.

a. If the acceleration of an object remains constant, then its velocity is constant.

b. If the acceleration of an object moving along a line is always 0, then its velocity is constant.

c. It is impossible for the instantaneous velocity at all times $a \le t \le b$ to equal the average velocity over the interval $a \le t \le b$.

d. A moving object can have negative acceleration and increasing speed.

36. **A feather dropped on the moon** On the moon, a feather will fall to the ground at the same rate as a heavy stone. Suppose a feather is dropped from a height of 40 m above the surface of the moon. Then, its height s (in meters) above the ground after t seconds is $s = 40 - 0.8\,t^2$. Determine the velocity and acceleration of the feather the moment it strikes the surface of the moon.

37. **Comparing velocities** A stone is thrown vertically into the air at an initial velocity of 96 ft/s. On Mars, the height s (in feet) of the stone above the ground after t seconds is $s = 96\,t - 6\,t^2$ and on Earth it is $s = 96\,t - 16\,t^2$. How much higher will the stone travel on Mars than on Earth?

38. **Comparing velocities** Two stones are thrown vertically upward, each with an initial velocity of 48 ft/s at time $t = 0$. One stone is thrown from the edge of a bridge that is 32 feet above the ground and the other stone is thrown from ground level. The height above the ground of the stone thrown from the bridge after t seconds is $f(t) = -16\,t^2 + 48\,t + 32$, and the height of the stone thrown from the ground after t seconds is $g(t) = -16\,t^2 + 48\,t$.

a. Show that the stones reach their high points at the same time.

b. How much higher does the stone thrown from the bridge go than the stone thrown from the ground?

c. When do the stones strike the ground and with what velocities?

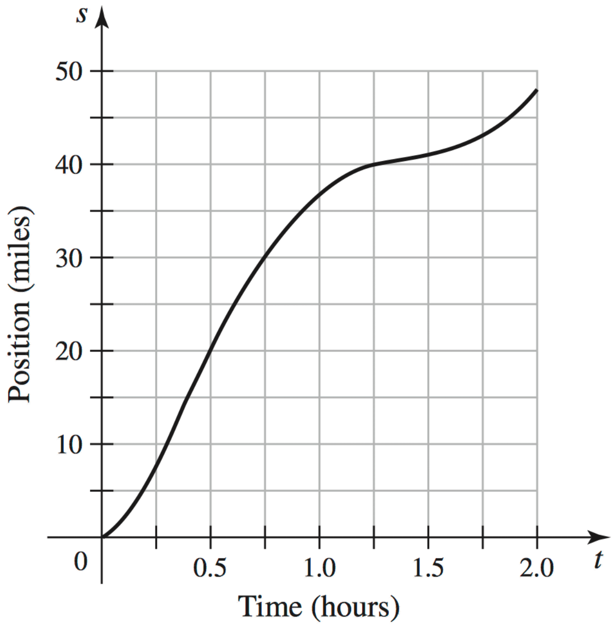
39. **Matching heights** A stone is thrown with an initial velocity of 32 ft/s from the edge of a bridge that is 48 ft above the ground. The height of this stone above the ground t seconds after it is thrown is $f(t) = -16\,t^2 + 32\,t + 48$. If a second stone is thrown from the ground, then its height above the ground after t seconds is given by $g(t) = -16\,t^2 + v_0\,t$, where v_0 is the initial velocity of the second stone. Determine the value of v_0 such that both stones reach the same high point.

40. **Velocity of a car** The graph shows the position $s = f(t)$ of a car t hours after 5:00 P.M. relative to its starting point $s = 0$, where s is measured in miles.

a. Describe the velocity of the car. Specifically, when is it speeding up and when is it slowing down?

b. At approximately what time is the car traveling the fastest? The slowest?

c. What is the approximate maximum velocity of the car? The approximate minimum velocity?



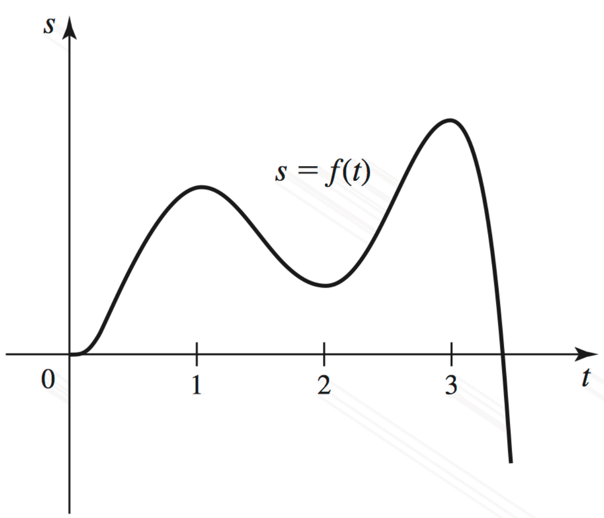
41. Velocity from position The graph of $s = f(t)$ represents the position of an object moving along a line at time $t \geq 0$.

a. Assume the velocity of the object is 0 when $t = 0$. For what other values of t is the velocity of the object zero?

b. When is the object moving in the positive direction and when is it moving in the negative direction?

c. Sketch a graph of the velocity function.

d. On what intervals is the speed increasing?

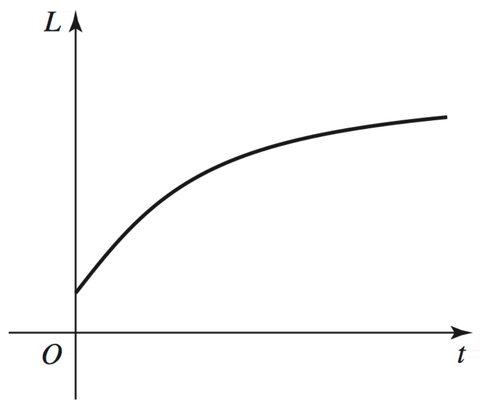


42. Fish length Assume the length L (in cm) of a particular species of fish after t years is modeled by the following graph.

a. What does dL/dt represent and what happens to this derivative as t increases?

b. What does the derivative tell you about how this species of fish grows?

c. Sketch a graph of L' and L''.



43–46. Average and marginal profit *Let $C(x)$ represent the cost of producing x items and $p(x)$ be the sale price per item if x items are sold. The profit $P(x)$ of selling x items is $P(x) = x\,p(x) - C(x)$ (revenue minus costs). The* ***average profit per item*** *when x items are sold is $P(x)/x$ and the* ***marginal profit*** *is dP/dx. The marginal profit approximates the profit obtained by selling one more item given that x items have already been sold. Consider the following cost functions C and price functions p.*

- **a.** *Find the profit function P.*
- **b.** *Find the average profit function and marginal profit function.*
- **c.** *Find the average profit and marginal profit if $x = a$ units are sold.*
- **d.** *Interpret the meaning of the values obtained in part (c).*

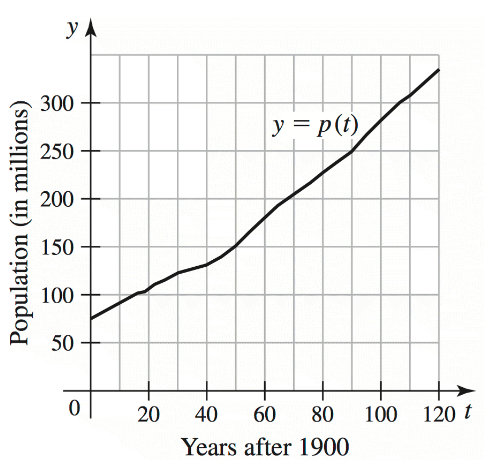
43. $C(x) = -0.02\,x^2 + 50\,x + 100$, $p(x) = 100$, $a = 500$

44. $C(x) = -0.02\,x^2 + 50\,x + 100$, $p(x) = 100 - 0.1\,x$, $a = 500$

45. $C(x) = -0.04\,x^2 + 100\,x + 800$, $p(x) = 200$, $a = 1000$

46. $C(x) = -0.04\,x^2 + 100\,x + 800$, $p(x) = 200 - 0.1\,x$, $a = 1000$

47. U.S. population growth The population $p(t)$ (in millions) of the United States t years after the year 1900 is shown in the figure. Approximately when (in what year) was the U.S. population growing most slowly between 1925 and 2020? Estimate the growth rate in that year.



T **48.** **Average of marginal production** Economists use *production functions* to describe how the output of a system varies with respect to another variable such as labor or capital. For example, the production function $P(L) = 200L + 10L^2 - L^3$ gives the output of a system as a function of the number of laborers L. The *average product* $A(L)$ is the average output per laborer when L laborers are working; that is $A(L) = P(L)/L$. The *marginal product* $M(L)$ is the approximate change in output when one additional laborer is added to L laborers; that is, $M(L) = dP/dL$.

a. For the given production function, compute and graph P, A, and L.

b. Suppose the peak of the average product curve occurs at $L = L_0$, so that $A'(L_0) = 0$. Show that for a general production function, $M(L_0) = A(L_0)$.

T **49.** **Velocity of a marble** The position (in meters) of a marble rolling up a long incline is given by $s = \dfrac{100t}{t+1}$, where t is measured in seconds and $s = 0$ is the starting point.

a. Graph the position function.

b. Find the velocity function for the marble.

c. Graph the velocity function and give a description of the motion of the marble.

d. At what time is the marble 80 m from its starting point?

e. At what time is the velocity 50 m/s?

T **50.** **Tree growth** Let b represent the base diameter of a conifer tree and let h represent the height of the tree, where b is measured in centimeters and h is measured in meters. Assume the height is related to the base diameter by the function $h = 5.67 + 0.70b + 0.0067b^2$.

a. Graph the height function.

b. Plot and interpret the meaning of dh/db.

Explorations and Challenges »

T **51.** **A different interpretation of marginal cost** Suppose a large company makes 25,000 gadgets per year in batches of x items at a time. After analyzing setup costs to produce each batch and taking into account storage costs, it has been determined that the total cost $C(x)$ of producing 25,000 gadgets in batches of x items at a time is given by

$$C(x) = 1{,}250{,}000 + \frac{125{,}000{,}000}{x} + 1.5x.$$

a. Determine the marginal cost and average cost functions. Graph and interpret these functions.

b. Determine the average cost and marginal cost when $x = 5000$.

c. The meaning of average cost and marginal cost here is different from earlier examples and exercises. Interpret the meaning of your answer in part (b).

52. **Diminishing returns** A cost function of the form $C(x) = \frac{1}{2}x^2$ reflects *diminishing returns to scale*. Find and graph the cost, average cost, and marginal cost functions. Interpret the graphs and explain the idea of diminishing returns.

T 53. Revenue function A store manager estimates that the demand for an energy drink decreases with increasing price according to the function $d(p) = \dfrac{100}{p^2 + 1}$, which means that at price p (in dollars), $d(p)$ units can be sold. The revenue generated at price p is $R(p) = p \cdot d(p)$ (price multiplied by number of units).

a. Find and graph the revenue function.

b. Find and graph the marginal revenue $R'(p)$.

c. From the graphs of R and R', estimate the price that should be charged to maximize the revenue.

T 54. Fuel economy Suppose you own a fuel-efficient hybrid automobile with a monitor on the dashboard that displays the mileage and gas consumption. The number of miles you can drive with g gallons of gas remaining in the tank on a particular stretch of highway is given by $m(g) = 50g - 25.8g^2 + 12.5g^3 - 1.6g^4$, for $0 \le g \le 4$.

a. Graph and interpret the mileage function.

b. Graph and interpret the gas mileage $m(g)/g$.

c. Graph and interpret dm/dg.

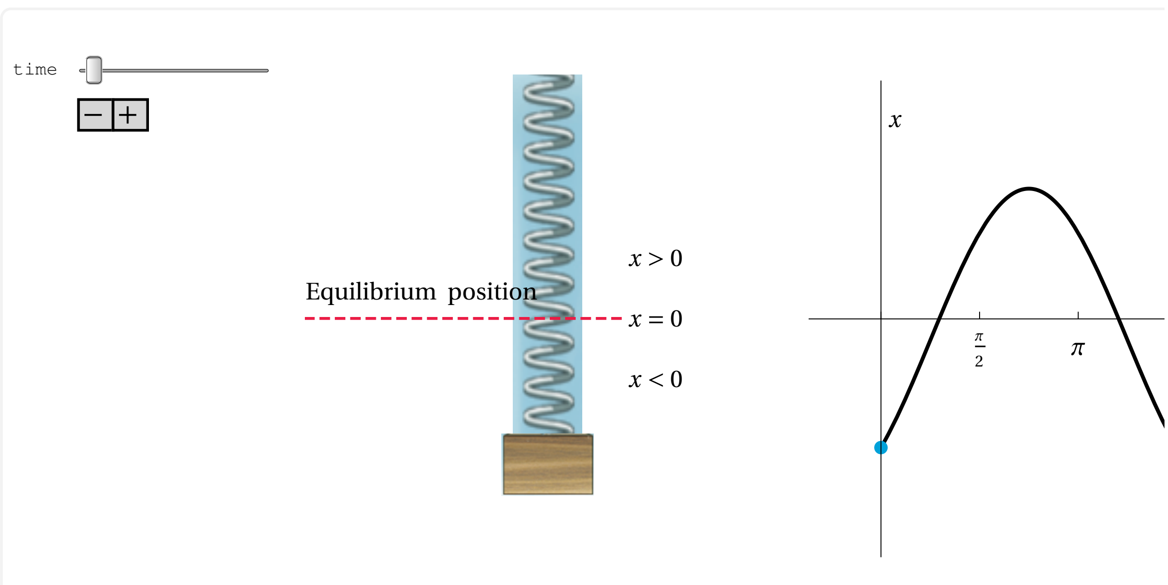
T 55. Spring oscillations A spring hangs from the ceiling at equilibrium with a mass attached to its end. Suppose you pull downward on the mass and release it 10 inches below its equilibrium position with an upward push. The distance x (in inches) of the mass from its equilibrium position after t seconds is given by the function $x(t) = 10 \sin t - 10 \cos t$, where x is positive when the mass is above the equilibrium position.

a. Graph and interpret this function.

b. Find dx/dt and interpret the meaning of this derivative.

c. At what times is the velocity of the mass zero?

d. The function given here is a model for the motion of an object on a spring. In what ways is this model unrealistic?



T **56.** **Power and energy** Power and energy are often used interchangeably, but they are quite different. **Energy** is what makes matter move or heat up. It is measured in units of **joules** or **Calories**, where 1 Cal = 4184 J. One hour of walking consumes roughly 10^6 J, or 240 Cal. On the other hand, **power** is the rate at which energy is used, which is measured in **watts**, where 1 W = 1 J/s. Other useful units of power are **kilowatts** (1 kW = 10^3 W) and **megawatts** (1 MW = 10^6 W). If energy is used at a rate of 1 kW for one hour, the total amount of energy used is 1 **kilowatt-hour** (1 kWh = 3.6×10^6 J). Suppose the cumulative energy used in a large building over a 24-hr period is given by

$E(t) = 100\,t + 4\,t^2 - \frac{t^3}{9}$ kWh, where $t = 0$ corresponds to midnight.

a. Graph the energy function.

b. The power is the rate of energy consumption; that is $P(t) = E'(t)$. Find the power over the interval $0 \le t \le 24$.

c. Graph the power function and interpret the graph. What are the units of power in this case?

57. **A race** Jean and Juan run a one-lap race on a circular track. Their angular positions on the track during the race are given by the functions $\theta(t)$ and $\varphi(t)$, respectively, where $0 \le t \le 4$ and t is measured in minutes (see figure). These angles are measured in radians, where $\theta = \varphi = 0$ represent the starting position and $\theta = \varphi = 2\pi$ represent the finish position. The angular velocities of the runners are $\theta'(t)$ and $\varphi'(t)$.

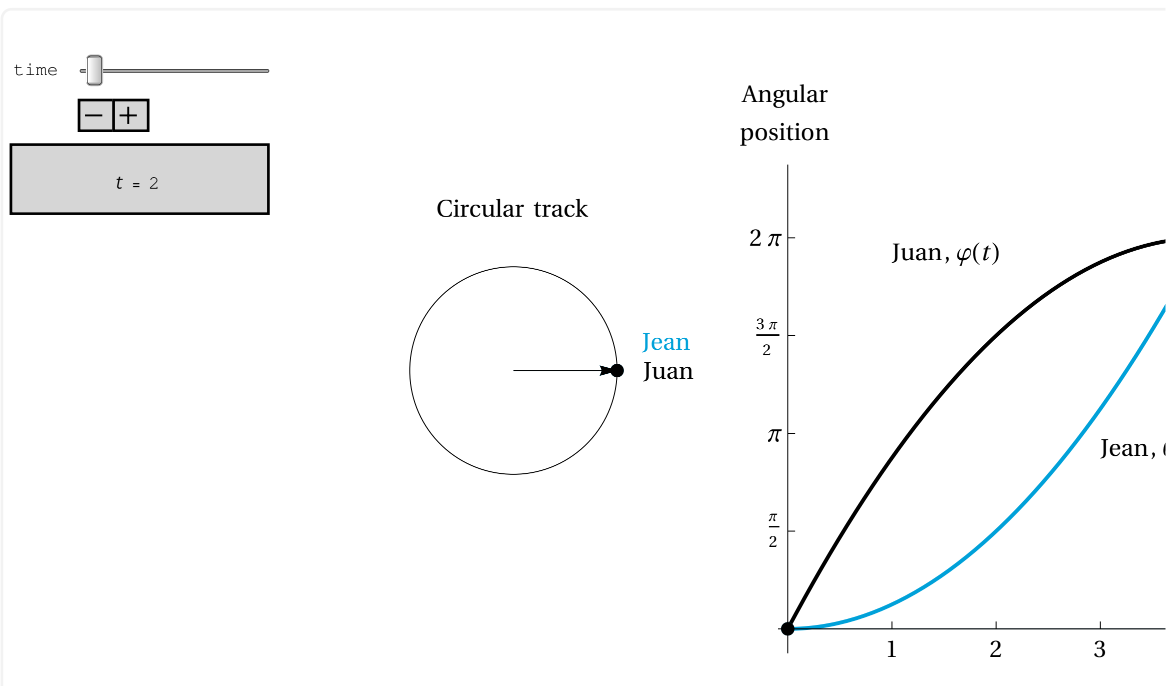
a. Compare in words the angular velocity of the two runners and the progress of the race.

b. Which runner has the greater average angular velocity?

c. Who wins the race?

d. Jean's position is given by $\theta(t) = \pi\,t^2/8$. What is her angular velocity at $t = 2$ and at what time is her angular velocity the greatest?

e. Juan's position is given by $\varphi(t) = \pi\,t\,(8 - t)/8$. What is his angular velocity at $t = 2$ and at what time is his angular velocity the greatest?



T **58.** **Flow from a tank** A cylindrical tank is full at time $t = 0$ when a valve in the bottom of the tank is opened. By Torricelli's law, the volume of water in the tank after t hours is $V = 100\,(200 - t)^2$, measured in cubic meters.

a. Graph the volume function. What is the volume of water in the tank before the valve is opened?

b. How long does it take for the tank to empty?

c. Find the rate at which water flows from the tank and plot the flow rate function.

d. At what time is the magnitude of the flow rate a minimum? A maximum?

59. **Temperature distribution** A thin copper rod, 4 m in length, is heated at its midpoint, and the ends are held at a constant temperature of $0°$. When the temperature reaches equilibrium, the temperature profile is given by $T(x) = 40\,x\,(4 - x)$, where $0 \le x \le 4$ is the position along the rod. The **heat flux** at a point on the rod equals $-k\,T'(x)$, where $k > 0$ is a constant. If the heat flux is positive at a point, heat moves in the positive x-direction at that point, and if the heat flux is negative, heat moves in the negative x-direction.

a. With $k = 1$, what is the heat flux at $x = 1$? At $x = 3$?

b. For what values of x is the heat flux negative? Positive?

c. Explain the statement that heat flows out of the rod at its ends.

T **60.** **Spring runoff** The flow of a small stream is monitored for 90 days between May 1 and August 1. The total water that flows past a gauging station is given by

$$V(t) = \begin{cases} \frac{4}{5}\,t^2 & \text{if } 0 \le t < 45 \\ -\frac{4}{5}\left(t^2 - 180\,t + 4050\right) & \text{if } 45 \le t < 90, \end{cases}$$

where V is measured in cubic feet and t is measured in days, with $t = 0$ corresponding to May 1.

a. Graph the volume function.

b. Find the flow rate function $V'(t)$ and graph it. What are the units of the flow rate?

c. Describe the flow of the stream over the 3-month period. Specifically, when is the flow rate a maximum?