3.5 Derivatives of Trigonometric Functions

From variations in market trends and ocean temperatures to daily fluctuations in tides and hormone levels, change is often cyclical or periodic. Trigonometric functions are well suited for describing such cyclical behavior. In this section, we investigate the derivatives of trigonometric functions and their many uses.

Two Special Limits »

Our principle goal is to determine derivative formulas for $\sin x$ and $\cos x$. In order to do this, we use two special limits.

Note »

THEOREM 3.9 Trigonometric Limits

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad \lim_{x\to 0} \frac{\cos x - 1}{x} = 0$$

Note that these limits cannot be evaluated by direct substitution because in both cases, the numerator and denominator approach zero as $x \to 0$. We first examine numerical and graphical evidence supporting Theorem 3.9, and then we offer an analytic proof.

The values of $\frac{\sin x}{x}$, rounded to 10 digits, appear in Table 3.1. As x approaches zero from both sides, it appears that $\frac{\sin x}{x}$ approaches 1. **Figure 3.33** shows a graph of $y = \frac{\sin x}{x}$, with a hole at $x = 0$, where the function is undefined. The graphical evidence also strongly suggests (but does not prove) that $\lim_{x\to 0} \frac{\sin x}{x} = 1$. Similar evidence also indicates that $\frac{\cos x - 1}{x}$ approaches 0 as x approaches 0.

Table 3.1

x	$\frac{\sin x}{x}$
± 0.1	0.9983341665
± 0.01	0.9999833334
± 0.001	0.9999998333

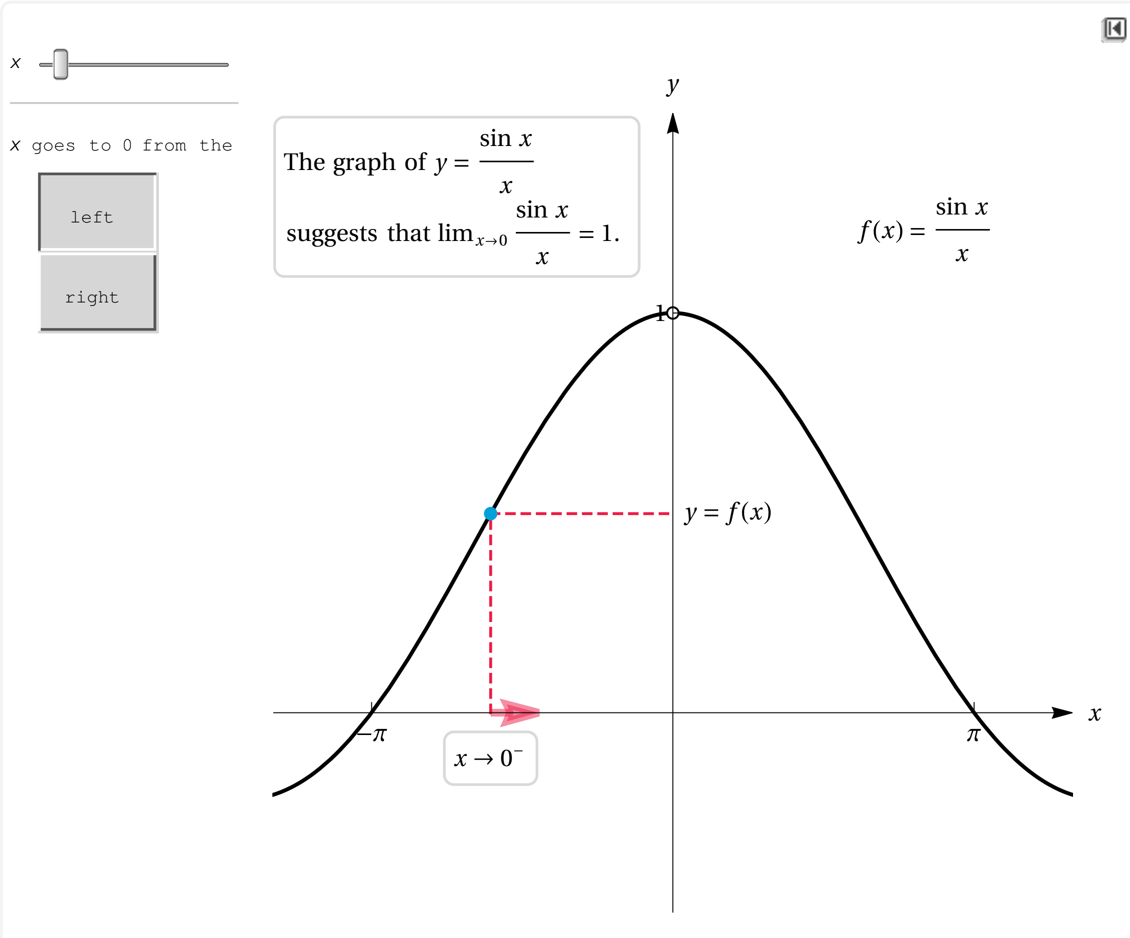


Figure 3.33

Using a geometric argument and the methods of Chapter 2, we now prove $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$. The proof that $\lim\limits_{x\to 0} \dfrac{\cos x - 1}{x} = 0$ is found in Exercise 81.

Proof: Consider **Figure 3.34**, in which $\triangle OAD$, $\triangle OBC$, and the sector OAC of the unit circle (with central angle x) are shown. Observe that with $0 < x < \dfrac{\pi}{2}$,

$$\text{area of } \triangle OAD < \text{area of sector } OAC < \text{area of } \triangle OBC. \tag{1}$$

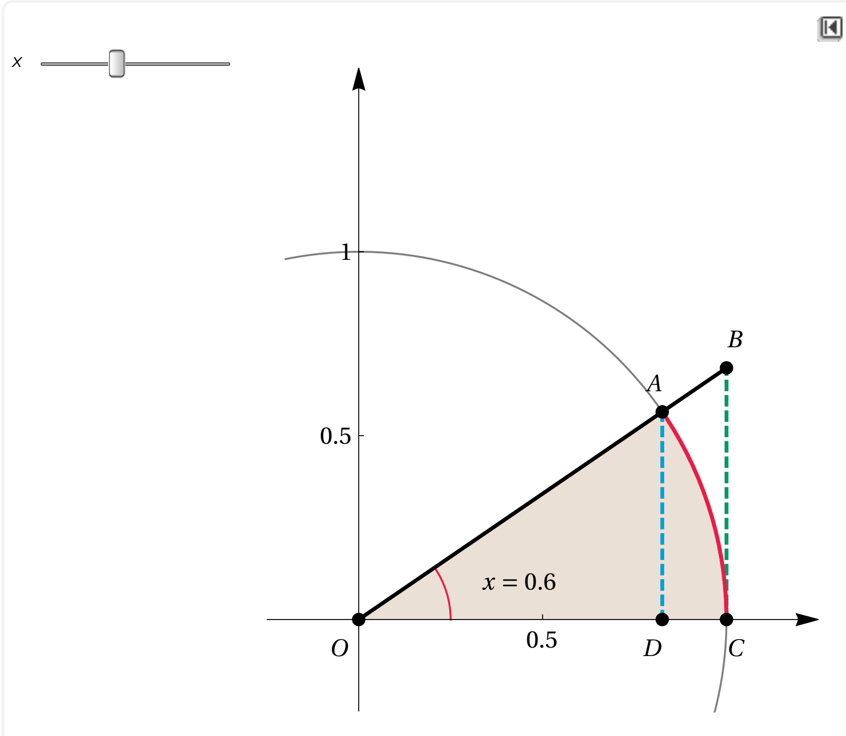


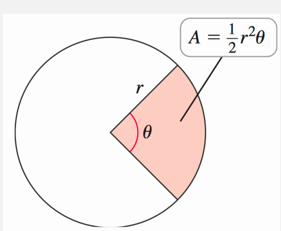
Figure 3.34

Because the circle in Figure 3.34 is a *unit* circle, $OA = OC = 1$. It follows that $\sin x = \dfrac{AD}{OA} = AD$, $\cos x = \dfrac{OD}{OA} = OD$, and $\tan x = \dfrac{BC}{OC} = BC$. From these observations, we conclude that

- the area of $\triangle OAD = \frac{1}{2}(OD)(AD) = \frac{1}{2}\cos x \sin x$,
- the area of sector $OAC = \frac{1}{2} \cdot 1^2 \cdot x = \frac{x}{2}$, and
- the area of $\triangle OBC = \frac{1}{2}(OC)(BC) = \frac{1}{2}\tan x$.

Note »

Area of the sector of a circle of radius r formed by a central angle θ:



Substituting these results into (1), we have

$$\frac{1}{2}\cos x \sin x < \frac{x}{2} < \frac{1}{2}\tan x.$$

Replacing $\tan x$ with $\dfrac{\sin x}{\cos x}$ and multiplying the inequalities by $\dfrac{2}{\sin x}$ (which is positive) leads to the inequalities

$$\cos x < \frac{x}{\sin x} < \frac{1}{\cos x}.$$

When we take reciprocals and reverse the inequalities, we have

$$\cos x < \frac{\sin x}{x} < \frac{1}{\cos x}, \qquad (2)$$

for $0 < x < \pi/2$.

A similar argument may be used to show that the inequalities in (2) also hold for $-\pi/2 < x < 0$. Taking the limit as $x \to 0$ in (2), we find that

$$\underbrace{\lim_{x\to 0} \cos x}_{1} \le \lim_{x\to 0} \frac{\sin x}{x} \le \underbrace{\lim_{x\to 0} \frac{1}{\cos x}}_{1}.$$

The Squeeze Theorem (Theorem 2.5) now implies that $\displaystyle\lim_{x\to 0} \frac{\sin x}{x} = 1$. ◆

Note »

> $\displaystyle\lim_{x\to 0} \frac{\sin x}{x} = 1$ implies that if $|x|$ is small, then $\sin x \approx x$.

EXAMPLE 1 Calculating trigonometric limits

Evaluate the following limits.

a. $\displaystyle\lim_{x\to 0} \frac{\sin 4x}{x}$

b. $\displaystyle\lim_{x\to 0} \frac{\sin 3x}{\sin 5x}$

SOLUTION »

a. To use the fact that $\displaystyle\lim_{x\to 0} \frac{\sin x}{x} = 1$, the argument of the sine function in the numerator must be the same as the denominator. Multiplying and dividing $\dfrac{\sin 4x}{x}$ by 4, we evaluate the limit as follows:

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin 4x}{x} &= \lim_{x\to 0} \frac{4 \sin 4x}{4x} && \text{Multiply and divide by 4.}\\
&= 4 \underbrace{\lim_{t\to 0} \frac{\sin t}{t}}_{1} && \text{Factor out 4 and let } t = 4x;\ t \to 0 \text{ as } x \to 0.\\
&= 4(1) = 4. && \text{Theorem 3.9}
\end{aligned}$$

b. The first step is to divide the numerator and denominator of $\dfrac{\sin 3x}{\sin 5x}$ by x:

$$\frac{\sin 3x}{\sin 5x} = \frac{(\sin 3x)/x}{(\sin 5x)/x}.$$

As in part (a), we now divide and multiply $\dfrac{\sin 3x}{x}$ by 3 and divide and multiply $\dfrac{\sin 5x}{x}$ by 5. In the numerator, we let $t = 3x$, and in the denominator, we let $u = 5x$. In each case, $t \to 0$ and $u \to 0$ as $x \to 0$. Therefore,

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin 3x}{\sin 5x} &= \lim_{x\to 0} \frac{\frac{3 \sin 3x}{3x}}{\frac{5 \sin 5x}{5x}} && \text{Multiply and divide by 3 and 5.}\\
&= \frac{3}{5}\, \frac{\lim_{t\to 0} (\sin t)/t}{\lim_{u\to 0} (\sin u)/u} && t = 3x \text{ in numerator and } u = 5x \text{ in denominator.}\\
&= \frac{3}{5} \cdot \frac{1}{1} = \frac{3}{5}. && \text{Both limits equal 1.}
\end{aligned}$$

Related Exercises 12–13 ◆

Quick Check 1 Evaluate $\lim_{x\to 0} \dfrac{\tan 2x}{x}$. ◆

Answer »

2

We now use the important limits of Theorem 3.9 to establish the derivatives of $\sin x$ and $\cos x$.

Derivatives of Sine and Cosine Functions »

We start with the definition of the derivative,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h},$$

with $f(x) = \sin x$, and then appeal to the sine addition identity

$$\sin(x+h) = \sin x \cos h + \cos x \sin h.$$

The derivative is

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} && \text{Definition of derivative} \\
&= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} && \text{Sine addition identity} \\
&= \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h} && \text{Factor } \sin x. \\
&= \lim_{h\to 0} \frac{\sin x(\cos h - 1)}{h} + \lim_{h\to 0} \frac{\cos x \sin h}{h} && \text{Theorem 2.3} \\
&= \sin x \underbrace{\left(\lim_{h\to 0} \frac{\cos h - 1}{h}\right)}_{0} + \cos x \underbrace{\left(\lim_{h\to 0} \frac{\sin h}{h}\right)}_{1} && \text{Both } \sin x \text{ and } \cos x \text{ are independent of } h. \\
&= (\sin x)(0) + \cos x(1) && \text{Theorem 3.9} \\
&= \cos x. && \text{Simplify.}
\end{aligned}
$$

We have proved the important result that $\frac{d}{dx}(\sin x) = \cos x$.

The fact that $\frac{d}{dx}(\cos x) = -\sin x$ is proved in a similar way using a cosine addition identity (Exercise 83).

THEOREM 3.10 **Derivatives of Sine and Cosine**

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$

From a geometric point of view, these derivative formulas make sense. Because $f(x) = \sin x$ is a periodic function, we expect its derivative to be periodic. Observe that the horizontal tangent lines on the graph of $f(x) = \sin x$ (**Figure 3.35a**) occur at the zeros of $f'(x) = \cos x$.

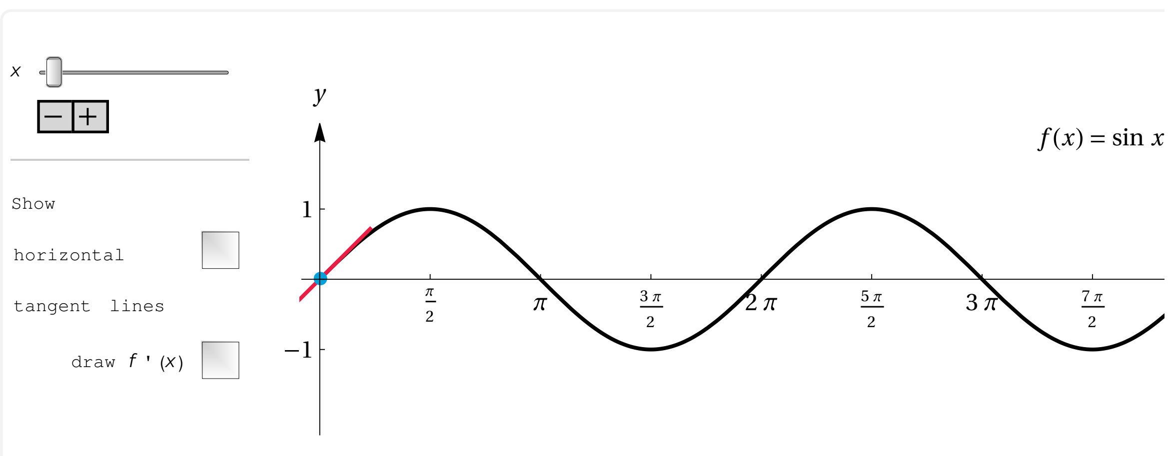


Figure 3.35 a

Similarly, the horizontal tangent lines on the graph of $f(x) = \cos x$ occur at the zeros of $f'(x) = -\sin x$ (**Figure 3.35b**).

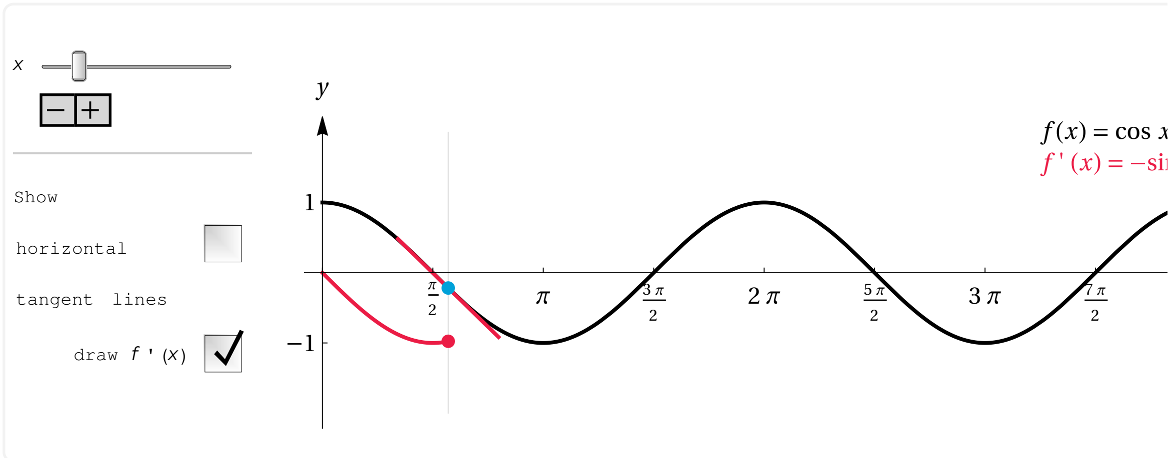


Figure 3.35 b

Quick Check 2 At what points on the interval $[0, 2\pi]$ does the graph of $f(x) = \sin x$ have tangent lines with positive slopes? At what points on the interval $[0, 2\pi]$ is $\cos x > 0$? Explain the connection. ◆

Answer »

> $0 < x < \frac{\pi}{2}$ and $\frac{3\pi}{2} < x < 2\pi$. The value of $\cos x$ is the slope of the line tangent to the curve $y = \sin x$.

EXAMPLE 2 Derivatives involving trigonometric functions

Calculate $\frac{dy}{dx}$ for the following functions.

a. $y = x^2 \cos x$

b. $y = \sin x - x \cos x$

c. $y = \frac{1 + \sin x}{1 - \sin x}$

SOLUTION »

a.

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^2 \cdot \cos x\right) = \overbrace{2x\cos x}^{(\text{derivative of } x^2)\cdot \cos x} + \overbrace{x^2(-\sin x)}^{x^2\cdot(\text{derivative of } \cos x)} \qquad \text{Product Rule}$$

$$= x(2\cos x - x\sin x) \qquad \text{Simplify.}$$

b.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(\sin x) - \frac{d}{dx}(x\cos x) && \text{Difference Rule}\\ &= \cos x - \Big[\underbrace{(1)\cos x}_{\substack{\text{(derivative of } x)\\ \cdot \cos x}} + \underbrace{x(-\sin x)}_{\substack{x\,\cdot\,\text{(derivative of}\\ \cos x)}}\Big] && \text{Product Rule}\\ &= x\sin x && \text{Simplify.}\end{aligned}$$

c.

$$\begin{aligned}\frac{dy}{dx} &= \frac{(1-\sin x)\overbrace{(\cos x)}^{\substack{\text{derivative of}\\ 1+\sin x}} - (1+\sin x)\overbrace{(-\cos x)}^{\substack{\text{derivative of}\\ 1-\sin x}}}{(1-\sin x)^2} && \text{Quotient Rule}\\ &= \frac{\cos x - \cos x\sin x + \cos x + \sin x\cos x}{(1-\sin x)^2} && \text{Expand.}\\ &= \frac{2\cos x}{(1-\sin x)^2} && \text{Simplify.}\end{aligned}$$

Related Exercises 25, 27, 29 ◆

Derivatives of Other Trigonometric Functions »

The derivatives of $\tan x$, $\cot x$, $\sec x$, and $\csc x$ are obtained using the derivatives of $\sin x$ and $\cos x$ together with the Quotient Rule and trigonometric identities.

Note »

Recall that $\tan x = \frac{\sin x}{\cos x}$, $\cot x = \frac{\cos x}{\sin x}$, $\sec x = \frac{1}{\cos x}$, and $\csc x = \frac{1}{\sin x}$.

EXAMPLE 3 Derivative of the tangent function

Calculate $\frac{d}{dx}(\tan x)$.

SOLUTION »

Using the identity $\tan x = \frac{\sin x}{\cos x}$ and the Quotient Rule, we have

$$\begin{aligned}\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)\\ &= \frac{\cos x\overbrace{\cos x}^{\substack{\text{derivative}\\ \text{of } \sin x}} - \sin x\overbrace{(-\sin x)}^{\substack{\text{derivative}\\ \text{of } \cos x}}}{\cos^2 x} && \text{Quotient Rule}\\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} && \text{Simplify numerator.}\\ &= \frac{1}{\cos^2 x} = \sec^2 x. && \cos^2 x + \sin^2 x = 1\end{aligned}$$

Therefore, $\dfrac{d}{dx}(\tan x) = \sec^2 x$.

Related Exercises 52–54 ◆

The derivatives of $\cot x$, $\sec x$, and $\csc x$ are given in Theorem 3.11 (Exercises 52–54).

THEOREM 3.11 **Derivatives of the Trigonometric Functions**

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$
$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$
$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

Note »

One way to remember Theorem 3.11 is to learn the derivatives of the sine, tangent, and secant functions. Then, replace each function by its corresponding **cofunction** and put a negative sign on the right-hand side of the new derivative formula.

$$\frac{d}{dx}(\sin x) = \cos x \leftrightarrow \frac{d}{dx}(\cos x) = -\sin x$$
$$\frac{d}{dx}(\tan x) = \sec^2 x \leftrightarrow \frac{d}{dx}(\cot x) = -\csc^2 x$$
$$\frac{d}{dx}(\sec x) = \sec x \tan x \leftrightarrow \frac{d}{dx}(\csc x) = -\csc x \cot x$$

Quick Check 3 The formulas for $\dfrac{d}{dx}(\cot x)$, $\dfrac{d}{dx}(\sec x)$, and $\dfrac{d}{dx}(\csc x)$ can be determined using the Quotient Rule. Why? ◆

Answer »

The Quotient Rule is used because each function is a quotient when written in terms of the sine and cosine functions.

EXAMPLE 4 Derivatives involving sec *x* and csc *x*

Find the derivative of $y = \sec x \csc x$.

SOLUTION »

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}(\sec x \cdot \csc x) \\
&= \underbrace{\sec x \tan x}_{\text{derivative of } \sec x} \csc x + \sec x \underbrace{(-\csc x \cot x)}_{\text{derivative of } \csc x} && \text{Product Rule} \\
&= \underbrace{\frac{1}{\cos x}}_{\sec x} \cdot \underbrace{\frac{\sin x}{\cos x}}_{\tan x} \cdot \underbrace{\frac{1}{\sin x}}_{\csc x} - \underbrace{\frac{1}{\cos x}}_{\sec x} \cdot \underbrace{\frac{1}{\sin x}}_{\csc x} \cdot \underbrace{\frac{\cos x}{\sin x}}_{\cot x} && \text{Write functions in terms of } \sin x \text{ and } \cos x. \\
&= \frac{1}{\cos^2 x} - \frac{1}{\sin^2 x} && \text{Cancel and simplify.} \\
&= \sec^2 x - \csc^2 x && \text{Definition of } \sec x \text{ and } \csc x
\end{aligned}$$

Related Exercises 43–44 ◆

Higher-Order Trigonometric Derivatives »

Higher-order derivatives of the sine and cosine functions are important in many applications. A few higher-order derivatives of $y = \sin x$ reveal a pattern.

$$\frac{dy}{dx} = \cos x \qquad \frac{d^2y}{dx^2} = \frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}(-\sin x) = -\cos x \qquad \frac{d^4y}{dx^4} = \frac{d}{dx}(-\cos x) = \sin x$$

We see that the higher-order derivatives of $\sin x$ cycle back periodically to $\pm \sin x$. In general, it can be shown that $\frac{d^{2n}y}{dx^{2n}} = (-1)^n \sin x$, with a similar result for $\cos x$ (Exercise 88). This cyclic behavior in the derivatives of $\sin x$ and $\cos x$ does not occur with the other trigonometric functions.

Quick Check 4 Find $\frac{d^2y}{dx^2}$ and $\frac{d^4y}{dx^4}$ when $y = \cos x$. Find $\frac{d^{40}y}{dx^{40}}$ and $\frac{d^{42}y}{dx^{42}}$ when $y = \sin x$. ◆

Answer »

$\frac{d^2y}{dx^2} = -\cos x$, $\frac{d^4y}{dx^4} = \cos x$, $\frac{d^{40}}{dx^{40}}(\sin x) = \sin x$, $\frac{d^{42}}{dx^{42}}(\sin x) = -\sin x$

EXAMPLE 5 Second-order derivatives

Find the second derivative of $y = \csc x$.

SOLUTION »

By Theorem 3.11, $\frac{dy}{dx} = -\csc x \cot x$. Applying the Product Rule gives the second derivative:

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx}(-\csc x \cot x) \\
&= \left(\frac{d}{dx}(-\csc x)\right)\cot x - \csc x \frac{d}{dx}(\cot x) && \text{Product Rule} \\
&= (\csc x \cot x)\cot x - \csc x\left(-\csc^2 x\right) && \text{Calculate derivatives.} \\
&= \csc x\left(\cot^2 x + \csc^2 x\right). && \text{Factor.}
\end{aligned}$$

Related Exercises 61–62 ◆

Exercises »

Getting Started »

Practice Exercises »

11–22. Trigonometric limits *Use Theorem 3.9 to evaluate the following limits.*

11. $\displaystyle\lim_{x\to 0} \frac{\sin 3x}{x}$

12. $\displaystyle\lim_{x\to 0} \frac{\sin 5x}{3x}$

13. $\displaystyle\lim_{x\to 0} \frac{\sin 7x}{\sin 3x}$

14. $\displaystyle\lim_{x\to 0} \frac{\sin 3x}{\tan 4x}$

15. $\displaystyle\lim_{x\to 0} \frac{\tan 5x}{x}$

16. $\displaystyle\lim_{\theta\to 0} \frac{\cos^2\theta - 1}{\theta}$

17. $\displaystyle\lim_{x\to 0} \frac{\tan 7x}{\sin x}$

18. $\displaystyle\lim_{\theta\to 0} \frac{\sec\theta - 1}{\theta}$

19. $\displaystyle\lim_{x\to 2} \frac{\sin(x-2)}{x^2-4}$

20. $\displaystyle\lim_{x\to -3} \frac{\sin(x+3)}{x^2+8x+15}$

21. $\displaystyle\lim_{x\to 0} \frac{\sin ax}{\sin bx}$, where a and b are constants with $b \neq 0$

22. $\lim_{x \to 0} \dfrac{\sin a\,x}{b\,x}$, where a and b are constants with $b \neq 0$

23–51. Calculating derivatives *Find the derivative of the following functions.*

23. $y = \sin x + \cos x$

24. $y = 5\,x^2 + \cos x$

25. $y = 3\,x^4 \sin x$

26. $y = \sin x + \dfrac{4 \cos x}{x}$

27. $y = x \sin x$

28. $y = \dfrac{x}{\sin x + 1}$

29. $y = \dfrac{\cos x}{\sin x + 1}$

30. $y = \dfrac{1 - \sin x}{1 + \sin x}$

31. $y = \sin x \cos x$

32. $y = \dfrac{a \sin x + b \cos x}{a \sin x - b \cos x}$; a and b are nonzero constants

33. $y = \cos^2 x$

34. $y = \dfrac{x \sin x}{1 + \cos x}$

35. $y = w^2 \sin w + 2\,w \cos w - 2 \sin w$

36. $y = -x^3 \cos x + 3\,x^2 \sin x + 6\,x \cos x - 6 \sin x$

37. $y = x \cos x \sin x$

38. $y = \dfrac{1}{2 + \sin x}$

39. $y = \dfrac{\sin x}{1 + \cos x}$

40. $y = \dfrac{1 - \sin x}{1 + \sin x}$

41. $y = \dfrac{1 - \cos x}{1 + \cos x}$

42. $y = \tan x + \cot x$

43. $y = \sec x + \csc x$

44. $y = \sec x \tan x$

45. $y = \sqrt{x} \csc x$

46. $y = \dfrac{\tan w}{1 + \tan w}$

47. $y = \dfrac{\cot x}{1 + \csc x}$

48. $y = \dfrac{\tan t}{1 + \sec t}$

49. $y = \dfrac{1}{\sec z \csc z}$

50. $y = \csc^2 \theta - 1$

51. $y = x - \cos x \sin x$

52–54. Verifying derivative formulas *Verify the following derivative formulas using the Quotient Rule.*

52. $\dfrac{d}{dx}(\cot x) = -\csc^2 x$

53. $\dfrac{d}{dx}(\sec x) = \sec x \tan x$

54. $\dfrac{d}{dx}(\csc x) = -\csc x \cot x$

T **55. Velocity of an oscillator** An object oscillates along a vertical line, and its position in centimeters is given by $y(t) = 30(\sin t - 1)$, where $t \geq 0$ is measured in seconds and y is positive in the upward direction.

- **a.** Graph the position function, for $0 \leq t \leq 10$.
- **b.** Find the velocity of the oscillator, $v(t) = y'(t)$.
- **c.** Graph the velocity function, for $0 \leq t \leq 10$.
- **d.** At what times and positions is the velocity zero?
- **e.** At what times and positions is the velocity a maximum?
- **f.** The acceleration of the oscillator is $a(t) = v'(t)$. Find and graph the acceleration function.

T **56. Resonance** An oscillator (such as a mass on a spring or a component in an electrical circuit) is subject to external forces that have the same frequency as the oscillator itself may undergo motion called *resonance* (at least for short periods of time). The position function of an oscillator in resonance has the form $y(t) = At \sin t$, where A is a constant.

a. Graph the position function with $A = \frac{1}{2}$, for $0 \le t \le 20$. How does the amplitude of the oscillation (the height of the peaks) change as t increases?

b. Compute and graph the velocity of the object, $v(t) = y'(t)$ (with $A = \frac{1}{2}$), for $0 \le t \le 20$.

c. Where do the zeros of the velocity function appear relative to the peaks and valleys of the position function?

d. If the oscillator were a suspension bridge, explain why resonance could be catastrophic.

57–64. Second derivatives *Find y'' for the following functions.*

57. $y = x \sin x$

58. $y = x^2 \cos x$

59. $y = \dfrac{\sin x}{x}$

60. $y = x^2 \cos x$

61. $y = \cot x$

62. $y = \tan x$

63. $y = \sec x \csc x$

64. $y = \cos\theta \sin\theta$

65. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. $\dfrac{d}{dx}\left(\sin^2 x\right) = \cos^2 x.$

b. $\dfrac{d^2}{dx^2}(\sin x) = \sin x.$

c. $\dfrac{d^4}{dx^4}(\cos x) = \cos x.$

d. The function $\sec x$ is not differentiable at $x = \pi/2$.

66–71. Trigonometric limits *Evaluate the following limits or state that they do not exist. (Hint: Identify each limit as the derivative of a function at a point.)*

66. $\displaystyle\lim_{x \to \pi/2} \frac{\cos x}{x - \pi/2}$

67. $\displaystyle\lim_{x \to \pi/4} \frac{\tan x - 1}{x - \pi/4}$

68. $\displaystyle\lim_{h \to 0} \frac{\sin\left(\frac{\pi}{6} + h\right) - \frac{1}{2}}{h}$

69. $\lim\limits_{h\to 0} \dfrac{\cos\left(\frac{\pi}{6}+h\right)-\frac{\sqrt{3}}{2}}{h}$

70. $\lim\limits_{x\to \pi/4} \dfrac{\cot x - 1}{x-\frac{\pi}{4}}$

71. $\lim\limits_{h\to 0} \dfrac{\tan\left(\frac{5\pi}{6}+h\right)+\frac{1}{\sqrt{3}}}{h}$

T **72–75. Equations of tangent lines**

a. *Find an equation of the line tangent to the following curves at the given value of x.*

b. *Use a graphing utility to plot the curve and the tangent line.*

72. $y = 4\sin x\cos x;\ x = \dfrac{\pi}{3}$

73. $y = 1 + 2\sin x;\ x = \dfrac{\pi}{6}$

74. $y = \csc x;\ x = \dfrac{\pi}{4}$

75. $y = \dfrac{\cos x}{1-\cos x};\ x = \dfrac{\pi}{3}$

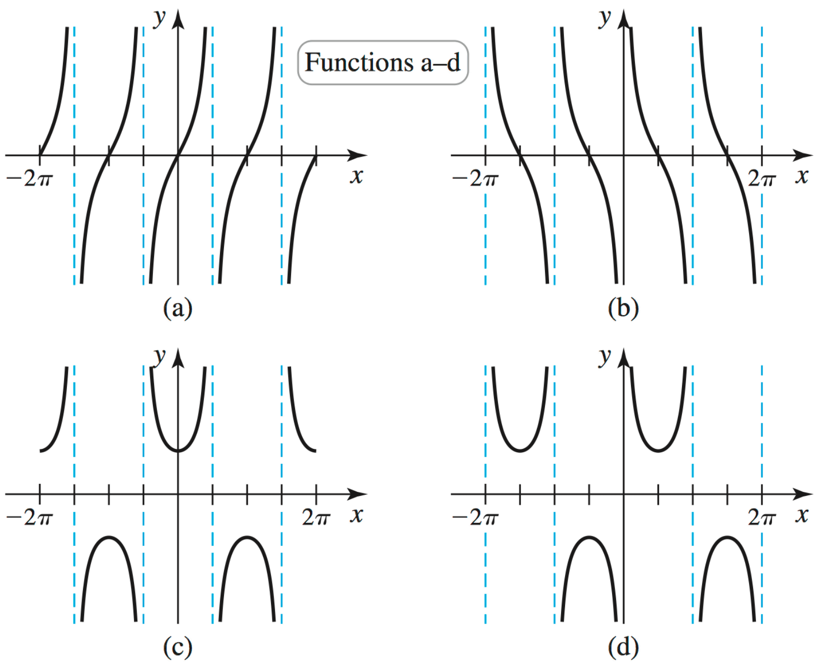
76. Locations of tangent lines

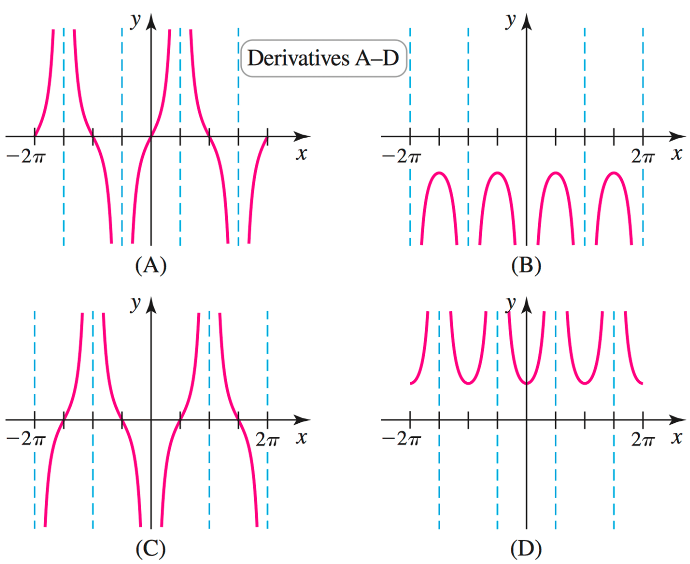
a. For what values of x does $g(x) = x - \sin x$ have a horizontal tangent line?

b. For what values of x does $g(x) = x - \sin x$ have a slope of 1?

77. Locations of horizontal tangent lines For what values of x does $f(x) = x - 2\cos x$ have a horizontal tangent line?

78. Matching Match the graphs of the functions in a–d with the graphs of their derivatives in A-D.





79. A differential equation A differential equation is an equation involving an unknown function and its derivatives. Consider the differential equation $y''(t) + y(t) = 0$.

a. Show that $y = A \sin t$ satisfies the equation for any constant A.

b. Show that $y = B \cos t$ satisfies the equation for any constant B.

c. Show that $y = A \sin t + B \cos t$ satisfies the equation for any constants A and B.

Explorations and Challenges »

80. Using identities Use the identity $\sin 2x = 2 \sin x \cos x$ to find $\dfrac{d}{dx}(\sin 2x)$. Then use the identity $\cos 2x = \cos^2 x - \sin^2 x$ to express the derivative of $\sin 2x$ in terms of $\cos 2x$.

81. Proof of $\lim_{x \to 0} \frac{\cos x - 1}{x} = 0$ Use the trigonometric identity $\cos^2 x + \sin^2 x = 1$ to prove that $\lim_{x \to 0} \frac{\cos x - 1}{x} = 0$. (*Hint:* Begin by multiplying the numerator and denominator by $\cos x + 1$.)

82. Another method for proving $\lim_{x \to 0} \frac{\cos x - 1}{x} = 0$ Use the half-angle formula $\sin^2 x = \frac{1 - \cos 2x}{2}$ to prove that $\lim_{x \to 0} \frac{\cos x - 1}{x} = 0$.

83. Proof of $\frac{d}{dx}(\cos x) = -\sin x$ Use the definition of the derivative and the trigonometric identity

$$\cos(x + h) = \cos x \cos h - \sin x \sin h$$

to prove that $\frac{d}{dx}(\cos x) = -\sin x$.

84. Continuity of a piecewise function Let

$$f(x) = \begin{cases} \frac{3 \sin x}{x} & \text{if } x \neq 0 \\ a & \text{if } x = 0. \end{cases}$$

For what values of a is f continuous?

85. Continuity of a piecewise function Let

$$g(x) = \begin{cases} \frac{1 - \cos x}{2x} & \text{if } x \neq 0 \\ a & \text{if } x = 0. \end{cases}$$

For what values of a is g continuous?

T **86. Computing limits with angles in degrees** Suppose your graphing calculator has two functions, one called $\sin x$, which calculates the sine of x when x is in radians and the other called $s(x)$, which calculates the sine of x when x is in degrees.

a. Explain why $s(x) = \sin\left(\frac{\pi}{180} x\right)$.

b. Evaluate $\lim_{x \to 0} \frac{s(x)}{x}$. Verify your answer by estimating the limit on your calculator.

87. Derivatives of $\sin^n x$ Calculate the following derivatives using the Product Rule.

a. $\frac{d}{dx}\left(\sin^2 x\right)$

b. $\frac{d}{dx}\left(\sin^3 x\right)$

c. $\frac{d}{dx}\left(\sin^4 x\right)$

d. Based upon your answers to parts (a)–(c), make a conjecture about $\frac{d}{dx}(\sin^n x)$, where n is a positive integer. Then prove the result by induction.

88. Prove that $\dfrac{d^{2n}}{dx^{2n}}(\sin x) = (-1)^n \sin x$ and $\dfrac{d^{2n}}{dx^{2n}}(\cos x) = (-1)^n \cos x$.

T **89–90. Difference quotients** *Suppose f is differentiable for all x and consider the function*

$$D(x) = \frac{f(x + 0.01) - f(x)}{0.01}.$$

For the following functions, graph D on the given interval, and explain why the graph appears as it does. What is the relationship between the functions f and D?

89. $f(x) = \sin x$ on $[-\pi, \pi]$

90. $f(x) = \dfrac{x^3}{3} + 1$ on $[-2, 2]$