3.4 The Product and Quotient Rules

The derivative of a sum of functions is the sum of the derivatives. So, you might assume that the derivative of a product is the product of the derivatives. Consider, however, the functions $f(x) = x^3$ and $g(x) = x^4$. In this case, $\frac{d}{dx}(f(x)\,g(x)) = \frac{d}{dx}\left(x^7\right) = 7\,x^6$, but $f'(x)\,g'(x) = 3\,x^2 \cdot 4\,x^3 = 12\,x^5$. Therefore, $\frac{d}{dx}(f \cdot g) \neq f' \cdot g'$. Similarly, the derivative of a quotient is *not* the quotient of the derivatives. The purpose of this section is to develop rules for differentiating products and quotients of functions.

Product Rule »

Here is an anecdote that suggests the formula for the Product Rule. Imagine running along a road at a constant speed. Your speed is determined by two factors: the length of your stride and the number of strides you take each second. Therefore,

$$\text{running speed} = \text{stride length} \cdot \text{stride rate}.$$

For example, if your stride length is 3 ft per stride and you take 2 strides/s, then your speed is 6 ft/s.

Now, suppose your stride length increases by 0.5 ft, from 3 to 3.5 ft. Then the change in speed is calculated as follows:

$$\begin{aligned}\textit{change} \text{ in speed} &= \text{change in stride length} \cdot \text{stride rate} \\ &= 0.5 \cdot 2 = 1 \text{ ft/s}.\end{aligned}$$

Alternatively, suppose your stride length remains constant but your stride rate increases by 0.25 stride/s, from 2 to 2.25 strides/s. Then

$$\begin{aligned}\textit{change} \text{ in speed} &= \text{stride length} \cdot \text{change in stride rate} \\ &= 3 \cdot 0.25 = 0.75 \text{ ft/s}.\end{aligned}$$

If both your stride rate and stride length change simultaneously, we expect two contributions to the change in your running speed:

$$\begin{aligned}\textit{change} \text{ in speed} &= (\text{change in stride length} \cdot \text{stride rate}) + (\text{stride length} \cdot \text{change in stride rate}) \\ &= 1 \text{ ft/s} + 0.75 \text{ ft/s} = 1.75 \text{ ft/s}.\end{aligned}$$

This argument correctly suggests that the derivative (or rate of change) of a product of two functions has *two components*, as shown by the following rule.

THEOREM 3.6 Product Rule

If f and g are differentiable at x, then

$$\frac{d}{dx}(f(x)\,g(x)) = f'(x)\,g(x) + f(x)\,g'(x).$$

Note »

Proof: We apply the definition of the derivative to the function $f\,g$:

$$\frac{d}{dx}(f(x)\,g(x)) = \lim_{h \to 0} \frac{f(x+h)\,g(x+h) - f(x)\,g(x)}{h}.$$

A useful tactic is to add $-f(x)\,g(x+h)+f(x)\,g(x+h)$ (which equals 0) to the numerator, so that

$$\frac{d}{dx}(f(x)\,g(x)) = \lim_{h\to 0}\frac{f(x+h)\,g(x+h)-f(x)\,g(x+h)+f(x)\,g(x+h)-f(x)\,g(x)}{h}$$

The fraction is now split and the numerators are factored:

$$\begin{aligned}\frac{d}{dx}(f(x)\,g(x)) &= \lim_{h\to 0}\frac{f(x+h)\,g(x+h)-f(x)\,g(x+h)}{h} + \lim_{h\to 0}\frac{f(x)\,g(x+h)-f(x)\,g(x)}{h}\\ &= \lim_{h\to 0}\underbrace{\frac{f(x+h)-f(x)}{h}}_{\substack{\text{approaches } f'(x)\\ \text{as } h\to 0}}\cdot\underbrace{g(x+h)}_{\substack{\text{approaches}\\ g(x)\\ \text{as } h\to 0}} + \lim_{h\to 0}\underbrace{f(x)}_{\substack{\text{equals}\\ f(x)\text{ as}\\ h\to 0}}\cdot\underbrace{\frac{g(x+h)-g(x)}{h}}_{\substack{\text{approaches } g'(x)\\ \text{as } h\to 0}}\\ &= f'(x)\cdot g(x)+f(x)\cdot g'(x).\end{aligned}$$

Note »

As $h\to 0$, $f(x)$ does not change in value; it is independent of h.

The continuity of g is used to conclude that $\lim_{h\to 0} g(x+h)=g(x)$. ◆

EXAMPLE 1 Using the Product Rule

Find and simplify the following derivatives.

a. $\dfrac{d}{dv}\left(v^2\left(2\sqrt{v}+1\right)\right)$

b. $\dfrac{d}{dx}\left(\left(x^3-8\right)\left(x^2+4\right)\right)$

SOLUTION »

a.

$$\begin{aligned}\frac{d}{dv}\left(v^2\left(2\sqrt{v}+1\right)\right) &= \left(\frac{d}{dv}\left(v^2\right)\right)\left(2\sqrt{v}+1\right)+v^2\left(\frac{d}{dv}\left(2\sqrt{v}+1\right)\right) && \text{Product Rule}\\ &= 2\,v\left(2\sqrt{v}+1\right)+v^2\left(2\cdot\frac{1}{2\sqrt{v}}\right) && \text{Evaluate the derivatives.}\\ &= \left(4\,v^{3/2}+2\,v\right)+v^{3/2} = 5\,v^{3/2}+2\,v && \text{Simplify.}\end{aligned}$$

Note »

In Example 2 of Section 3.2, we proved that $\dfrac{d}{dv}\left(\sqrt{v}\right)=\dfrac{1}{2\sqrt{v}}$.

b.

$$\frac{d}{dx}\left(\left(x^3-8\right)\left(x^2+4\right)\right) = \underbrace{3\,x^2}_{\frac{d}{dx}\left(x^3-8\right)}\cdot\left(x^2+4\right)+\left(x^3-8\right)\cdot\underbrace{2\,x}_{\frac{d}{dx}\left(x^2+4\right)} = x\left(5\,x^3+12\,x-16\right)$$

Related Exercises 19–20 ◆

Quick Check 1 Find the derivative of $f(x) = x^5$. Then find the same derivative using the Product Rule with $f(x) = x^2\,x^3$. ◆

Answer »

Quotient Rule »

Consider the quotient $q(x) = \dfrac{f(x)}{g(x)}$ and note that $f(x) = g(x)\,q(x)$. By the Product Rule, we have

$$f'(x) = g'(x)\,q(x) + g(x)\,q'(x).$$

Solving for $q'(x)$, we find that

$$q'(x) = \frac{f'(x) - g'(x)\,q(x)}{g(x)}.$$

Substituting $q(x) = \dfrac{f(x)}{g(x)}$ produces a rule for finding $q'(x)$:

$$q'(x) = \frac{f'(x) - g'(x)\frac{f(x)}{g(x)}}{g(x)} \qquad \text{Replace } q(x) \text{ with } \frac{f(x)}{g(x)}.$$

$$= \frac{g(x)\left(f'(x) - g'(x)\frac{f(x)}{g(x)}\right)}{g(x)\cdot g(x)} \qquad \text{Multiply numerator and denominator by } g(x).$$

$$= \frac{g(x)\,f'(x) - f(x)\,g'(x)}{(g(x))^2}. \qquad \text{Simplify.}$$

This calculation produces the correct result for the derivative of a quotient. However, there is one subtle point: How do we know that the derivative of $\dfrac{f}{g}$ exists in the first place? A complete proof of the Quotient Rule is outlined in Exercise 92.

THEOREM 3.7 The Quotient Rule

If f and g are differentiable at x and $g(x) \neq 0$, then the derivative of $\dfrac{f}{g}$ at x exists and

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\,f'(x) - f(x)\,g'(x)}{(g(x))^2}.$$

Note »

> In words, Theorem 3.7 states that the derivative of the quotient of two functions equals the denominator multiplied by the derivative of the numerator minus the numerator multiplied by the derivative of the denominator, all divided by the denominator squared. An easy way to remember the Quotient Rule is with
>
> $$\frac{Lo\,D(Hi) - Hi\,D(Lo)}{(Lo)^2}.$$

EXAMPLE 2 Using the Quotient Rule

Find and simplify the following derivatives.

a. $\dfrac{d}{dx}\left(\dfrac{x^2+3x+4}{x^2-1}\right)$

b. $\dfrac{d}{dx}\left(2x^{-3}\right)$

SOLUTION »

a.

$$\frac{d}{dx}\left(\frac{x^2+3x+4}{x^2-1}\right) = \frac{\overbrace{\left(x^2-1\right)(2x+3)}^{\left(x^2-1\right)\cdot\text{ the derivative of }\left(x^2+3x+4\right)} - \overbrace{\left(x^2+3x+4\right)2x}^{\left(x^2+3x+4\right)\cdot\text{ the derivative of }\left(x^2-1\right)}}{\underbrace{\left(x^2-1\right)^2}_{\text{the denominator }\left(x^2-1\right)\text{ squared}}} \quad \text{Quotient Rule}$$

$$= \frac{2x^3 - 2x + 3x^2 - 3 - 2x^3 - 6x^2 - 8x}{\left(x^2-1\right)^2} \quad \text{Expand.}$$

$$= \frac{-3x^2 - 10x - 3}{\left(x^2-1\right)^2} \quad \text{Simplify.}$$

Note »

> The Product and Quotient Rules are used on a regular basis throughout this text. Therefore, it is a good idea to memorize these rules (along with the other derivative rules and formulas presented in this chapter) so that you can evaluate derivatives quickly.

b. We rewrite $2x^{-3}$ as $\dfrac{2}{x^3}$, and use the Quotient Rule:

$$\frac{d}{dx}\left(\frac{2}{x^3}\right) = \frac{x^3\cdot 0 - 2\cdot 3x^2}{\left(x^3\right)^2} = -\frac{6}{x^4} = -6x^{-4}.$$

Related Exercises 22, 25 ◆

Quick Check 2 Find the derivative of $f(x) = x^5$. Then find the same derivative using the Quotient Rule with $f(x) = \frac{x^8}{x^3}$. ◆

Answer »

$f'(x) = 5x^4$ by either method

EXAMPLE 3 Finding tangent lines

Find an equation of the line tangent to the graph of $f(x) = \frac{x^2+1}{x^2-4}$ at the point (3, 2). Plot the curve and the tangent line.

SOLUTION »

To find the slope of the tangent line, we compute f' using the Quotient Rule:

$$f'(x) = \frac{(x^2-4)\,2x - (x^2+1)\,2x}{(x^2-4)^2} \qquad \text{Quotient Rule}$$

$$= \frac{2x^3 - 8x - 2x^3 - 2x}{(x^2-4)^2} = -\frac{10x}{(x^2-4)^2}. \qquad \text{Simplify.}$$

The slope of the tangent line at (3, 2) is

$$m_{\tan} = f'(3) = -\frac{10(3)}{(3^2-4)^2} = -\frac{6}{5}.$$

Therefore, an equation of the tangent line is

$$y - 2 = -\frac{6}{5}(x-3), \quad \text{or} \quad y = -\frac{6}{5}x + \frac{28}{5}.$$

The graphs of f and the tangent line are shown in **Figure 3.31**.

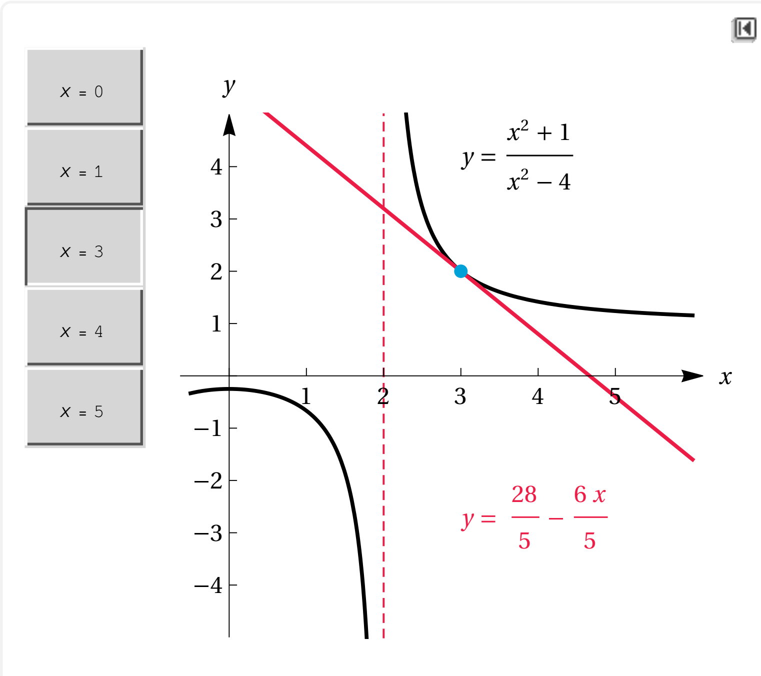


Figure 3.31

Related Exercises 57–58 ◆

Extending the Power Rule to Negative Integers »

The Power Rule in Section 3.3 says that $\frac{d}{dx}(x^n) = n\,x^{n-1}$, for nonnegative integers n. Using the Quotient Rule, we show that the Power Rule also holds if n is a negative integer. Assume n is a negative integer and let $m = -n$, so that $m > 0$. Then

$$\frac{d}{dx}(x^n) = \frac{d}{dx}\left(\frac{1}{x^m}\right) \qquad x^n = \frac{1}{x^{-n}} = \frac{1}{x^m}$$

$$= \frac{x^m \overbrace{\left(\frac{d}{dx}(1)\right)}^{\text{derivative of } a \text{ constant is } 0} - 1\overbrace{\left(\frac{d}{dx}x^m\right)}^{\text{equals } m\,x^{m-1}}}{(x^m)^2} \qquad \text{Quotient Rule}$$

$$= -\frac{m\,x^{m-1}}{x^{2m}} \qquad \text{Simplify.}$$

$$= -m\,x^{-m-1} \qquad \frac{x^{m-1}}{x^{2m}} = x^{m-1-2m}$$

$$= n\,x^{n-1}. \qquad \text{Replace } -m \text{ with } n.$$

This calculation leads to the first extension of the Power Rule; the rule now applies to all integers. In Theorem 3.8, we assert that, in fact, the Power Rule is valid for all real powers. A proof of this theorem appears in Chapter 7.

THEOREM 3.8 Power Rule (general form)

If n is any real number, then

$$\frac{d}{dx}(x^n) = n\,x^{n-1}.$$

Note »

Quick Check 3 Find the derivative of $f(x) = 1/x^5$ in two different ways: using the Power Rule and using the Quotient Rule. ◆

Answer »

EXAMPLE 4 Using the Power Rule

Find the following derivatives.

a. $\frac{d}{dx}\left(\frac{9}{x^5}\right)$

b. $\frac{d}{dt}\left(\frac{3\,t^{16} - 4}{t^6}\right)$

c. $\frac{d}{dz}\left(6\,\sqrt[3]{z}\right)$

d. $\frac{d}{dx}\left(\frac{3\,x^{5/2}}{2\,x^2 + 4}\right)$

SOLUTION »

Rates of Change »

Remember that the derivative has multiple uses and interpretations. The following example illustrates the derivative as a rate of change of a population. Specifically, the derivative tells us when the population is growing most rapidly and how the population behaves in the long run.

EXAMPLE 5 Population growth rates

The population of a culture of cells increases and approaches a constant level (often called a *steady state* or *carrying capacity*) and is modeled by the function $p(t) = 400\left(\frac{t^2 + 1}{t^2 + 4}\right)$, where $t \geq 0$ is measured in hours (**Figure 3.32**).

a. Compute and graph the instantaneous growth rate of the population for $t \geq 0$.

b. At approximately what time is the instantaneous growth rate the greatest?

c. What is the steady-state population?

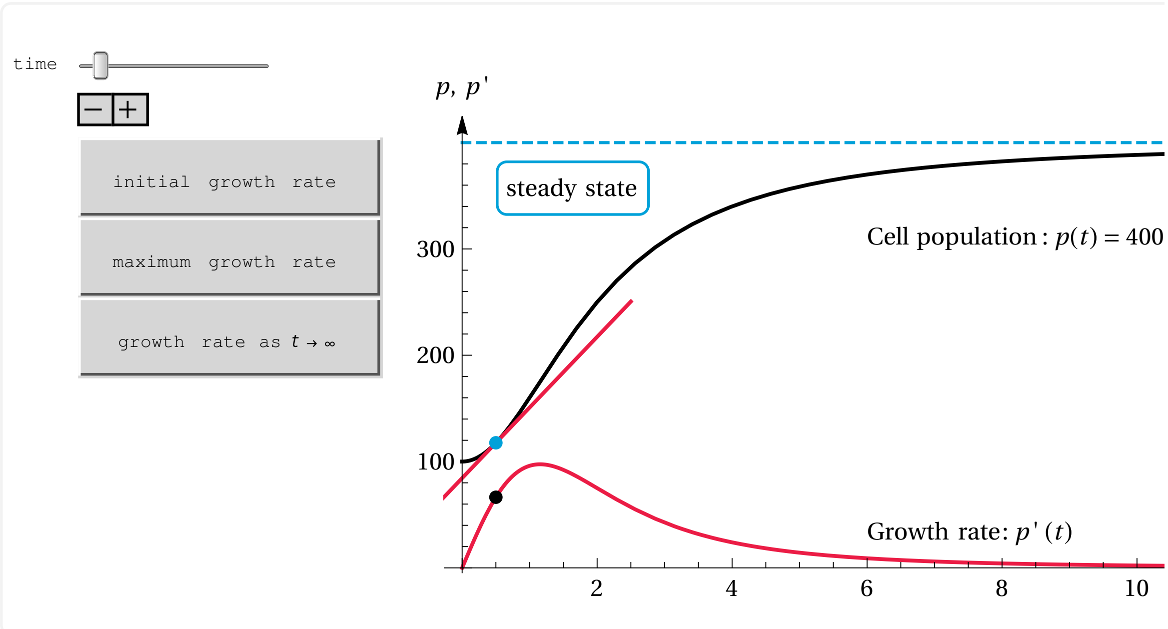


Figure 3.32

SOLUTION

a. The instantaneous growth rate is given by the derivative of the population function:

$$\begin{aligned} p'(t) &= \frac{d}{dt}\left(400\left(\frac{t^2+1}{t^2+4}\right)\right) \\ &= 400\,\frac{\left(t^2+4\right)(2\,t)-\left(t^2+1\right)(2\,t)}{\left(t^2+4\right)^2} && \text{Quotient Rule} \\ &= \frac{2400\,t}{\left(t^2+4\right)^2}. && \text{Simplify.} \end{aligned}$$

The growth rate has units of cells per hour; its graph is shown in Figure 3.32.

b. The growth rate $p'(t)$ has a maximum value at the point at which the population curve is steepest. Using a graphing utility, this point corresponds to $t \approx 1.15$ hr and the growth rate has a value of $p'(1.15) \approx 97$ cells/hr.

Note »

Methods for determining exactly when the growth rate is a maximum are discussed in Chapter 4.

c. To determine whether the population approaches a fixed value after a long period of time (the steady-state population), we investigate the limit of the population function as $t \to \infty$. In this case, the steady-state population exists and is

$$\lim_{t\to\infty} p(t) = \lim_{t\to\infty} 400 \underbrace{\left(\frac{t^2+1}{t^2+4}\right)}_{\text{approaches } 1} = 400,$$

which is confirmed by the population curve in Figure 3.32. Notice that as the population approaches its steady state, the growth rate p' approaches zero.

Related Exercises 61–62 ◆

Combining Derivative Rules

Some situations call for the use of multiple differentiation rules. This section concludes with one such example.

EXAMPLE 6 Combining derivative rules

Find the derivative of

$$y = \frac{4x\left(2x^3 - 3x^{-1}\right)}{x^2 + 1}.$$

SOLUTION

In this case, we have the quotient of two functions, with a product in the numerator:

$$\frac{dy}{dx} = \frac{\left(x^2+1\right)\cdot \frac{d}{dx}\left(4x\left(2x^3-3x^{-1}\right)\right) - \left(4x\left(2x^3-3x^{-1}\right)\right)\cdot \frac{d}{dx}\left(x^2+1\right)}{\left(x^2+1\right)^2} \quad \text{Quotient Rule}$$

$$= \frac{\left(x^2+1\right)\left(4\left(2x^3-3x^{-1}\right)+4x\left(6x^2+3x^{-2}\right)\right) - \left(4x\left(2x^3-3x^{-1}\right)\right)(2x)}{\left(x^2+1\right)^2} \quad \text{Product Rule in the numerator}$$

$$= \frac{8x\left(2x^4+4x^2+3\right)}{\left(x^2+1\right)^2}. \quad \text{Simplify.}$$

Related Exercises 43–43 ◆

Exercises »

Getting Started »

Practice Exercises »

19–56. Derivatives *Find and simplify the derivative of the following functions.*

19. $f(x) = 3x^4\left(2x^2 - 1\right)$

20. $g(x) = 6x - 2x\left(x^{10} - 3x^3\right)$

21. $f(x) = \dfrac{x}{x+1}$

22. $f(x) = \dfrac{x^3 - 4x^2 + x}{x - 2}$

23. $f(t) = t^5\left(\sqrt{t} + 1\right)$

24. $g(w) = \left(w^8 + 3\right)\left(5w^2 + 3w + 1\right)$

25. $y = (3t - 1)(2t - 2)^{-1}$

26. $h(w) = \dfrac{w^2 - 1}{w^2 + 1}$

27. $g(w) = \left(w^3 + 4\right)\left(w^3 - 1\right)$

28. $s(t) = 4\left(3\,t^2 + 2\,t - 1\right)\sqrt{t}$

29. $h(x) = (x - 1)\left(x^3 + x^2 + x + 1\right)$

30. $f(x) = \left(1 + \dfrac{1}{x^2}\right)\left(x^2 + 1\right)$

31. $g(x) = \dfrac{w - 1}{\sqrt{w} + 1}$

32. $f(x) = \dfrac{2\,x + 1}{x - 1}$

33. $h(t) = \dfrac{t}{t^2 + 1}$

34. $f(t) = \dfrac{t + \sqrt{t} + 1}{t - \sqrt{t} - 1}$

35. $g(x) = \dfrac{x^4 + 1}{x^2 - 1}$

36. $y = \left(2\sqrt{x} - 1\right)(4\,x + 1)^{-1}$

37. $f(x) = 3\,x^{-9}$

38. $y = \dfrac{4}{p^3}$

39. $g(t) = 3\,t^2 + \dfrac{6}{t^7}$

40. $y = \dfrac{w^4 + 5\,w^2 + w}{w^2}$

41. $g(t) = \dfrac{t^3 + 3\,t^2 + t}{t^3}$

42. $p(x) = \dfrac{4\,x^3 + 3\,x + 1}{2\,x^5}$

43. $g(x) = \dfrac{x\,(3 - x)}{2\,x^2}$

44. $h(x) = \dfrac{(x-1)\left(2x^2-1\right)}{\left(x^3-1\right)}$

45. $g(x) = \dfrac{4x}{\left(x^2+x\right)(1-x)}$

46. $h(x) = \dfrac{x+1}{x^2\left(2x^3+1\right)}$

47. $g(w) = \dfrac{\sqrt{w}+w}{\sqrt{w}-w}$

48. $f(x) = \dfrac{4-x^2}{x-2}$

49. $h(w) = \dfrac{w^{5/3}}{w^{5/3}+1}$

50. $g(x) = \dfrac{x^{4/3}-1}{x^{4/3}+1}$

51. $f(x) = 4x^2 - \dfrac{2x}{5x+1}$

52. $f(w) = \dfrac{2+2\sqrt{w}-3}{\sqrt{w}+3}$

53. $h(r) = \dfrac{2-r-\sqrt{r}}{r+1}$

54. $y = \dfrac{x-a}{\sqrt{x}-\sqrt{a}}$, where a is a positive constant

55. $h(x) = \left(5x^7+5x\right)\left(6x^3+3x^2+3\right)$

56. $s(t) = (t+1)(t+2)(t+3)$

T 57–60. Equations of tangent lines

a. *Find an equation of the line tangent to the given curve at a.*

b. *Use a graphing utility to graph the curve and the tangent line on the same set of axes.*

57. $y = \dfrac{x+5}{x-1}$; $a = 3$

58. $y = \dfrac{2x^2}{3x-1}$; $a = 1$

59. $y = x\left(2x^{-2} + 1\right);\ a = -1$

60. $y = \dfrac{x-2}{x+1};\ a = 1$

T **61–62. Population growth** *Consider the following population functions.*

a. *Find the instantaneous growth rate of the population, for* $t \geq 0$.

b. *What is the instantaneous growth rate at* $t = 5$*?*

c. *Estimate the time when the instantaneous growth rate is greatest.*

d. *Evaluate and interpret* $\lim\limits_{t\to\infty} p(t)$.

e. *Use a graphing utility to graph the population and its growth rate.*

61. $p(t) = \dfrac{200t}{t+2}$

62. $p(t) = 600\left(\dfrac{t^2+3}{t^2+9}\right)$

63. Electrostatic force The magnitude of the electrostatic force between two point charges Q and q of the same sign is given by $F(x) = \dfrac{kQq}{x^2}$, where x is the distance (measured in meters) between the charges and $k = 9\times 10^9$ N-m$^2/C^2$ is a physical constant (C stands for coulomb, the unit of charge; N stands for newton, the unit of force).

a. Find the instantaneous rate of change of the force with respect to the distance between the charges.

b. For two identical charges with $Q = q = 1$ C, what is the instantaneous rate of change of the force at a separation of $x = 0.001$ m?

c. Does the magnitude of the instantaneous rate of change of the force increase or decrease with the separation? Explain.

64. Gravitational force The magnitude of the gravitational force between two objects of mass M and m is given by $F(x) = -\dfrac{GMm}{x^2}$, where x is the distance between the centers of mass of the objects and $G = 6.7\times 10^{-11}$ N-m^2/kg^2 is the gravitational constant (N stands for newton, the unit of force; the negative sign indicates an attractive force).

a. Find the instantaneous rate of change of the force with respect to the distance between the objects.

b. For two identical objects of mass $M = m = 0.1$ kg, what is the instantaneous rate of change of the force at a separation of $x = 0.01$ m?

c. Does the magnitude of the instantaneous rate of change of the force increase or decrease with the separation? Explain.

65. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. $\dfrac{d}{dx}\left(\pi^5\right) = 5\pi^4$.

b. The Quotient Rule must be used to evaluate $\dfrac{d}{dx}\left(\dfrac{x^2+3x+2}{x}\right)$.

c. $\dfrac{d}{dx}\left(\dfrac{1}{x^5}\right)=\dfrac{1}{5x^4}$.

66–67. Higher-order derivatives *Find $f'(x)$, $f''(x)$, and $f'''(x)$.*

66. $f(x)=\dfrac{1}{x}$

67. $f(x)=x^2\left(2+x^{-3}\right)$

68–69. First and second derivatives *Find $f'(x)$ and $f''(x)$.*

68. $f(x)=\dfrac{x}{x+2}$

69. $f(x)=\dfrac{x^2-7x}{x+1}$

70. Tangent lines Suppose $f(2)=2$ and $f'(2)=3$. Let $g(x)=x^2 f(x)$ and $h(x)=\dfrac{f(x)}{x-3}$.

a. Find an equation of the line tangent to $y=g(x)$ at $x=2$.

b. Find an equation of the line tangent to $y=h(x)$ at $x=2$.

T **71. The Witch of Agnesi** The graph of $y=\dfrac{a^3}{x^2+a^2}$, where a is a constant, is called the *witch of Agnesi* (named after the 18th-century Italian mathematician Maria Agnesi).

a. Let $a=3$ and find an equation of the line tangent to $y=\dfrac{27}{x^2+9}$ at $x=2$.

b. Plot the function and the tangent line found in part (a).

72–77. Derivatives from a table *Use the following table to find the given derivatives.*

x	1	2	3	4
$f(x)$	5	4	3	2
$f'(x)$	3	5	2	1
$g(x)$	4	2	5	3
$g'(x)$	2	4	3	1

72. $\dfrac{d}{dx}(f(x)\,g(x))\Big|_{x=1}$

73. $\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right)\Big|_{x=2}$

74. $\dfrac{d}{dx}(x\,f(x))\Big|_{x=3}$

75. $\frac{d}{dx}\left(\frac{f(x)}{x+2}\right)\bigg|_{x=4}$

76. $\frac{d}{dx}\left(\frac{x\,f(x)}{g(x)}\right)\bigg|_{x=4}$

77. $\frac{d}{dx}\left(\frac{f(x)\,g(x)}{x}\right)\bigg|_{x=4}$

78–79. Flight formula for Indian spotted owlets *The following table shows the average body mass $m(t)$ (in grams) and average wing chord length $w(t)$ (in millimeters), along with the derivatives $m'(t)$ and $w'(t)$, of t-week-old Indian spotted owlets. The* ***flight formula*** *function $f(t) = w(t)/m(t)$, which is the ratio of wing chord length to mass, is used to predict when these fledglings are old enough to fly. The values of f are less than 1, but approach 1 as t increases. When f is close to 1, the fledglings are capable of flying, which is important for determining when rescued fledglings can be released back into the wild. (Source: ZooKeys,* 132, 2011*)*

t	$m(t)$	$m'(t)$	$w(t)$	$w'(t)$
1	23.32	41.45	10.14	14.5
1.5	50.59	64.94	20.13	26.17
2	82.83	57.95	36.7	39.86
2.5	105.13	31.08	58.92	47.11
3	115.48	12.48	81.55	41.38
3.5	119.4	4.44	98.99	27.94
4	120.76	1.51	109.75	15.74
4.5	121.22	0.51	115.5	7.99
5	121.37	0.17	118.34	3.85
5.5	121.42	0.06	119.69	1.8
6	121.44	0.02	120.32	0.84
6.5	121.45	0.01	120.61	0.38

78. State the units associated with the following derivatives and state the physical meaning of each derivative.

a. $m'(t)$

b. $w'(t)$

c. $f'(t)$

T 79. Complete the following steps to examine the behavior of the flight formula.

a. Sketch an approximate graph of $y = f(t)$ by plotting and connecting the points $(1, f(1))$, $(1.5, f(1.5))$, ..., $(6.5, f(6.5))$ with a smooth curve.

b. For what value of t does f appear to be changing most rapidly? Round t to the nearest whole number.

c. For the value of t found in part (b), use the table and the Quotient Rule to find $f'(t)$. Describe what is happening to the bird at this stage in its life.

d. Use your graph of f to predict what happens to $f'(t)$ as t grows larger and confirm your prediction by evaluating $f'(6.5)$ using the Quotient Rule. Describe what is happening in the physical development of the fledglings as t grows larger.

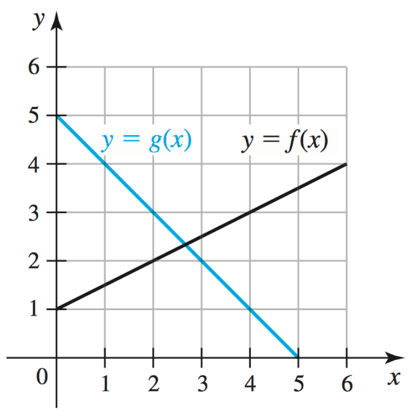
80–81. *Assume both the graphs of f and g pass through the point* $(3, 2)$, $f'(3) = 5$, *and* $g'(3) = -10$. *If* $p(x) = f(x)\,g(x)$ *and* $q(x) = f(x)/g(x)$, *find the following derivatives.*

80. $p'(3)$

81. $q'(3)$

82. Given that $f(1) = 2$ and $f'(1) = 2$, find the slope of the curve $y = x f(x)$ at the point $(1, 2)$.

83–86. Derivatives from graphs *Use the figure to find the following derivatives.*



83. $\left.\dfrac{d}{dx}(f(x)\,g(x))\right|_{x=4}$

84. $\left.\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right)\right|_{x=4}$

85. $\left.\dfrac{d}{dx}(x\,g(x))\right|_{x=2}$

86. $\left.\dfrac{d}{dx}\left(\dfrac{x^2}{f(x)}\right)\right|_{x=2}$

87. **Tangent Lines** The line tangent to the curve $y = h(x)$ at $x = 4$ is $y = -3x + 14$. Find an equation of the line tangent to the following curves at $x = 4$.

a. $y = \left(x^2 - 3x\right) h(x)$

b. $y = h(x)/(x + 2)$

88. **Derivatives from tangent lines** Suppose the line tangent to the graph of f at $x = 2$ is $y = 4x + 1$ and suppose $y = 3x - 2$ is the line tangent to the graph of g at $x = 2$. Find an equation of the line tangent to the following curves at $x = 2$.

a. $y = f(x)\,g(x)$

b. $y = f(x)/g(x)$

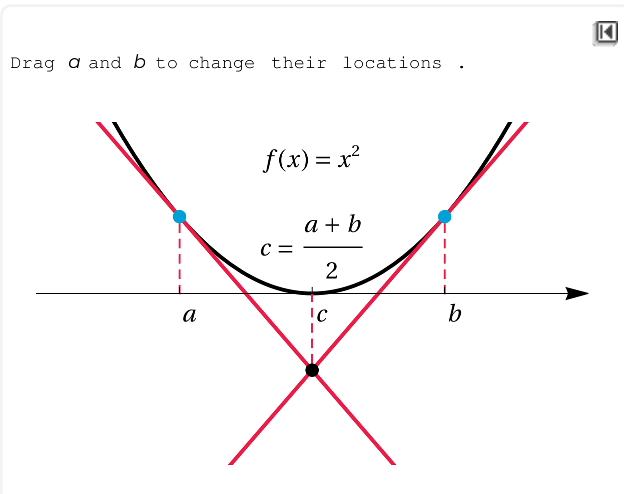
Explorations and Challenges »

89. **Avoiding tedious work** Given that $q(x) = \dfrac{5x^8 + 6x^5 + 5x^4 + 3x^2 + 20x + 100}{10x^{10} + 8x^9 + 6x^5 + 6x^2 + 4x + 2}$, find $q'(0)$ without computing $q'(x)$. (*Hint:* Evaluate $f(0)$, $f'(0)$, $g(0)$, and $g'(0)$ where f equals the numerator of q and g equals the denominator of q.)

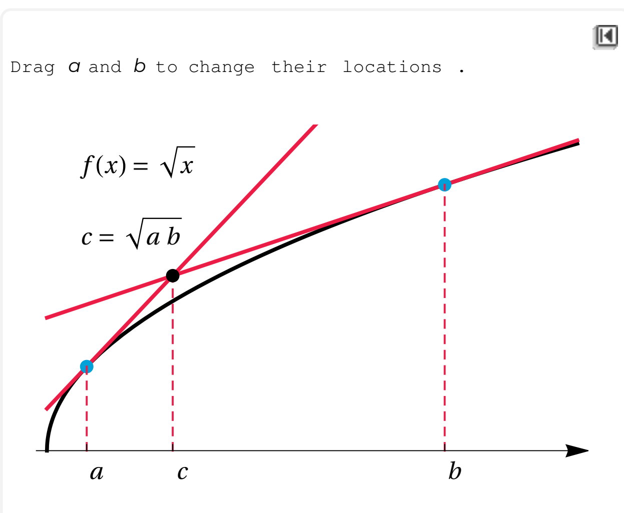
90. Given that $p(x) = (10x^5 + 20x^3 + 100x^2 + 5x + 20)\cdot(10x^5 + 40x^3 + 20x^2 + 4x + 10)$, find $p'(0)$ without computing $p'(x)$.

91. **Means and tangents** Suppose f is differentiable on an interval containing a and b, and let $P(a, f(a))$ and $Q(b, f(b))$ be distinct points on the graph of f. Let c be the x-coordinate of the point at which the lines tangent to the curve at P and Q intersect, assuming the tangent lines are not parallel (see figure).

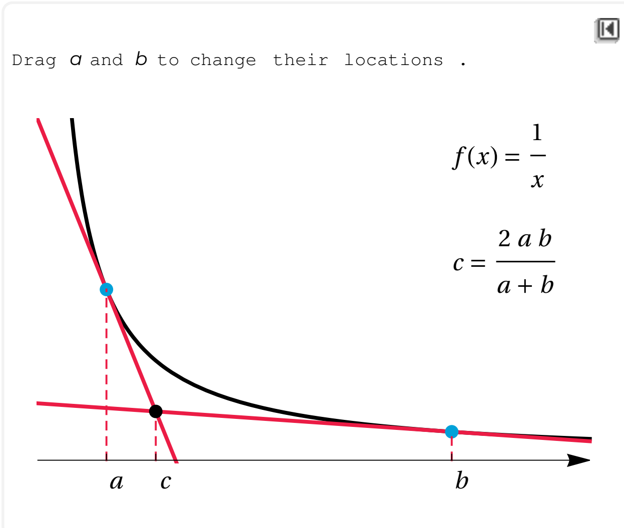
a. If $f(x) = x^2$, show that $c = (a + b)/2$, the arithmetic mean of a and b, for real numbers a and b.



b. If $f(x) = \sqrt{x}$, show that $c = \sqrt{ab}$, the geometric mean of a and b, for $a > 0$ and $b > 0$.



c. If $f(x) = \frac{1}{x}$, show that $c = 2\,a\,b/(a+b)$, the harmonic mean of a and b, for $a > 0$ and $b > 0$.



d. Find an expression for c in terms of a and b for any (differentiable) function f whenever c exists.

92. Proof of the Quotient Rule Let $F = f/g$ be the quotient of two functions that are differentiable at x.

a. Use the definition of F' to show that $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \lim_{h\to 0} \frac{f(x+h)\,g(x) - f(x)\,g(x+h)}{h\,g(x+h)\,g(x)}$.

b. Now add $-f(x)\,g(x) + f(x)\,g(x)$ (which equals 0) to the numerator in the preceding limit to obtain $\lim_{h\to 0} \frac{f(x+h)\,g(x) - f(x)\,g(x) + f(x)\,g(x) - f(x)\,g(x+h)}{h\,g(x+h)\,g(x)}$. Use this limit to obtain the Quotient Rule.

c. Explain why $F' = (f/g)'$ exists, whenever $g(x) \neq 0$.

93. Product Rule for the second derivative Assuming the first and second derivatives of f and g exist at x, find a formula for $\frac{d^2}{dx^2}(f(x)\,g(x))$.

94. One of the Leibniz Rules One of several Leibniz Rules in calculus deals with higher-order derivatives of products. Let $(f\,g)^{(n)}$ denote the nth derivative of the product $f\,g$, for $n \geq 1$.

a. Prove that $(f\,g)^{(2)} = g\,f'' + 2\,f'\,g' + f\,g''$.

b. Prove that, in general,

$$(f\,g)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}\,g^{(n-k)},$$

where $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$ are the binomial coefficients.

c. Compare the result of (b) to the expansion of $(a+b)^n$.

95. Product Rule for three functions Assume that f, g, and h are differentiable at x.

a. Use the Product Rule (twice) to find a formula for $\frac{d}{dx}(f(x)\,g(x)\,h(x))$.

b. Use the formula in (a) to find $\frac{d}{dx}\left(\sqrt{x}\,(x-1)(x+3)\right)$.