3.3 Rules of Differentiation

If you always had to use limits to evaluate derivatives, as we did in Section 3.2, calculus would be a tedious affair. The goal of this section is to establish rules and formulas for quickly evaluating derivatives—not just for individual functions but for entire families of functions. By the end of the chapter, you will have learned many derivative rules and formulas, all of which are listed in the endpapers of the book.

The Constant and Power Rules for Derivatives »

The graph of the **constant function** $f(x) = c$ is a horizontal line with a slope of 0 at every point (**Figure 3.29**). It follows that $f'(x) = 0$ or, equivalently, $\frac{d}{dx}(c) = 0$ (Exercise 82). This observation leads to the *Constant Rule* for derivatives.

Note »

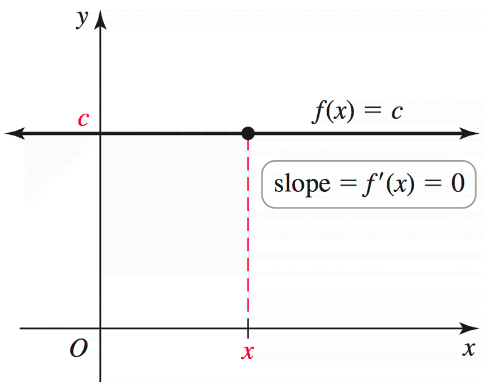


Figure 3.29

> **THEOREM 3.2 Constant Rule**
>
> If c is a real number, then $\frac{d}{dx}(c) = 0$.

Quick Check 1 Find the values of $\frac{d}{dx}(11)$ and $\frac{d}{dx}(\pi)$. ◆

Answer »

$\frac{d}{dx}(11) = 0$ and $\frac{d}{dx}(\pi) = 0$ because 11 and π are constants.

Next, consider power functions of the form $f(x) = x^n$, where n is a nonnegative integer. If you completed Exercise 58 in Section 3.2, you used the limit definition of the derivative to discover that

$$\frac{d}{dx}\left(x^2\right) = 2x, \quad \frac{d}{dx}\left(x^3\right) = 3x^2, \quad \text{and} \quad \frac{d}{dx}\left(x^4\right) = 4x^3.$$

In each case, the derivative of x^n appears to be evaluated by placing the exponent n in front of x as a coefficient and decreasing the exponent by 1. Based on these observations, we state and prove the following theorem.

THEOREM 3.3 Power Rule

If n is a positive integer, then $\frac{d}{dx}(x^n) = n\,x^{n-1}$.

Proof: We let $f(x) = x^n$ and use the definition of the derivative in the form

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}.$$

With $n = 1$ and $f(x) = x$, we have

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{x-a}{x-a} = 1,$$

as given by the Power Rule.

With $n \ge 2$ and $f(x) = x^n$, note that $f(x) - f(a) = x^n - a^n$. A factoring formula gives

$$x^n - a^n = (x-a)\left(x^{n-1} + x^{n-2}\,a + \cdots + x\,a^{n-2} + a^{n-1}\right).$$

Note »

Note that this formula agrees with familiar factoring formulas for differences of perfect squares and cubes:

$$x^2 - a^2 = (x-a)(x+a)$$
$$x^3 - a^3 = (x-a)\left(x^2 + x\,a + a^2\right)$$

Therefore,

$$\begin{aligned} f'(a) &= \lim_{x\to a} \frac{x^n - a^n}{x-a} && \text{Definition of } f'(a) \\ &= \lim_{x\to a} \frac{(x-a)\left(x^{n-1} + x^{n-2}\,a + \cdots + x\,a^{n-2} + a^{n-1}\right)}{x-a} && \text{Factor } x^n - a^n. \\ &= \lim_{x\to a} \left(x^{n-1} + x^{n-2}\,a + \cdots + x\,a^{n-2} + a^{n-1}\right) && \text{Cancel common factors.} \\ &= \underbrace{a^{n-1} + a^{n-2}\cdot a + \cdots + a\cdot a^{n-2} + a^{n-1}}_{n \text{ terms}} = n\,a^{n-1}. && \text{Evaluate the limit.} \end{aligned}$$

Replacing a by the variable x in $f'(a) = n\,a^{n-1}$, we obtain the result given in the Power Rule for $n \ge 2$. Finally, note that the Constant Rule is consistent with the Power Rule with $n = 0$. ◆

EXAMPLE 1 Derivatives of power and constant functions

Evaluate the following derivatives.

a. $\frac{d}{dx}\left(x^9\right)$

b. $\frac{d}{dx}(x)$

c. $\frac{d}{dx}\left(2^8\right)$

SOLUTION »

a. $\dfrac{d}{dx}\left(x^9\right) = 9\,x^{9-1} = 9\,x^8$ Power Rule

b. $\dfrac{d}{dx}(x) = \dfrac{d}{dx}\left(x^1\right) = 1\,x^0 = 1$ Power Rule

c. You might be tempted to use the Power Rule here, but $2^8 = 256$ is a constant. So, by the Constant Rule, $\dfrac{d}{dx}\left(2^8\right) = 0$.

Related Exercises 19–22 ◆

Quick Check 2 Use the graph of $y = x$ to give a geometric explanation of why $\dfrac{d}{dx}(x) = 1$. ◆

Answer »

The slope of the curve $y = x$ is 1 at any point; therefore, $\dfrac{d}{dx}(x) = 1$.

Constant Multiple Rule »

Consider the problem of finding the derivative of a constant c multiplied by a function f (assuming that f' exists). We apply the definition of the derivative in the form

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

to the function $c\,f$:

$$\begin{aligned}
\frac{d}{dx}(c\,f(x)) &= \lim_{h\to 0} \frac{c f(x+h) - c f(x)}{h} && \text{Definition of the derivative of } cf \\
&= \lim_{h\to 0} \frac{c\,(f(x+h) - f(x))}{h} && \text{Factor out } c. \\
&= c \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} && \text{Theorem 2.3} \\
&= c f'(x). && \text{Definition of } f'(x)
\end{aligned}$$

This calculation leads to the *Constant Multiple Rule* for derivatives.

THEOREM 3.4 Constant Multiple Rule

If f is differentiable at x and c is a constant, then

$$\frac{d}{dx}(c f(x)) = c f'(x).$$

Note »

Theorem 3.4 says that the derivative of a constant multiplied by a function is the constant multiplied by the derivative of the function.

EXAMPLE 2 Derivatives of constant multiples of functions

Evaluate the following derivatives.

a. $\frac{d}{dx}\left(-\frac{7x^{11}}{8}\right)$

b. $\frac{d}{dt}\left(\frac{3}{8}\sqrt{t}\right)$

SOLUTION »

a.

$$\begin{aligned}\frac{d}{dx}\left(-\frac{7x^{11}}{8}\right) &= -\frac{7}{8}\cdot\frac{d}{dx}\left(x^{11}\right) && \text{Constant Multiple Rule}\\ &= -\frac{7}{8}\cdot 11x^{10} && \text{Power Rule}\\ &= -\frac{77}{8}x^{10} && \text{Simplify.}\end{aligned}$$

b.

$$\begin{aligned}\frac{d}{dt}\left(\frac{3}{8}\sqrt{t}\right) &= \frac{3}{8}\cdot\frac{d}{dt}\left(\sqrt{t}\right) && \text{Constant Multiple Rule}\\ &= \frac{3}{8}\cdot\frac{1}{2\sqrt{t}} && \text{Replace } \frac{d}{dt}\left(\sqrt{t}\right) \text{ by } \frac{1}{2\sqrt{t}}.\\ &= \frac{3}{16\sqrt{t}}\end{aligned}$$

Note »

Related Exercises 23, 24, 28 ◆

Sum Rule »

Many functions are sums of simpler functions. Therefore, it is useful to establish a rule for calculating the derivative of the sum of two or more functions.

THEOREM 3.5 Sum Rule

If f and g are differentiable at x, then

$$\frac{d}{dx}\left(f(x)+g(x)\right) = f'(x) + g'(x).$$

Note »

Proof: Let $F = f + g$, where f and g are differentiable at x, and use the definition of the derivative:

$$\begin{aligned}
\frac{d}{dx}(f(x)+g(x)) &= F'(x) \\
&= \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} && \text{Definition of derivative} \\
&= \lim_{h\to 0}\frac{(f(x+h)+g(x+h))-(f(x)+g(x))}{h} && F=f+g \\
&= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}\right) && \text{Regroup.} \\
&= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} && \text{Theorem 2.3} \\
&= f'(x)+g'(x). && \text{Definition of } f' \text{ and } g' \ \blacklozenge
\end{aligned}$$

Quick Check 3 If $f(x)=x^2$ and $g(x)=2x$, what is the derivative of $f(x)+g(x)$? ◆

Answer »

$2x+2$

The Sum Rule can be extended to three or more differentiable functions, $f_1, f_2, \ldots, f_n$, to obtain the **Generalized Sum Rule**:

$$\frac{d}{dx}(f_1(x)+f_2(x)+\cdots+f_n(x))=f_1'(x)+f_2'(x)+\cdots+f_n'(x).$$

The difference of two functions $f-g$ can be rewritten as the sum $f+(-g)$. By combining the Sum Rule with the Constant Multiple Rule, the **Difference Rule** is established:

$$\frac{d}{dx}(f(x)-g(x))=f'(x)-g'(x).$$

Let's put the Sum and Difference Rules to work on one of the more common problems: differentiating polynomials.

EXAMPLE 3 Derivative of a polynomial

Determine $\frac{d}{dw}\left(2w^3+9w^2-6w+4\right)$.

SOLUTION »

$$\begin{aligned}
\frac{d}{dw}\left(2w^3+9w^2-6w+4\right) &= \frac{d}{dw}\left(2w^3\right)+\frac{d}{dw}\left(9w^2\right)-\frac{d}{dw}(6w)+\frac{d}{dw}(4) && \text{Generalized Sum Rule and Difference Rule} \\
&= 2\frac{d}{dw}\left(w^3\right)+9\frac{d}{dw}\left(w^2\right)-6\frac{d}{dw}(w)+\frac{d}{dw}(4) && \text{Constant Multiple Rule} \\
&= 2\cdot 3w^2+9\cdot 2w-6\cdot 1+0 && \text{Power Rule and Constant Rule} \\
&= 6w^2+18w-6 && \text{Simplify.}
\end{aligned}$$

Related Exercises 31–33 ◆

The technique used to differentiate the polynomial in Example 3 may be used for *any* polynomial. Much of the remainder of this chapter is devoted to discovering differentiation rules for rational, algebraic, and trigonometric functions.

EXAMPLE 4 **Slope of a tangent line**

Let $f(x) = 2x^3 - 15x^2 + 24x$. For what values of x does the line tangent to the graph of f have a slope of 6?

SOLUTION »

The tangent line has a slope of 6 when

$$f'(x) = 6x^2 - 30x + 24 = 6.$$

Subtracting 6 from both sides of the equation and factoring, we have

$$6\left(x^2 - 5x + 3\right) = 0.$$

Using the quadratic formula, the roots are

$$x = \frac{5 - \sqrt{13}}{2} \approx 0.697 \quad \text{and} \quad x = \frac{5 + \sqrt{13}}{2} \approx 4.303.$$

Therefore, the slope of the curve at these points is 6 (**Figure 3.30**).

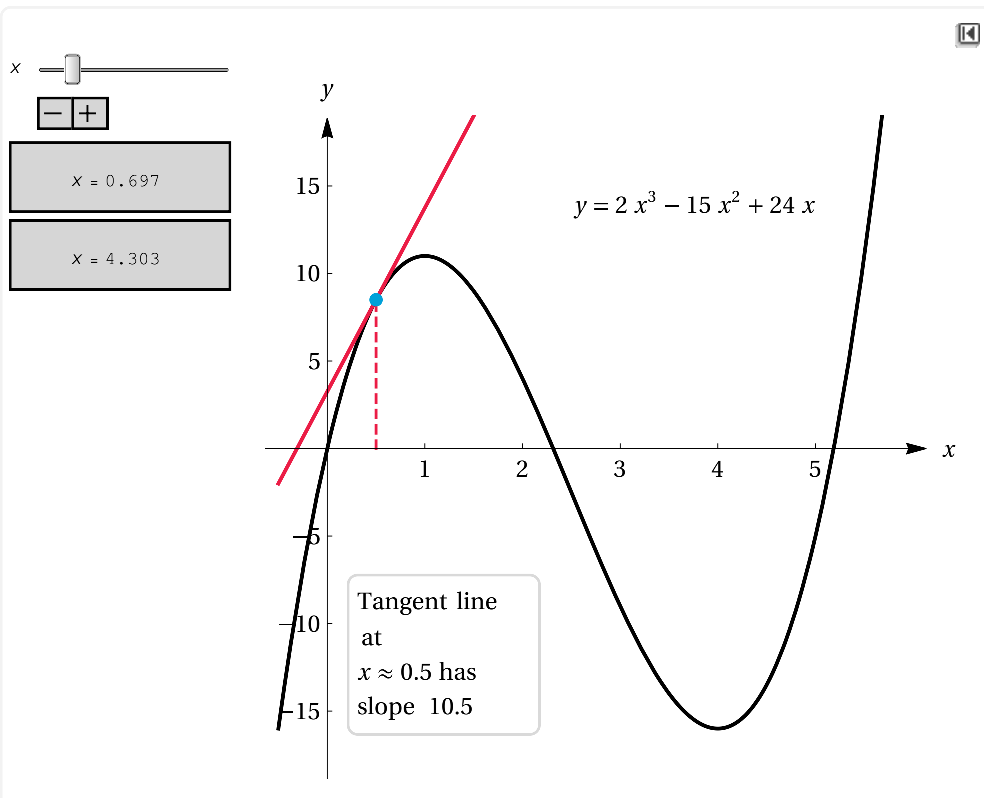


Figure 3.30

Related Exercises 59–60 ◆

Quick Check 4 Determine the point(s) at which $f(x) = x^3 - 12x$ has a horizontal tangent line. ◆

Answer »

$x = 2$ and $x = -2$

Higher-Order Derivatives »

Because the derivative of a function f is a function in its own right, we can take the derivative of f'. The result is the *second derivative of* f, denoted f'' (read f *double prime*). The derivative of the second derivative is the *third derivative of* f, denoted f''' or $f^{(3)}$ (read f *triple prime*). In general, derivatives of order $n \geq 2$ are called *higher-order derivatives*.

Note »

The prime notation, f', f'', and f''', is used only for the first, second, and third derivatives.

DEFINITION **Higher-Order Derivatives**

Assuming f can be differentiated as often as necessary, the **second derivative** of f is

$$f''(x) = \frac{d}{dx}(f'(x)).$$

For integers $n \geq 2$, the ***n*th derivative** is

$$f^{(n)}(x) = \frac{d}{dx}\left(f^{(n-1)}(x)\right).$$

Note »

Parentheses are placed around n to distinguish a derivative from a power. Therefore $f^{(n)}$ is the nth derivative of f and f^n is the function f raised to the nth power.

Other common notations for the second derivative of $y = f(x)$ include $\frac{d^2y}{dx^2}$ and $\frac{d^2f}{dx^2}$; the notations $\frac{d^ny}{dx^n}$, $\frac{d^nf}{dx^n}$, and $y^{(n)}$ are used for the nth derivative of f.

Note »

The notation $\frac{d^2f}{dx^2}$ comes from $\frac{d}{dx}\left(\frac{df}{dx}\right)$ and is read d 2 f d x *squared*.

EXAMPLE 5 **Finding higher-order derivatives**

Find the third derivative of the following functions.

a. $f(x) = 3x^3 - 5x + 12$

b. $y = 3t + 2t^{10}$

SOLUTION »

a.

$$f'(x) = 9x^2 - 5$$
$$f''(x) = \frac{d}{dx}\left(9x^2 - 5\right) = 18x$$
$$f'''(x) = 18$$

Note »

b. Here we use an alternative notation for higher-order derivatives:

$$\frac{dy}{dt} = \frac{d}{dt}\left(3t + 2t^{10}\right) = 3 + 20t^9$$
$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(3 + 20t^9\right) = 180t^8$$
$$\frac{d^3y}{dt^3} = \frac{d}{dt}\left(180t^8\right) = 1440t^7.$$

Related Exercises 65–66 ◆

Quick Check 5 With $f(x) = x^5$, find $f^{(5)}(x)$ and $f^{(6)}(x)$. ◆

Answer »

$f^{(5)}(x) = 120$, $f^{(6)}(x) = 0$

Exercises »

Getting Started »

Practice Exercises »

19–38. Derivatives *Find the derivative of the following functions. See Example 2 of Section 3.2 for the derivative of* $\sqrt{x}$.

19. $y = x^5$

20. $f(t) = t$

21. $f(x) = 5$

22. $g(x) = 2^3$

23. $f(x) = 5x^3$

24. $g(w) = \frac{5}{6}w^{12}$

25. $h(t) = \frac{t^2}{2} + 1$

26. $f(v) = v^{100} + v + 10$

27. $p(x) = 8x$

28. $g(t) = 6\sqrt{t}$

29. $g(t) = 100\,t^2$

30. $f(s) = \dfrac{\sqrt{s}}{4}$

31. $f(x) = 3\,x^4 + 7\,x$

32. $g(x) = 6\,x^5 - \dfrac{5}{2}x^2 + x + 5$

33. $f(x) = 10\,x^4 - 32\,x + 4^3$

34. $f(t) = 6\sqrt{t} - 4\,t^3 + 9$

35. $g(w) = 2\,w^3 + 3\,w^2 + 10\,w$

36. $s(t) = 4\sqrt{t} - \dfrac{1}{4}t^4 + t + 1$

37. $f(x) = \begin{cases} x^2 + 1 & \text{if } x \le 0 \\ 2\,x^2 + x + 1 & \text{if } x > 0 \end{cases}$

38. $g(w) = \begin{cases} w & \text{if } w \le 1 \\ 2\,w^3 + 4\,w + 5 & \text{if } w > 1 \end{cases}$

39. Height estimate The distance an object falls (when released from rest, under the influence of Earth's gravity, and with no air resistance) is given by $d(t) = 16\,t^2$, where d is measured in feet and t is measured in seconds. A rock climber sits on a ledge on a vertical wall and carefully observes the time it takes for a small stone to fall from the ledge to the ground.

a. Compute $d'(t)$. What units are associated with the derivative and what does it measure?

b. If it takes 6 s for a stone to fall to the ground, how high is the ledge? How fast is the stone moving when it strikes the ground (in miles per hour)?

T 40. Projectile trajectory The position of a small rocket that is launched vertically upward is given by $s(t) = -5\,t^2 + 40\,t + 100$, for $0 \le t \le 10$, where t is measured in seconds and s is measured in meters above the ground.

a. Find the rate of change in the position (instantaneous velocity) of the rocket, for $0 \le t \le 10$.

b. At what time is the instantaneous velocity zero?

c. At what time does the instantaneous velocity have the greatest magnitude, for $0 \le t \le 10$?

d. Graph the position and instantaneous velocity, for $0 \le t \le 10$.

41. City urbanization City planners model the size of their city using the function $A(t) = -\dfrac{1}{50}t^2 + 2\,t + 20$, for $0 \le t \le 50$, where A is measured in square miles and t is the number of years after 2010.

a. Compute $A'(t)$. What units are associated with this derivative and what does the derivative measure?

b. How fast will the city be growing when it reaches a size of 38 mi^2?

c. Suppose the population density of the city remains constant from year to year at 1000 people $/\text{mi}^2$. Determine the growth rate of the population in 2030.

42. **Cell growth** When observations begin at $t = 0$, a cell culture has 1200 cells and continues to grow according to the function $p(t) = 1200 + 24\,t^4$, where p is the number of cells and t is measured in days.

a. Compute $p'(t)$. What units are associated with the derivative and what does it measure?

b. On the interval $[0, 4]$, when is the growth rate $p'(t)$ the least? When is it the greatest?

43. **Weight of Atlantic salmon** The weight $w(x)$ (in pounds) of an Atlantic salmon that is x inches long can be estimated by the function

$$w(x) = \begin{cases} 0.4\,x - 5 & \text{if } 19 \le x \le 21 \\ 0.8\,x - 13.4 & \text{if } 21 < x \le 32 \\ 1.5\,x - 35.8 & \text{if } x > 32. \end{cases}$$

Calculate $w'(x)$ and explain the physical meaning of this derivative. (*Source*: www.atlanticsalmonfederation.org)

44–54. Derivatives of products and quotients *Find the derivative of the following functions by first expanding or simplifying the expression. Simplify your answers.*

44. $f(x) = \left(\sqrt{x} + 1\right)\left(\sqrt{x} - 1\right)$

45. $f(x) = (2\,x + 1)\left(3\,x^2 + 2\right)$

46. $g(r) = \left(5\,r^3 + 3\,r + 1\right)\left(r^2 + 3\right)$

47. $f(w) = \dfrac{w^3 - w}{w}$

48. $y = \dfrac{12\,s^3 - 8\,s^2 + 12\,s}{4\,s}$

49. $h(x) = \left(x^2 + 1\right)^2$

50. $h(x) = \sqrt{x}\,\left(\sqrt{x} - x^{3/2}\right)$

51. $g(x) = \dfrac{x^2 - 1}{x - 1}$

52. $h(x) = \dfrac{x^3 - 6\,x^2 + 8\,x}{x^2 - 2\,x}$

53. $y = \dfrac{x - a}{\sqrt{x} - \sqrt{a}}$, where a is a positive constant

54. $y = \dfrac{x^2 - 2\,a\,x + a^2}{x - a}$, where a is a constant

T 55–58. Equations of tangent lines

a. *Find an equation of the line tangent to the given curve at a.*

b. *Use a graphing utility to graph the curve and the tangent line on the same set of axes.*

55. $y = -3x^2 + 2;\ a = 1$

56. $y = x^3 - 4x^2 + 2x - 1;\ a = 2$

57. $y = \sqrt{x};\ a = 4$

58. $y = \frac{1}{2}x^4 + x;\ a = 2$

59. Finding slope locations Let $f(x) = x^2 - 6x + 5$.

a. Find the values of x for which the slope of the curve $y = f(x)$ is 0.

b. Find the values of x for which the slope of the curve $y = f(x)$ is 2.

60. Finding slope locations Let $f(t) = t^3 - 27t + 5$.

a. Find the values of t for which the slope of the curve $y = f(t)$ is 0.

b. Find the values of t for which the slope of the curve $y = f(t)$ is 21.

61. Finding slope locations Let $f(x) = 2x^3 - 3x^2 - 12x + 4$.

a. Find all points on the graph of f at which the tangent line is horizontal.

b. Find all points on the graph of f at which the tangent line has slope 60.

62. Finding slope locations Let $f(x) = \sqrt{x}$.

a. Use $f'(x)$ to explain why the graph of f has no horizontal tangent line.

b. Find all points on the graph of f at which the tangent line has slope $\frac{1}{4}$.

63. Finding slope locations Let $f(x) = 4\sqrt{x} - x$.

a. Find all points on the graph of f at which the tangent line is horizontal.

b. Find all points on the graph of f at which the tangent line has slope $-\frac{1}{2}$.

64–68. Higher-order derivatives *Find $f'(x)$, $f''(x)$, and $f'''(x)$ for the following functions.*

64. $f(x) = 3x^3 + 5x^2 + 6x$

65. $f(x) = 5x^4 + 10x^3 + 3x + 6$

66. $f(x) = 3x^{12} + 4x^3$

67. $f(x) = \dfrac{x^2 - 7x - 8}{x + 1}$

68. $f(x) = \frac{1}{8}x^4 - 3x^2 + 1$

69. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. $\frac{d}{dx}\left(10^5\right) = 5 \cdot 10^4$.

b. The slope of a line tangent to $f(x) = 4x + 1$ is never 0.

c. $\frac{d^n}{dx^n}\left(5x^3 + 2x + 5\right) = 0$, for any integer $n \geq 3$.

70. Tangent lines Suppose $f(3) = 1$ and $f'(3) = 4$. Let $g(x) = x^2 + f(x)$ and $h(x) = 3f(x)$.

a. Find an equation of the line tangent to $y = g(x)$ at $x = 3$.

b. Find an equation of the line tangent to $y = h(x)$ at $x = 3$.

71. Derivatives from tangent lines Suppose the line tangent to the graph of f at $x = 2$ is $y = 4x + 1$ and suppose the line tangent to the graph of g at $x = 2$ has slope 3 and passes through $(0, -2)$. Find an equation of the line tangent to the following curves at $x = 2$.

a. $y = f(x) + g(x)$

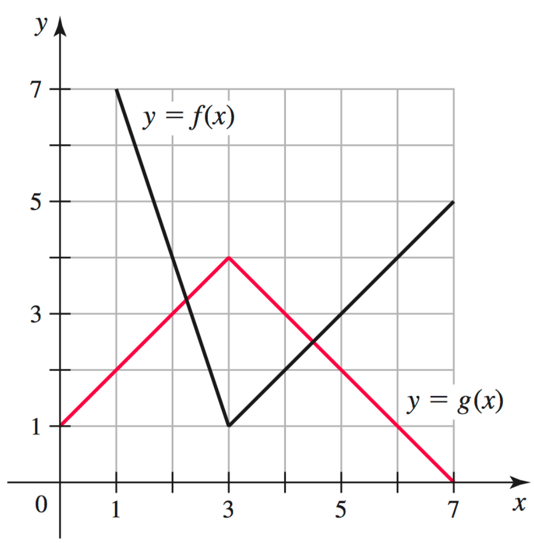
b. $y = f(x) - 2g(x)$

c. $y = 4f(x)$

72. Tangent line Find the equation of the line tangent to the curve $y = x + \sqrt{x}$ that has slope 2.

73. Tangent line given Determine the constants b and c such that the line tangent to $f(x) = x^2 + bx + c$ at $x = 1$ is $y = 4x + 2$.

74–77. Derivatives from a graph *Let $F = f + g$ and $G = 3f - g$, where the graphs of f and g are shown in the figure. Find the following derivatives.*



74. $F'(2)$

75. $G'(2)$

76. $F'(5)$

77. $G'(5)$

78–81. Derivatives from limits *The following limits represent $f'(a)$ for some function f and some real number a.*

- **a.** *Find a possible function f and number a.*
- **b.** *Evaluate the limit by computing $f'(a)$.*

78. $\lim_{x \to 0} \frac{x^{100} - 1}{x - 1}$

79. $\lim_{x \to 0} \frac{x^7 + x^6 - 2}{x - 1}$

80. $\lim_{h \to 0} \frac{(1 + h)^8 + (1 + h)^3 - 2}{h}$

81. $\lim_{h \to 0} \frac{\sqrt{9 + h} - \sqrt{9}}{h}$

Explorations and Challenges »

82. Constant Rule proof For the constant function $f(x) = c$, use the definition of the derivative to show that $f'(x) = 0$.

83. Alternative proof of the Power Rule The Binomial Theorem states that for any positive integer n,

$$(a + b)^n = a^n + n\,a^{n-1}\,b + \frac{n(n-1)}{2 \cdot 1}\,a^{n-2}\,b^2 + \frac{n(n-1)(n-2)}{3 \cdot 2 \cdot 1}\,a^{n-3}\,b^3 + \cdots + n\,a\,b^{n-1} + b^n.$$

Use this formula and the definition $f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$ to show that $\frac{d}{dx}(x^n) = n\,x^{n-1}$, for any positive integer n.

84. Power Rule for negative integers Suppose n is a negative integer and $f(x) = x^n$. Use the following steps to prove that $f'(a) = n\,a^{n-1}$, which means the Power Rule for positive integers extends to all integers. This result is proved in Section 3.4 by a different method.

a. Assume $m = -n$, so that $m > 0$. Use the definition

$$f'(a) = \lim_{x \to a} \frac{x^n - a^n}{x - a} = \lim_{x \to a} \frac{x^{-m} - a^{-m}}{x - a}.$$

Simplify using the factoring rule (which is valid for $n > 0$)

$$x^n - a^n = (x - a)\left(x^{n-1} + x^{n-2}\,a + \cdots + x\,a^{n-2} + a^{n-1}\right)$$

until it is possible to take the limit.

b. Use this result to find $\frac{d}{dx}\left(x^{-7}\right)$ and $\frac{d}{dx}\left(\frac{1}{x^{10}}\right)$.

85. Extending the Power Rule to $n = \frac{1}{2}, \frac{3}{2}$, and $\frac{5}{2}$ With Theorem 3.3 and Exercise 84, we have shown that the Power Rule, $\frac{d}{dx}(x^n) = n\,x^{n-1}$, applies to any integer n. Later in the chapter, we extend this rule so that it applies to any real number n.

a. Explain why the Power Rule is consistent with the formula $\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$.

b. Prove that the Power Rule holds for $n = \frac{3}{2}$. (*Hint:* Use the definition of the derivative: $\frac{d}{dx}\left(x^{3/2}\right) = \lim_{h\to 0} \frac{(x+h)^{3/2} - x^{3/2}}{h}$.)

c. Prove that the Power Rule holds for $n = \frac{5}{2}$.

d. Propose a formula for $\frac{d}{dx}\left(x^{n/2}\right)$ for any positive integer n.