3.2 The Derivative as a Function

In Section 3.1, we learned that the derivative of a function f at a point a is the slope of the line tangent to the graph of f that passes through the point $(a, f(a))$. We now extend the concept of a derivative at a point to all (suitable) points in the domain of f to create a new function called the derivative of f.

The Derivative Function »

So far we have computed the derivative of a function (or, equivalently, slope of the tangent line) at one fixed point on the curve. If this point is moved along the curve, the tangent line also moves, and, in general, its slope changes (**Figure 3.14**). For this reason, the slope of the tangent line for the function f is itself a function of x, called the *derivative* of f.

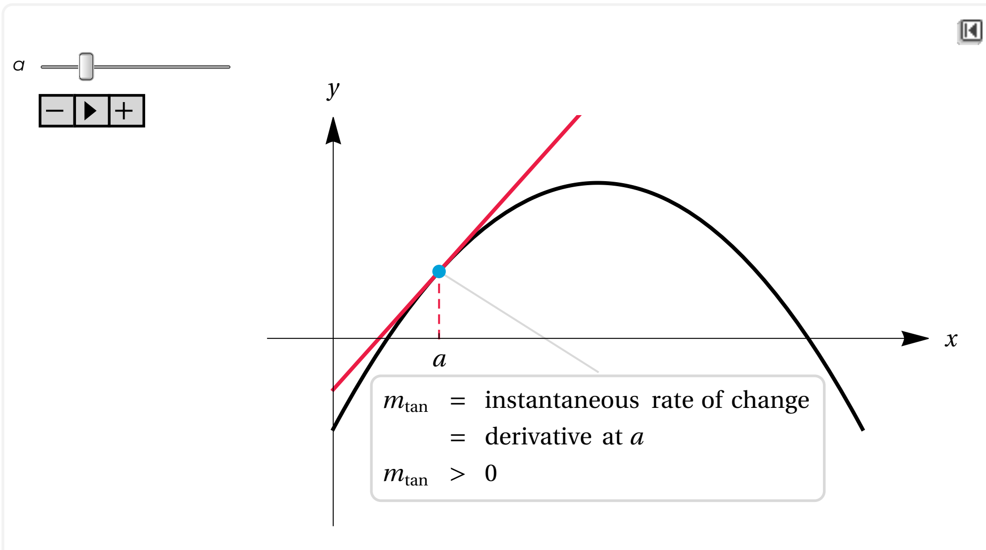


Figure 3.14

We let f' (read f *prime*) denote the derivative function for f, which means that $f'(a)$, when it exists, is the slope of the line tangent to the graph of f at $(a, f(a))$. Using definition (2) for the derivative of f at the point a from Section 3.1, we have

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

We now take an important step. The derivative is a special function, but it works just like any other function. For example, if the graph of f is smooth and 2 is in the domain of f, then $f'(2)$ is the slope of the line tangent to the graph of f at the point $(2, f(2))$. Similarly, if -2 is in the domain of f, then $f'(-2)$ is the slope of the tangent line at the point $(-2, f(-2))$. In fact, if x is *any* point in the domain of f, then $f'(x)$ is the slope of the tangent line at the point $(x, f(x))$. When we introduce a variable point x, the expression $f'(x)$ becomes the *derivative function*.

Note »

> To emphasize an important point, $f'(2)$ or $f'(-2)$ or $f'(a)$, for a real number a, are a real numbers, whereas f' or $f'(x)$ refer to the derivative *function*.

DEFINITION **The Derivative**

The **derivative** of f is the function

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists and x is in the domain of f. If $f'(x)$ exists, we say f is **differentiable** at x. If f is differentiable at every point of an open interval I, we say that f is differentiable on I.

Note »

The process of finding f' is called *differentiation*, and to *differentiate* f means to find f'.

Notice that the definition of f' applies only at points in the domain of f. Therefore, the domain of f' is no larger than the domain of f. If the limit in the definition of f' fails to exist at some points, then the domain of f' is smaller than the domain of f. Let's use this definition to compute a derivative function.

EXAMPLE 1 Computing a derivative

Find the derivative of $f(x) = -x^2 + 6x$.

SOLUTION »

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} && \text{Definition of } f'(x)\\
&= \lim_{h\to 0} \frac{\overbrace{-(x+h)^2 + 6(x+h)}^{f(x+h)} - \overbrace{\left(-x^2 + 6x\right)}^{f(x)}}{h} && \text{Substitute.}\\
&= \lim_{h\to 0} \frac{-\left(x^2 + 2xh + h^2\right) + 6x + 6h + x^2 - 6x}{h} && \text{Expand the numerator.}\\
&= \lim_{h\to 0} \frac{h(-2x - h + 6)}{h} && \text{Simplify and factor out } h.\\
&= \lim_{h\to 0} (-2x - h + 6) = -2x + 6 && \text{Cancel } h \text{ and evaluate the limit.}
\end{aligned}$$

Note »

Notice that this argument applies for $h > 0$ and for $h < 0$; that is, the limit as $h \to 0^+$ and the limit as $h \to 0^-$ are equal.

The derivative is $f'(x) = -2x + 6$, which gives the slope of the tangent line (equivalently, the instantaneous rate of change) at *any* point on the curve. For example, at the point $(1, 5)$, the slope of the tangent line is $f'(1) = -2(1) + 6 = 4$, confirming the calculation in Example 1. The slope of the tangent line at $(3, 9)$ is $f'(3) = -2(3) + 6 = 0$, which means the tangent line is horizontal at that point (**Figure 3.15**).

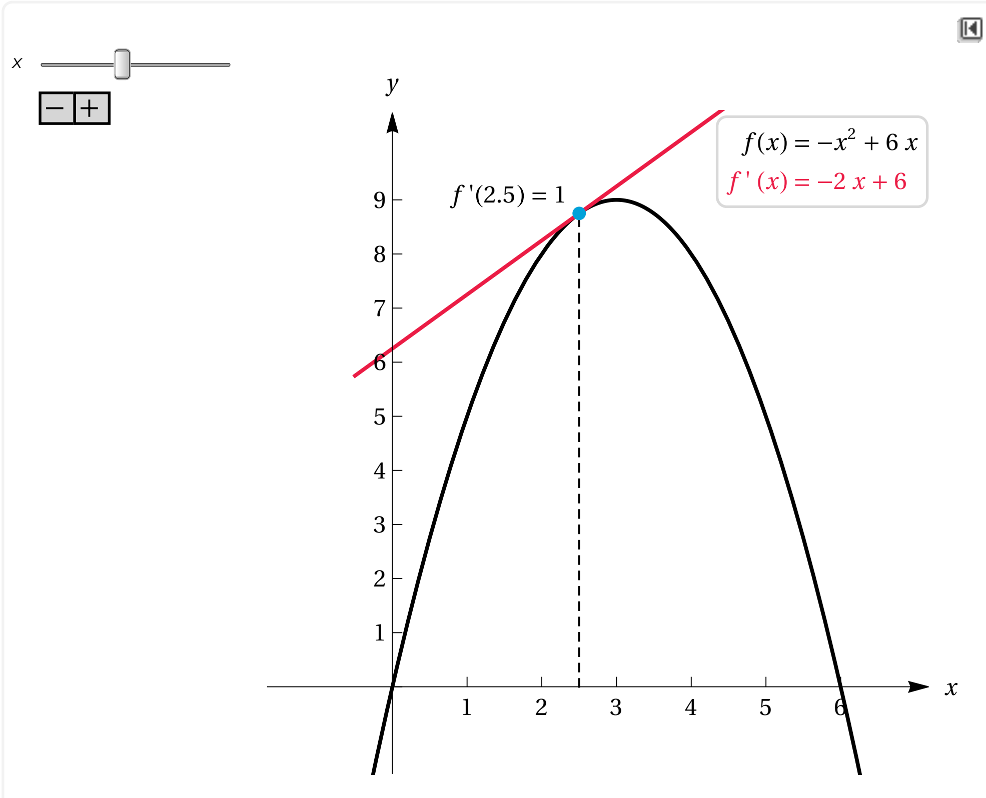


Figure 3.15

Related Exercises 23–24 ◆

Quick Check 1 In Example 1, determine the slope of the tangent line at $x = 2$. ◆

Answer »

2

Derivative Notation »

For historical and practical reasons, several notations for the derivative are used. To see the origin of one notation, recall that the slope of the secant line between two points $P(x, f(x))$ and $Q(x+h, f(x+h))$ on the curve $y = f(x)$ is $\dfrac{f(x+h) - f(x)}{h}$. The quantity h is the change in the x-coordinates in moving from P to Q. A standard notation for change is the symbol Δ (uppercase Greek letter delta). So, we replace h by Δx to represent the change in x. Similarly, $f(x+h) - f(x)$ is the change in y, denoted Δy (**Figure 3.16**). Therefore, the slope of the secant line is

$$m_{\text{sec}} = \frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{\Delta y}{\Delta x}.$$

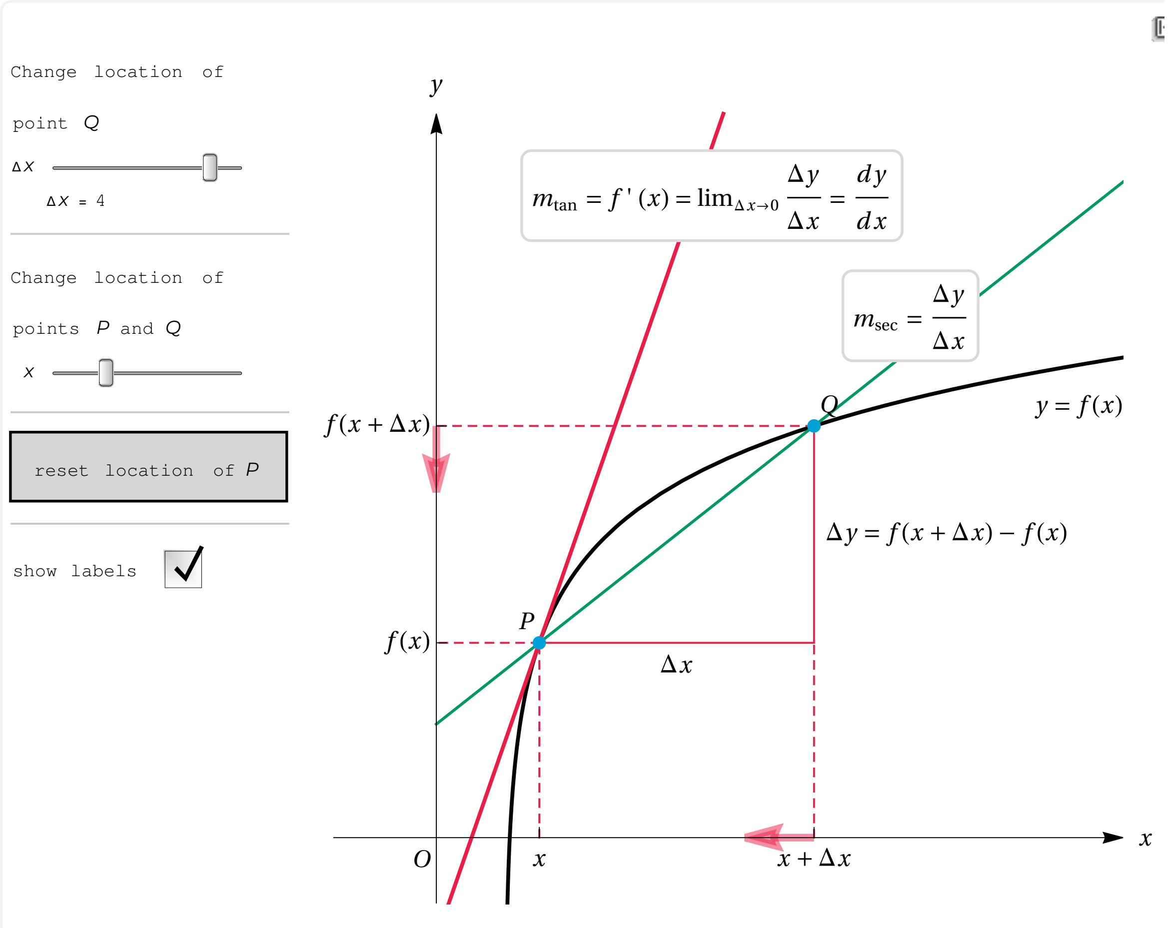


Figure 3.16

By letting $\Delta x \to 0$, the slope of the tangent line at $(x, f(x))$ is

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}.$$

The new notation for the derivative is $\frac{dy}{dx}$; it reminds us that $f'(x)$ is the limit of $\frac{\Delta y}{\Delta x}$ as $\Delta x \to 0$.

Note »

The notation $\frac{dy}{dx}$ is read *the derivative of y with respect to x* or *dy dx*. It does not mean *dy* divided by *dx*, but it is a reminder of the limit of the quotient $\frac{\Delta y}{\Delta x}$.

In addition to the notation $f'(x)$ and $\frac{dy}{dx}$, other common ways of writing the derivative include

$$\frac{df}{dx}, \quad \frac{d}{dx}(f(x)), \quad D_x(f(x)), \quad \text{and} \quad y'(x).$$

Note »

> The derivative notation $\dfrac{dy}{dx}$ was introduced by Gottfried Wilhelm von Leibniz (1646-1716), one of the coinventors of calculus. His notation is used today in its original form. The notation used by Sir Isaac Newton (1642-1727), the other coinventor of calculus, is used less frequently.

The following notations represent the derivative of f evaluated at a.

$$f'(a), \quad y'(a), \quad \left.\frac{df}{dx}\right|_{x=a}, \quad \text{and} \quad \left.\frac{dy}{dx}\right|_{x=a}$$

Quick Check 2 What are some other ways to write $f'(3)$, where $y = f(x)$? ◆

Answer »

> $\left.\dfrac{df}{dx}\right|_{x=3}, \left.\dfrac{dy}{dx}\right|_{x=3}, y'(3)$

EXAMPLE 2 A derivative calculation

Let $y = f(x) = \sqrt{x}$.

a. Compute $\dfrac{dy}{dx}$.

b. Find an equation of the line tangent to the graph of f at $(4, 2)$.

SOLUTION »

a.

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} && \text{Definition of } \frac{dy}{dx} = f'(x) \\
&= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} && \text{Substitute } f(x) = \sqrt{x}. \\
&= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right) && \text{Multiply the numerator and denominator by } \sqrt{x+h} + \sqrt{x}. \\
&= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}} && \text{Simplify and evaluate the limit.}
\end{aligned}$$

Note »

b. The slope of the tangent line at $(4, 2)$ is

$$\left.\frac{dy}{dx}\right|_{x=4} = \frac{1}{2\sqrt{4}} = \frac{1}{4}.$$

Therefore, an equation of the tangent line (**Figure 3.17**) is

$$y - 2 = \frac{1}{4}(x - 4) \text{ or } y = \frac{1}{4}x + 1.$$

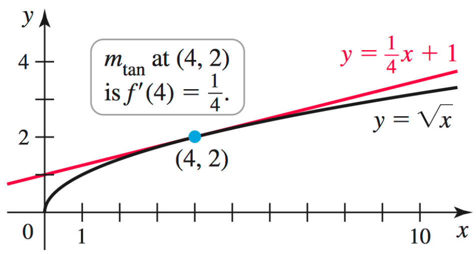


Figure 3.17

Related Exercises 37–38 ◆

Quick Check 3 In Example 2, do the slopes of the tangent lines increase or decrease as x increases? Explain. ◆

Answer »

> The slopes of tangent lines decrease as x increases because the values of $f'(x) = \dfrac{1}{2\sqrt{x}}$ decrease as x increases.

If a function is given in terms of variables other than x and y, we make an adjustment to the derivative definition. For example, if $y = g(t)$, we replace f by g and x by t to obtain the *derivative of g with respect to t*:

$$g'(t) = \lim_{h \to 0} \frac{g(t+h) - g(t)}{h}.$$

Other notation for $g'(t)$ includes $\dfrac{dg}{dt}$, $\dfrac{d}{dt}(g(t))$, $D_t(g(t))$, and $y'(t)$.

Quick Check 4 Express the derivative of $p = q(r)$ in three ways. ◆

Answer »

> $\dfrac{dq}{dr}$, $\dfrac{dp}{dr}$, $D_r(q(r))$, $q'(r)$, $p'(r)$

EXAMPLE 3 Another derivative calculation

Let $g(t) = \dfrac{1}{t^2}$ and compute $g'(t)$.

SOLUTION »

$$\begin{aligned}
g'(t) &= \lim_{h\to 0}\frac{g(t+h)-g(t)}{h} && \text{Definition of } g' \\
&= \lim_{h\to 0}\frac{1}{h}\left(\frac{1}{(t+h)^2}-\frac{1}{t^2}\right) && \text{Substitute } g(t)=\frac{1}{t^2}. \\
&= \lim_{h\to 0}\frac{1}{h}\left(\frac{t^2-(t+h)^2}{t^2(t+h)^2}\right) && \text{Common denominator} \\
&= \lim_{h\to 0}\frac{1}{h}\left(\frac{-2ht-h^2}{t^2(t+h)^2}\right) && \text{Expand the numerator and simplify.} \\
&= \lim_{h\to 0}\left(\frac{-2t-h}{t^2(t+h)^2}\right) && h\neq 0;\ \text{cancel } h. \\
&= -\frac{2}{t^3} && \text{Evaluate the limit.}
\end{aligned}$$

Related Exercises 25–27 ◆

Graphs of Derivatives »

The function f' is called the derivative of f because it is *derived* from f. The following examples illustrate how to *derive* the graph of f' from the graph of f.

EXAMPLE 4 Graph of the derivative

Sketch the graph of f' from the graph of f (**Figure 3.18**).

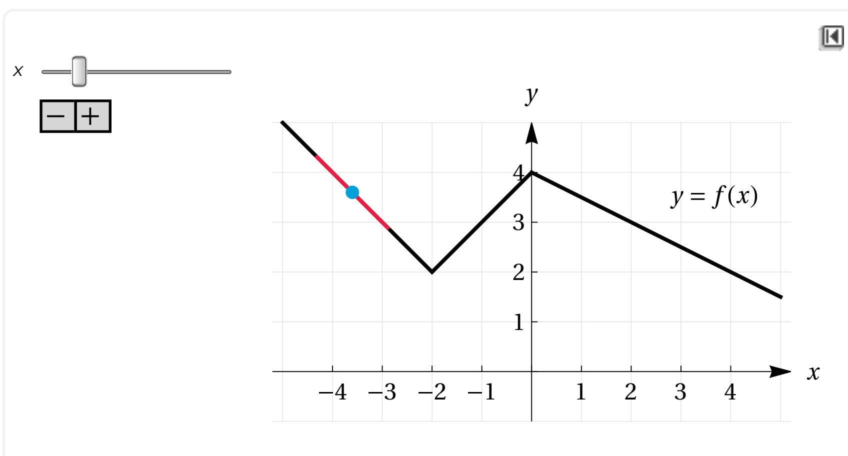


Figure 3.18

SOLUTION »

The graph of f consists of line segments, which are their own tangent lines. Therefore, the slope of the curve $y=f(x)$, for $x<-2$, is -1; that is, $f'(x)=-1$, for $x<-2$. Similarly, $f'(x)=1$, for $-2<x<0$, and $f'(x)=-\frac{1}{2}$, for $x>0$. **Figure 3.19** shows the graph of f in black and the graph of f' in red.

Note »

In terms of limits at $x = -2$, we can write

$$\lim_{h \to 0^-} \frac{f(-2+h) - f(-2)}{h} = -1 \text{ and}$$

$$\lim_{h \to 0^+} \frac{f(-2+h) - f(-2)}{h} = 1.$$

Because the one-sided limits are not equal, $f'(-2)$ does not exist. The analogous one-sided limits at $x = 0$ are also unequal.

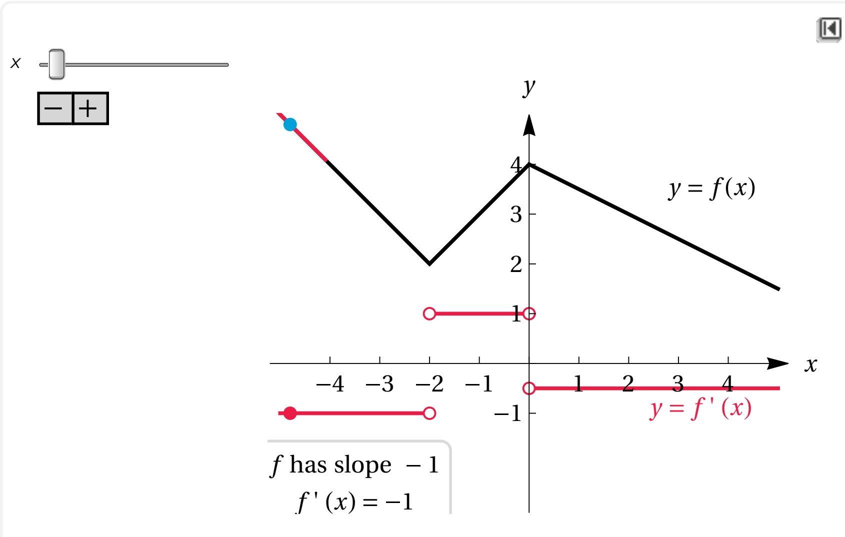


Figure 3.19

Notice that the slopes of the tangent lines change abruptly at $x = -2$ and $x = 0$. As a result, $f'(-2)$ and $f'(0)$ are undefined and the graph of the derivative is discontinuous at these points.

Related Exercises 17–18 ◆

EXAMPLE 5 Graph of the derivative

Sketch the graph of g' using the graph of g (**Figure 3.20**).

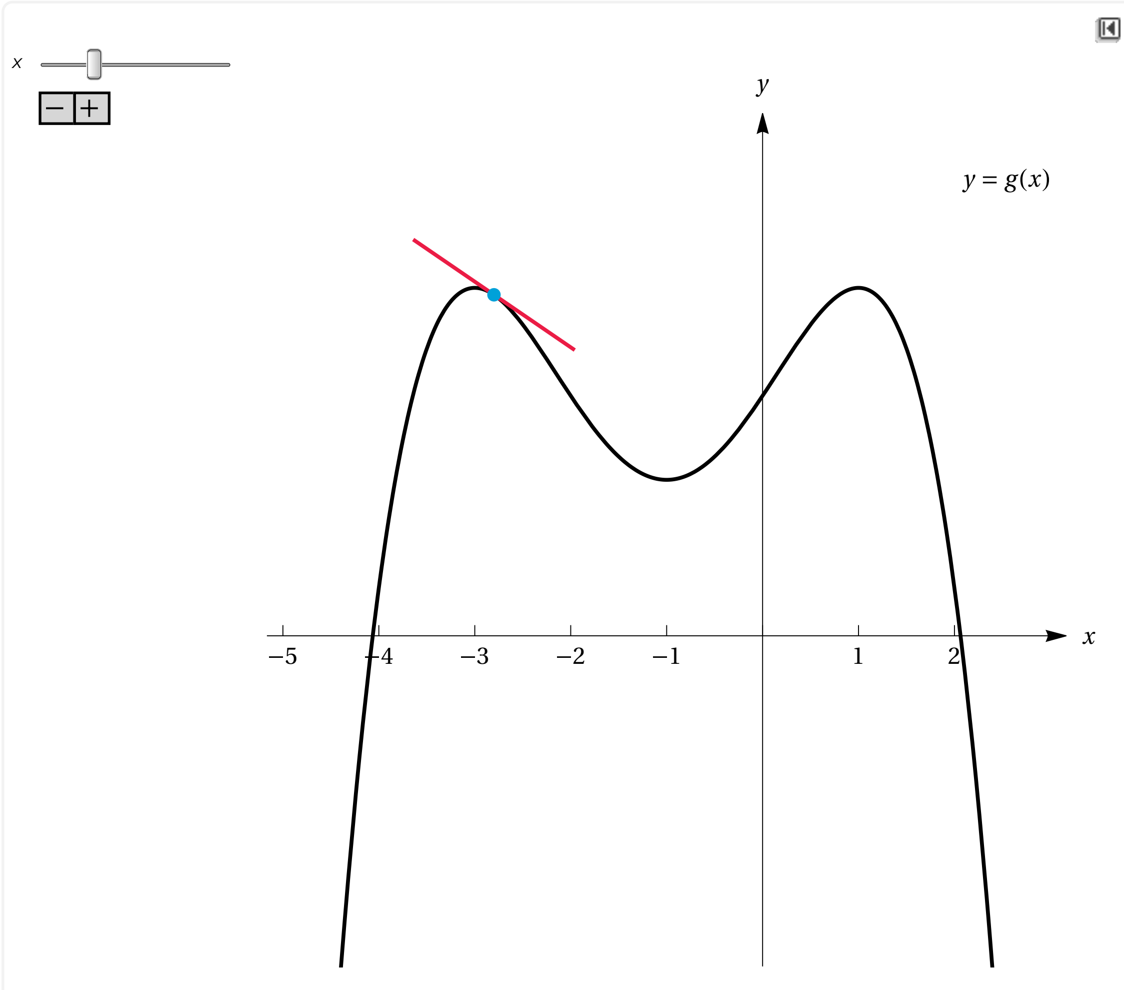


Figure 3.20

SOLUTION »

Without an equation for g, the best we can do is to find the general shape of the graph of g'. Here are the key observations.

1. First note that the lines tangent to the graph of g at $x = -3, -1,$ and 1 have a slope of 0. Therefore,

$$g'(-3) = g'(-1) = g'(1) = 0,$$

which means the graph of g' has x-intercepts at these points (**Figure 3.21**).

2. For $x < -3$, the slopes of the tangent lines are positive and decrease to 0 as x approaches -3 from the left. Therefore, $g'(x)$ is positive for $x < -3$ and decreases to 0 as x approaches -3.

3. For $-3 < x < -1$, $g'(x)$ is negative; it initially decreases as x increases and then increases to 0 at $x = -1$. For $-1 < x < 1$, $g'(x)$ is positive; it initially increases as x increases and then returns to 0 at $x = 1$.

4. Finally, $g'(x)$ is negative and decreasing for $x > 1$. Because the slope of g changes gradually, the graph of g' is continuous with no jumps or breaks.

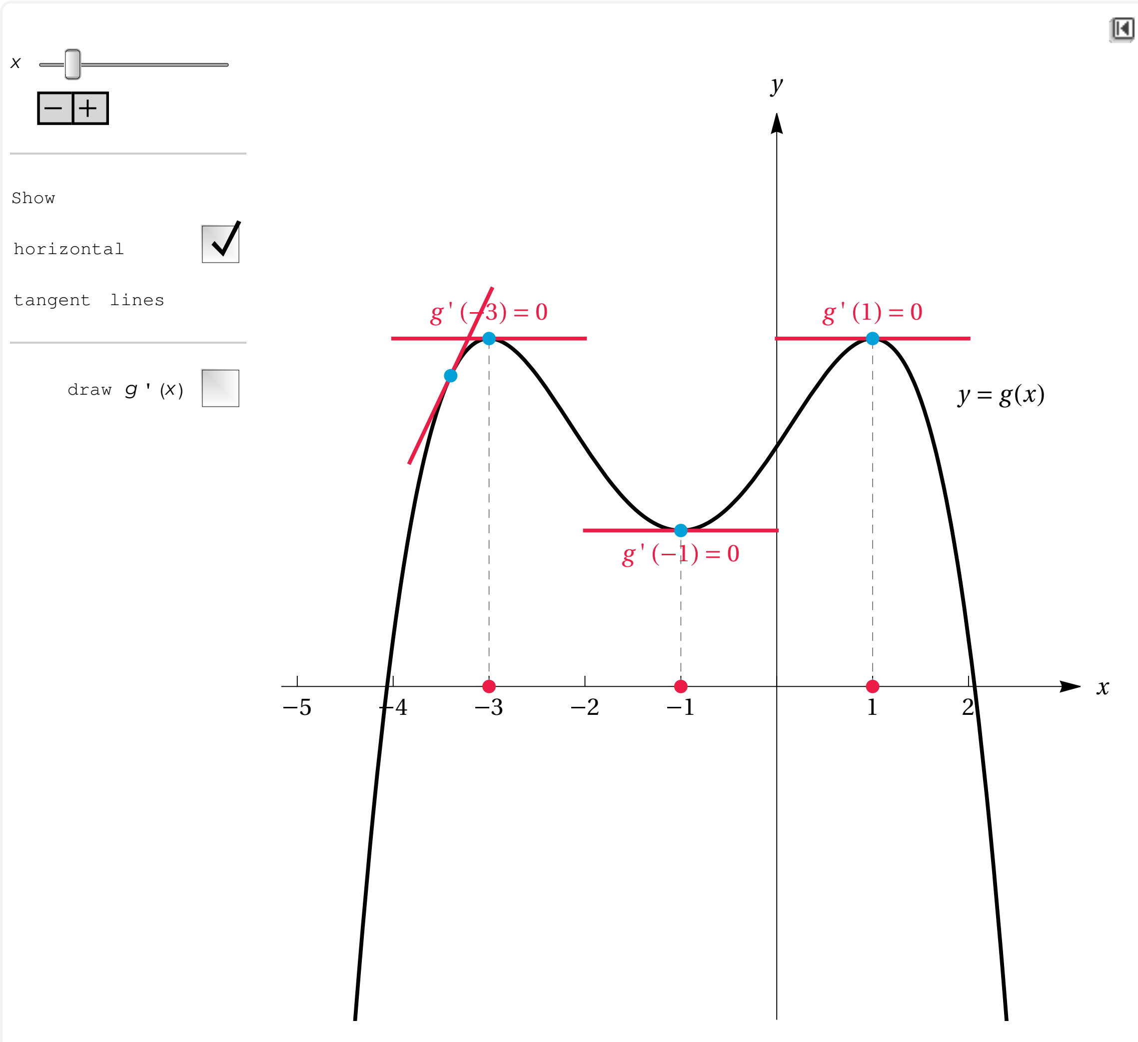


Figure 3.21

Related Exercises 48–50 ◆

Quick Check 5 Is it true that if $f(x) > 0$ at a point, then $f'(x) > 0$ at that point? Is it true that if $f'(x) > 0$ at a point, then $f(x) > 0$ at that point? Explain. ◆

Answer »

No. No.

EXAMPLE 6 Graphing the derivative with asymptotes

The graph of the function f is shown in **Figure 3.22**. Sketch a graph of its derivative.

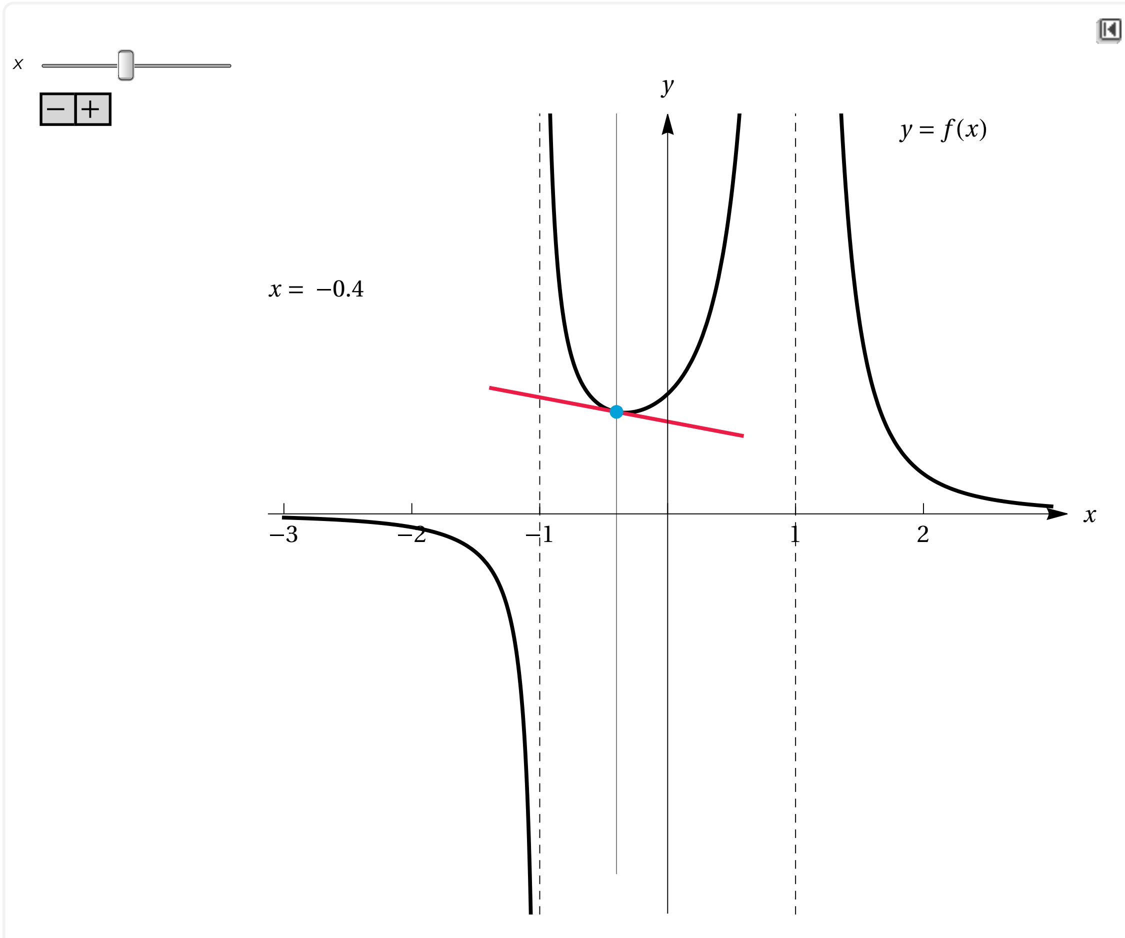


Figure 3.22

SOLUTION »

Identifying intervals on which the slopes of tangent lines are zero, positive, and negative, we make the following observations:

- A horizontal tangent line occurs at approximately $\left(-\frac{1}{3}, f\left(-\frac{1}{3}\right)\right)$. Therefore, $f'\left(-\frac{1}{3}\right) = 0$.

Assembling all this information, we obtain a graph of f' shown in **Figure 3.23**. Notice that f and f' have the same vertical asymptotes. However, as we pass through -1, the sign of f changes, while the sign of f' does not. As we pass through 1, the sign of f does not change, while the sign of f' does.

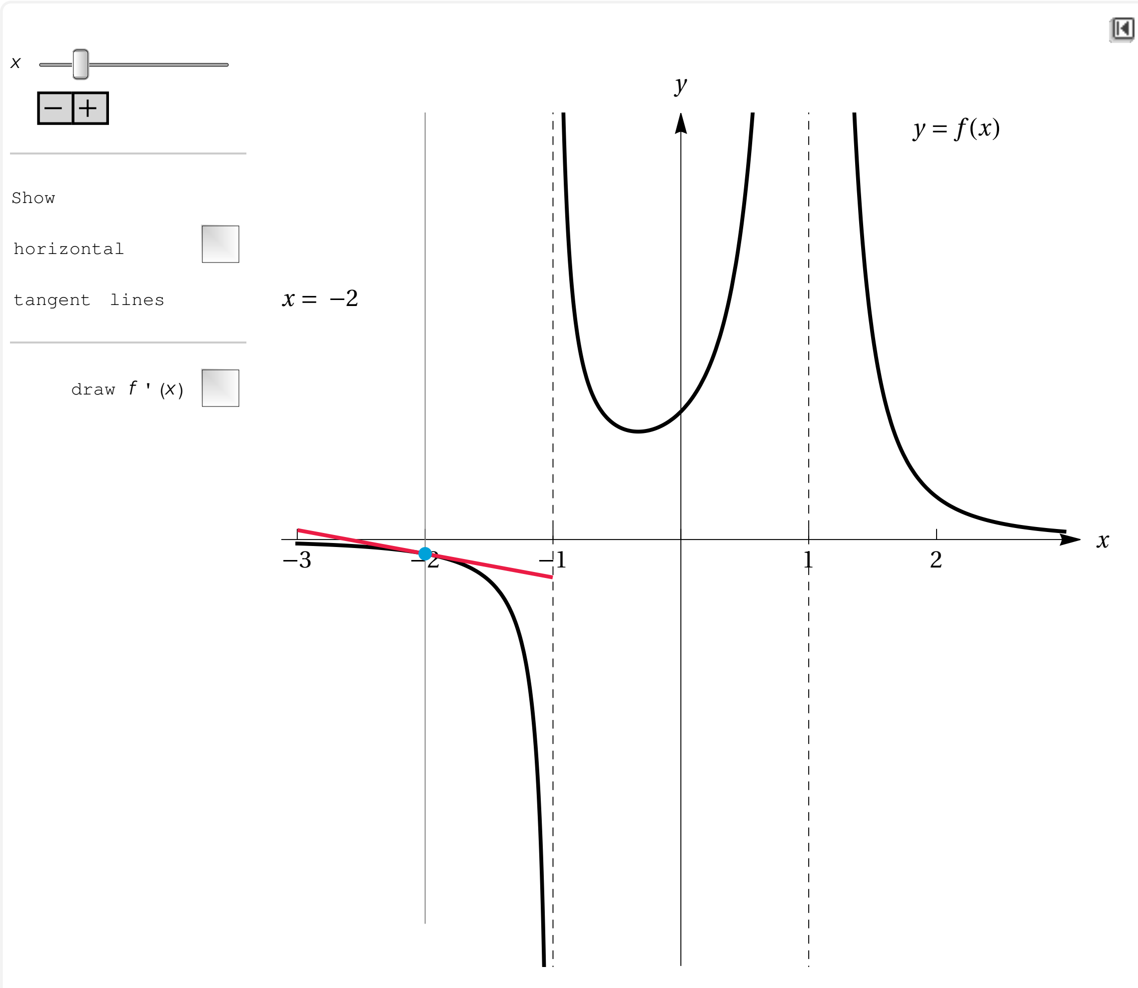


Figure 3.23

Note »

Although it is the case in Example 6, a function and its derivative do not always share the same vertical asymptotes.

Related Exercises 51–52 ◆

Continuity »

We now return to the discussion of continuity (Section 2.6) and investigate the relationship between continuity and differentiability. Specifically, we show that if a function is differentiable at a point, then it is also continuous at that point.

THEOREM 3.1 Differentiable Implies Continuous

If f is differentiable at a, then f is continuous at a.

Proof: Because f is differentiable at a point a, we know that

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists. To show that f is continuous at a, we must show that $\lim\limits_{x \to a} f(x) = f(a)$. The key is the identity

$$f(x) = \frac{f(x) - f(a)}{x - a}(x - a) + f(a), \quad \text{for } x \neq a. \tag{1}$$

Note »

> Expression (1) is an identity because it holds for all values of $x \neq a$, which can be seen by canceling $x - a$ and simplifying.

Taking the limit as x approaches a on both sides of (1) and simplifying, we have

$$\begin{aligned}
\lim_{x \to a} f(x) &= \lim_{x \to a}\left[\frac{f(x) - f(a)}{x - a}(x - a) + f(a)\right] && \text{Use identity.}\\
&= \underbrace{\lim_{x \to a}\left(\frac{f(x) - f(a)}{x - a}\right)}_{f'(a)} \underbrace{\lim_{x \to a}(x - a)}_{0} + \underbrace{\lim_{x \to a} f(a)}_{f(a)} && \text{Theorem 2.3}\\
&= f'(a) \cdot 0 + f(a) && \text{Evaluate limits.}\\
&= f(a). && \text{Simplify.}
\end{aligned}$$

Therefore, $\lim\limits_{x \to a} f(x) = f(a)$, which means that f is continuous at a. ◆

Quick Check 6 Verify that the right-hand side of (1) equals $f(x)$ if $x \neq a$. ◆

Theorem 3.1 tell us that if f is differentiable at a point, then it is necessarily continuous at that point. Therefore, if f is *not* continuous at a point, then f is *not* differentiable there (**Figure 3.24**). So Theorem 3.1 can be stated in another way.

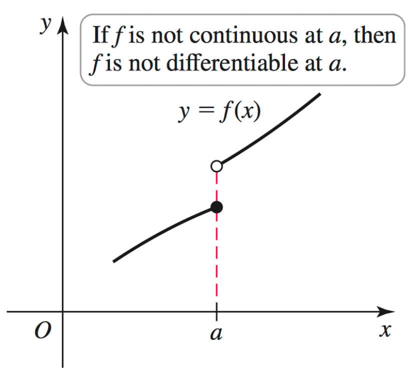


Figure 3.24

> **THEOREM 3.1 (ALTERNATIVE VERSION) Not Continuous Implies Not Differentiable**
>
> If f is not continuous at a, then f is not differentiable at a.

Note »

The alternative version of Theorem 3.1 is called the *contrapositive* of the first statement of Theorem 3.1. A statement and its contrapositive are two equivalent ways of expressing the same statement.

For example, the statement

If I live in Denver, then I live in Colorado

is logically equivalent to its contrapositive:

If I do not live in Colorado, then I do not live in Denver.

It is tempting to read more into Theorem 3.1 than what it actually states. If f is continuous at a point, f is *not* necessarily differentiable at that point. For example, consider the continuous function in **Figure 3.25** and note the **corner point** at a. Ignoring the portion of the graph for $x > a$, we might be tempted to conclude that ℓ_1 is the line tangent to the curve at a. Ignoring the part of the graph for $x < a$, we might incorrectly conclude that ℓ_2 is the line tangent to the curve at a. The slopes of ℓ_1 and ℓ_2 are not equal. Because of the abrupt change in the slope of the curve at a, f is not differentiable at a: The limit that defines f' does not exist at a.

Note »

To avoid confusion about continuity and differentiability, it helps to think about the function $f(x) = |x|$: It is continuous everywhere but not differentiable at 0.

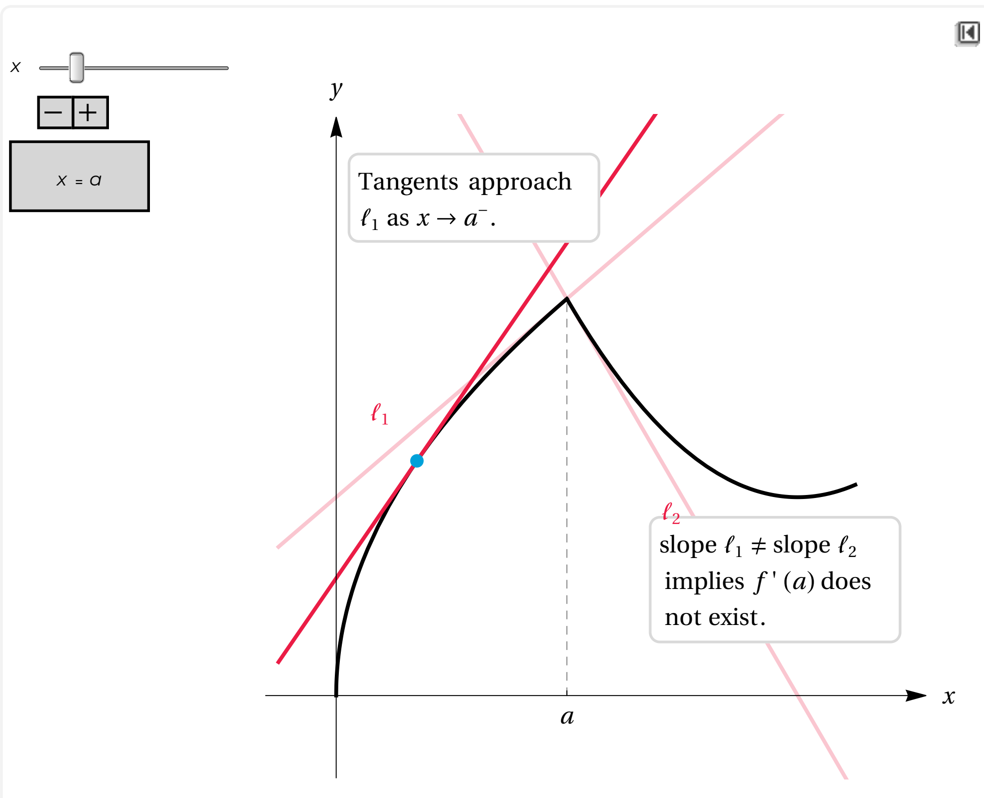


Figure 3.25

Note »

Continuity requires that $\lim_{x \to a} (f(x) - f(a)) = 0$. Differentiability requires more: $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ must exist.

Another common situation occurs when the graph of a function f has a vertical tangent line at a. In this case, $f'(a)$ is undefined because the slope of a vertical line is undefined. A vertical tangent line may occur at a sharp point on the curve called a **cusp** (for example, the function $f(x) = \sqrt{|x|}$ in **Figure 3.26a**). In other cases, a vertical tangent line may occur without a cusp (for example, the function $f(x) = \sqrt[3]{x}$ in **Figure 3.26b**).

Note »

See Exercises 73–76 for a formal definition of a vertical tangent line.

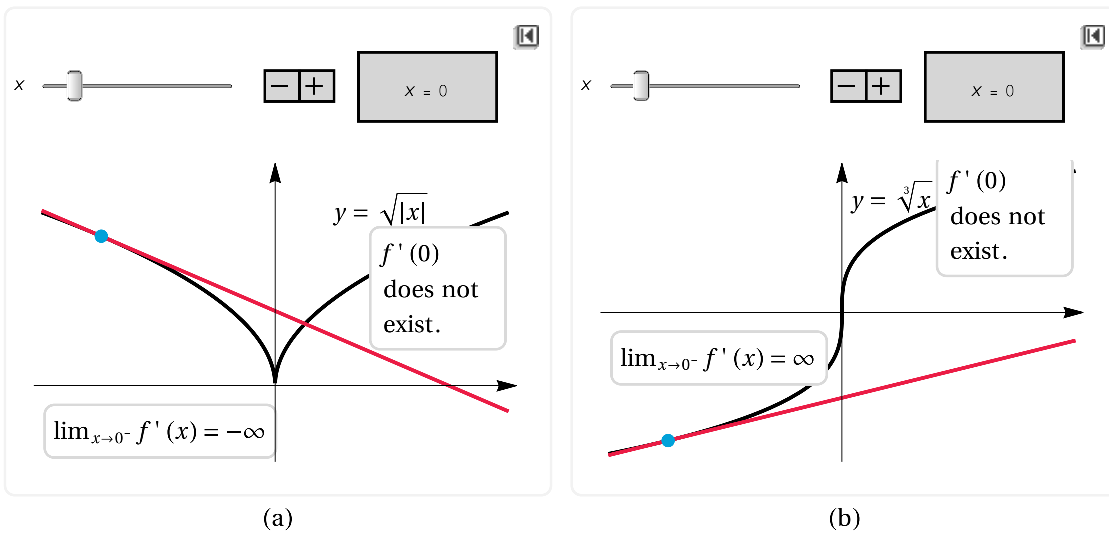


Figure 3.26

When Is a Function Not Differentiable at a Point?

A function f is *not* differentiable at a if at least one of the following conditions holds:

a. f is not continuous at a (Figure 3.24).

b. f has a corner at a (Figure 3.25).

c. f has a vertical tangent at a (Figure 3.26).

EXAMPLE 7 Continuous and differentiable

Consider the graph of g in **Figure 3.27**.

a. Find the values of x in the interval $(-4, 4)$ at which g is not continuous.
b. Find the values of x in the interval $(-4, 4)$ at which g is not differentiable.
c. Sketch a graph of the derivative of g.

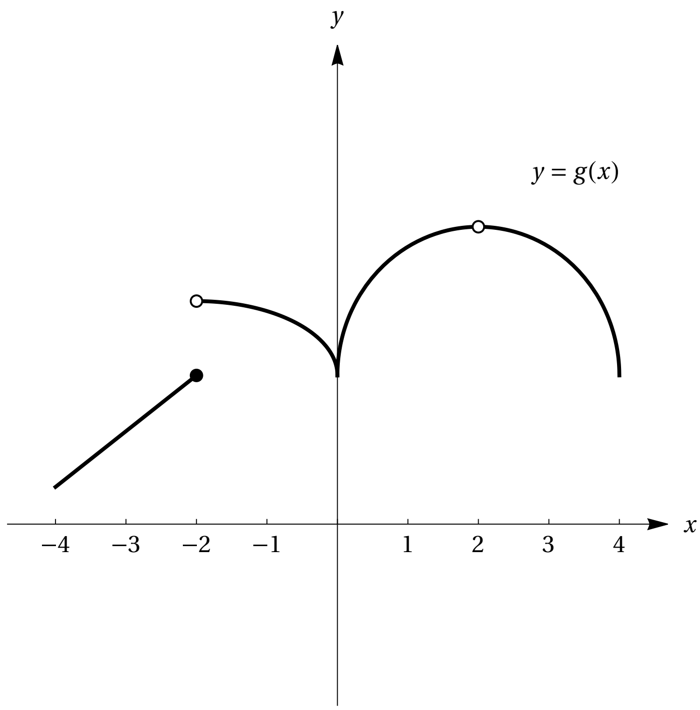


Figure 3.27

SOLUTION »

a. The function g fails to be continuous at -2 (where the one-sided limits are not equal) and at 2 (where g is not defined).

b. Because it is not continuous at ± 2, g is not differentiable at those points. Furthermore, g is not differentiable at 0, because the graph has a cusp at that point.

c. A sketch of the derivative (**Figure 3.28**) has the following features:

- $g'(x) > 0$, for $-4 < x < -2$ and $0 < x < 2$

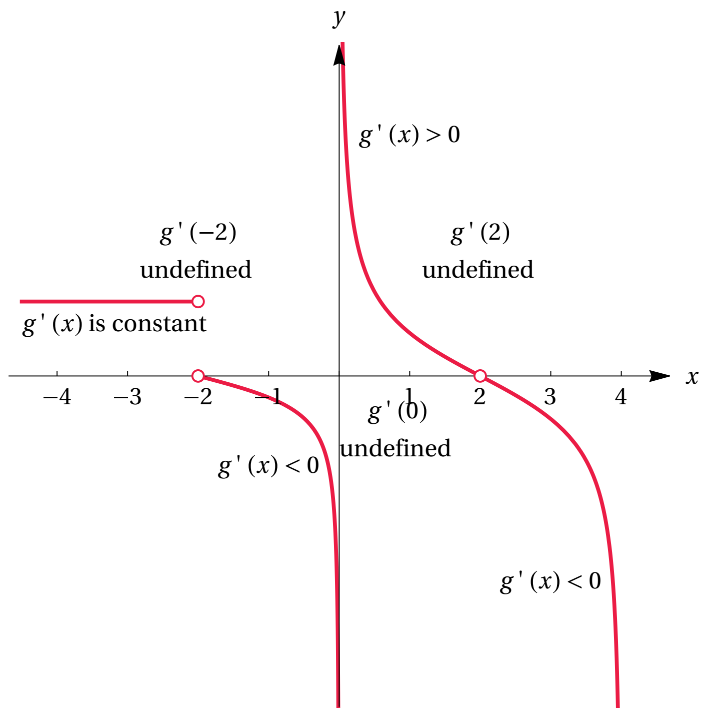


Figure 3.28

Related Exercises 53–54 ◆

Exercises »

Getting Started »

Practice Exercises »

21–30. Derivatives

a. *Use limits to find the derivative function f' for the following functions f.*

b. *Evaluate $f'(a)$ for the given values of a.*

21. $f(x) = 5x + 2$; $a = 1,\ 2$

22. $f(x) = 7$; $a = -1,\ 2$

23. $f(x) = 4x^2 + 1$; $a = 2,\ 4$

24. $f(x) = x^2 + 3x$; $a = -1,\ 4$

25. $f(x) = \dfrac{1}{x+1}$; $a = -\dfrac{1}{2},\ 5$

26. $f(x) = \dfrac{x}{x+2}$; $a = -1,\ 0$

27. $f(t) = \dfrac{1}{\sqrt{t}}$; $a = 9,\ 1/4$

28. $f(w) = \sqrt{4w - 3}$; $a = 1, 3$

29. $f(s) = 4s^3 + 3s$; $a = -3, \ -1$

30. $f(t) = 3t^4$; $a = -2, \ 2$

31–32. Velocity functions *A projectile is fired vertically upward into the air and its position (in feet) above the ground after t seconds is given by the function* $s(t)$.

a. *For the following functions* $s(t)$, *find the instantaneous velocity function* $v(t)$. *(Recall that the velocity function v is the derivative of the position function s.)*

b. *Determine the instantaneous velocity of the projectile at t = 1 and t = 2 seconds.*

31. $s(t) = -16t^2 + 100t$

32. $s(t) = -16t^2 + 128t + 192$

33. Evaluate $\dfrac{dy}{dx}$ and $\left.\dfrac{dy}{dx}\right|_{x=2}$ if $y = \dfrac{x+1}{x+2}$.

34. Evaluate $\dfrac{ds}{dt}$ and $\left.\dfrac{ds}{dt}\right|_{t=-1}$ if $s = 11t^3 + t + 1$.

35–40. Tangent lines

a. *Find the derivative function* f' *for the following functions* f.

b. *Find an equation of the line tangent to the graph of f at* $(a, f(a))$ *for the given value of* a.

35. $f(x) = 3x^2 + 2x - 10$; $a = 1$

36. $f(x) = 5x^2 - 6x + 1$; $a = 2$

37. $f(x) = \sqrt{3x + 1}$; $a = 8$

38. $f(x) = \sqrt{x + 2}$; $a = 7$

39. $f(x) = \dfrac{2}{3x + 1}$; $a = -1$

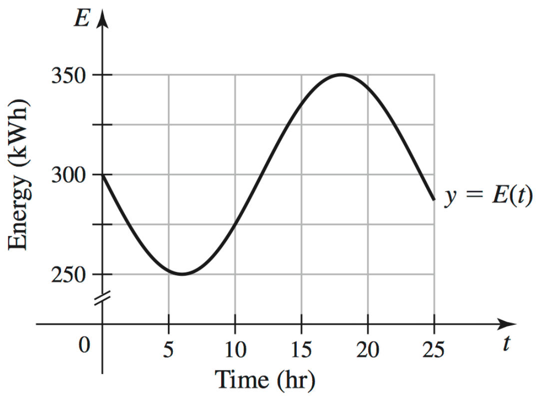
40. $f(x) = \dfrac{1}{x}$; $a = -5$

41. Power and energy Energy is the capacity to do work, and power is the rate at which energy is used or consumed. Therefore, if $E(t)$ is the energy function for a system, then $P(t) = E'(t)$ is the power function. A unit of energy is the kilowatt-hour (1 kWh is the amount of energy needed to light ten 100-W light bulbs for an hour); the corresponding units for power are kilowatts. The following figure shows the energy function for a small community over a 25-hour period.

a. Estimate the power at $t = 10$ and $t = 20$ hr. Be sure to include units in your calculation.

b. At what times on the interval [0, 25] is the power zero?

c. At what times on the interval [0, 25] is the power a maximum?



42. **Slope of a line** Consider the line $f(x) = mx + b$, where m and b are constants. Show that $f'(x) = m$ for all x. Interpret this result.

43. **A derivative formula**

- **a.** Use the definition of the derivative to determine $\frac{d}{dx}\left(ax^2 + bx + c\right)$, where a, b, and c are constants.
- **b.** Let $f(x) = 4x^2 - 3x + 10$ and use part (a) to find $f'(x)$.
- **c.** Use part (b) to find $f'(1)$.

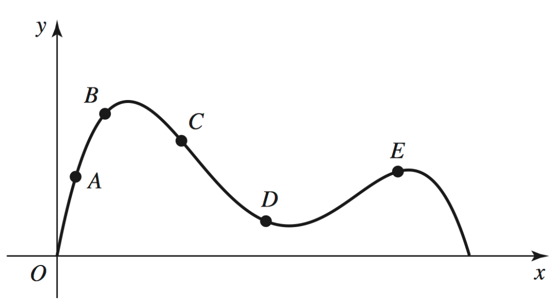
44. **A derivative formula**

- **a.** Use the definition of the derivative to determine $\frac{d}{dx}\left(\sqrt{ax + b}\right)$, where a and b are constants.
- **b.** Let $f(x) = \sqrt{5x + 9}$ and use part (a) to find $f'(x)$.
- **c.** Use part (b) to find $f'(-1)$.

45–46. Analyzing slopes *Use the points A, B, C, D, and E in the following graphs to answer these questions.*

- ***a.*** *At which points is the slope of the curve negative?*
- ***b.*** *At which points is the slope of the curve positive?*
- ***c.*** *Using A-E, list the slopes in decreasing order.*

45.



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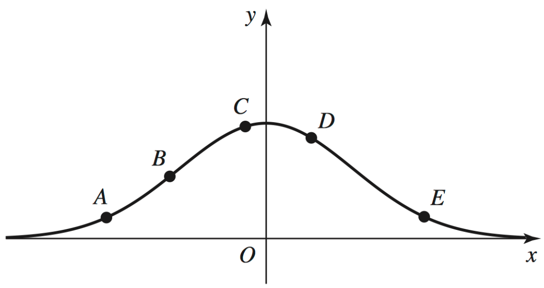
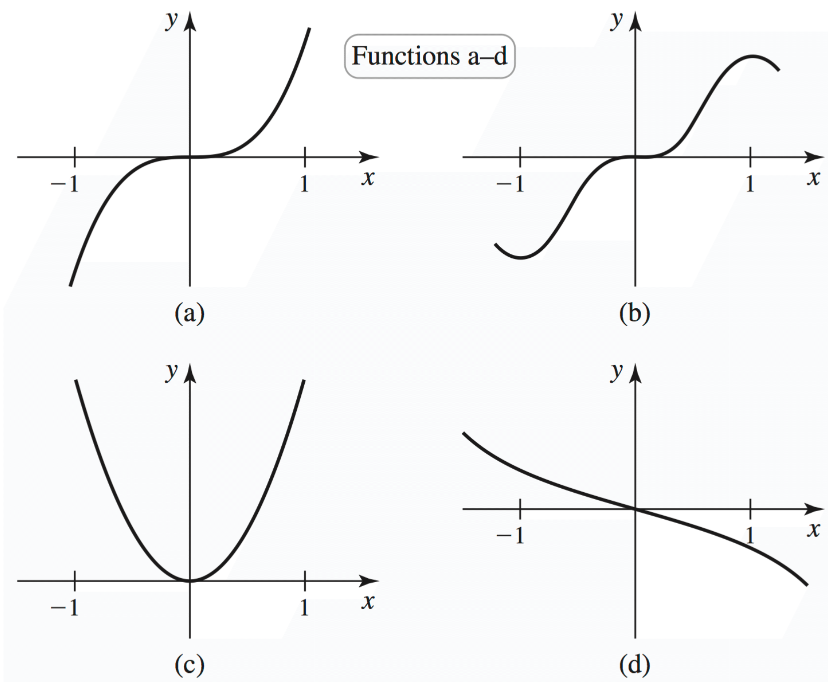
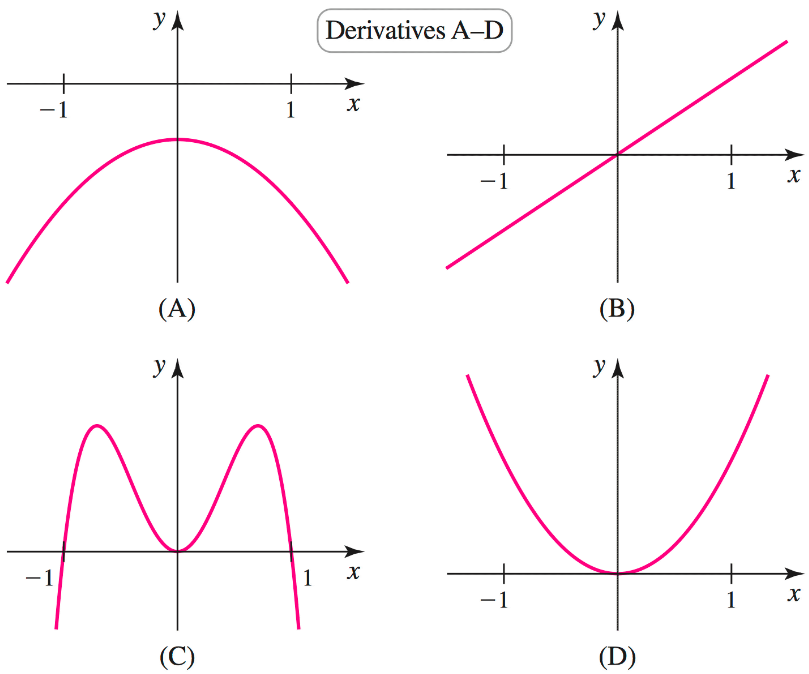


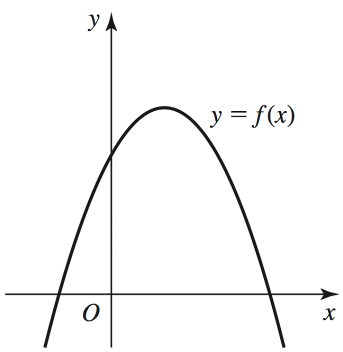
47. Matching functions with derivatives Match the functions a-d in the first set of figures with the derivative functions A-D in the next set of figures.





48–52. Sketching derivatives *Reproduce the graph of f and then plot a graph of f' on the same axes.*

48.



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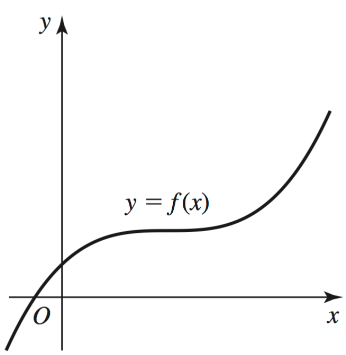

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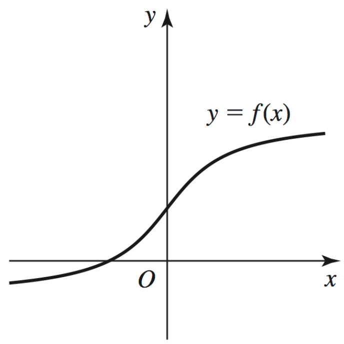

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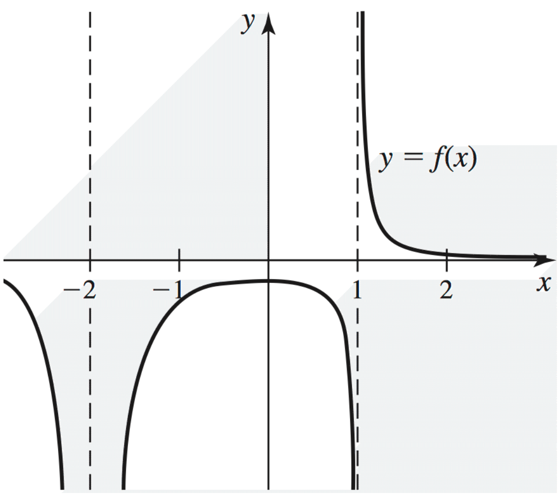

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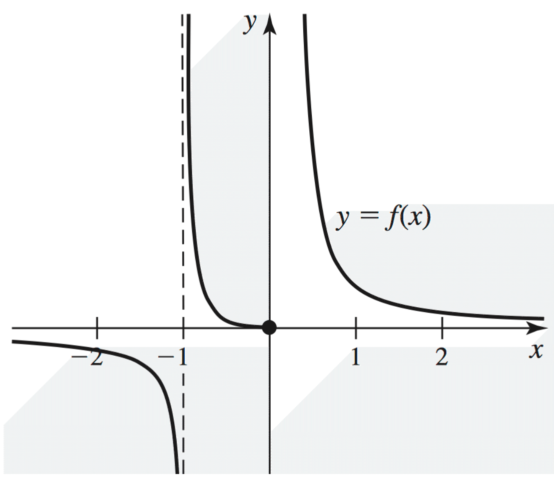
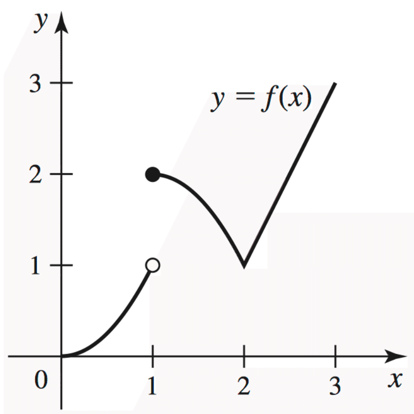


53. Where is the function continuous? Differentiable? Use the graph of f in the figure to do the following.

a. Find the values of x in $(0, 3)$ at which f is not continuous.

b. Find the values of x in $(0, 3)$ at which f is not differentiable.

c. Sketch a graph of f'.

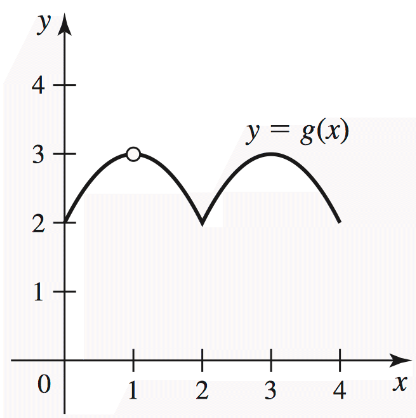


54. Where is the function continuous? Differentiable? Use the graph of g in the figure to do the following.

a. Find the values of x in $(0, 4)$ at which g is not continuous.

b. Find the values of x in $(0, 4)$ at which g is not differentiable.

c. Sketch a graph of g'.



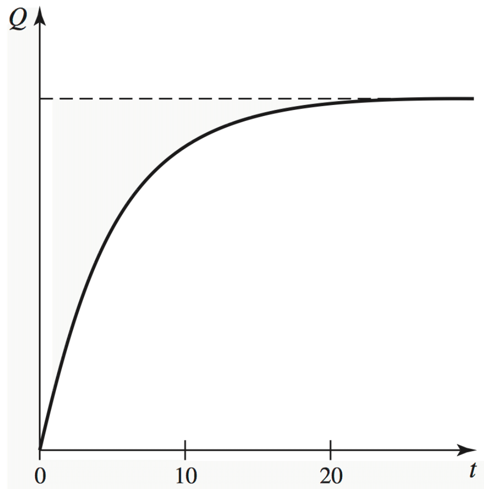
55. Voltage on a capacitor A capacitor is a device in an electrical circuit that stores charge. In one particular circuit, the charge on the capacitor Q varies in time as shown in the figure.

a. At what time is the rate of change of the charge Q' the greatest?

b. Is Q' positive or negative for $t \geq 0$?

c. Is Q' an increasing or decreasing function of time (or neither)?

d. Sketch the graph of Q'. You do not need a scale on the vertical axis.



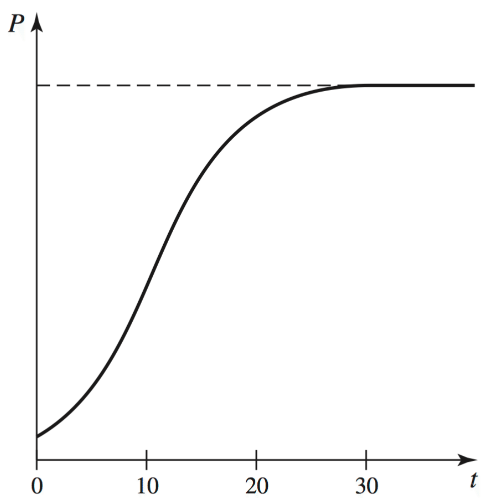
56. Logistic growth A common model for population growth uses the logistic (or sigmoid) curve. Consider the logistic curve in the figure, where $P(t)$ is the population at time $t \geq 0$.

a. At approximately what time is the rate of growth P' the greatest?

b. Is P' positive or negative for $t \geq 0$?

c. Is P' an increasing or decreasing function of time (or neither)?

d. Sketch the graph of P'. You do not need a scale on the vertical axis.



57. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. If the function f is differentiable for all values of x, then f is continuous for all values of x.

b. The function $f(x) = |x + 1|$ is continuous for all x, but not differentiable for all x.

c. It is possible for the domain of f to be (a, b) and the domain of f' to be $[a, b]$.

Explorations and Challenges »

58. Looking ahead: Derivative of x^n Use the definition $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ to find $f'(x)$ for the following functions.

a. $f(x) = x^2$

b. $f(x) = x^3$

c. $f(x) = x^4$

d. Based upon your answers to parts (a)–(c), propose a formula for $f'(x)$ if $f(x) = x^n$, where n is a positive integer.

59. Determining the unknown constant Let

$$f(x) = \begin{cases} 2x^2 & \text{if } x \le 1 \\ ax - 2 & \text{if } x > 1. \end{cases}$$

Determine a value of a (if possible) for which f' is continuous at $x = 1$.

60. Finding *f* from *f'* Sketch the graph of $f'(x) = 2$. Then sketch three possible graphs of f.

61. Finding *f* from *f'* Sketch the graph of $f'(x) = x$. Then sketch a possible graph of f. Is more than one graph possible?

62. Finding *f* from *f'* Create the graph of a continuous function f such that

$$f'(x) = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } 0 < x < 1 \\ -1 & \text{if } x > 1. \end{cases}$$

Is more than one graph possible?

63–66. Normal lines *A line perpendicular to another line or to a tangent line is often called a* ***normal line****. Find an equation of the line perpendicular to the line that is tangent to the following curves at the given point P.*

63. $y = 3x - 4$; $P(1, -1)$

64. $y = \sqrt{x}$; $P(4, 2)$

65. $y = \dfrac{2}{x}$; $P(1, 2)$

66. $y = x^2 - 3x$; $P(3, 0)$

67–70. Aiming a tangent line *Given the function f and the point Q, find all points P on the graph of f such that the line tangent to f at P passes through Q. Check your work by graphing f and the tangent lines.*

67. $f(x) = x^2 + 1$; $Q(3, 6)$

68. $f(x) = -x^2 + 4x - 3$; $Q(0, 6)$

69. $f(x) = \dfrac{1}{x}$; $Q(-2, 4)$

70. $f(x) = 3\sqrt{4x + 1}$; $Q(0, 5)$

71–72. One-sided derivatives *The* ***right-sided*** *and* ***left-sided derivatives*** *of a function at a point a are given by* $f_+'(a) = \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}$ *and* $f_-'(a) = \lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h}$, *respectively, provided these limits exist. The derivative* $f'(a)$ *exists if and only if* $f_+'(a) = f_-'(a)$.

- ***a.*** *Sketch the following functions.*
- ***b.*** *Compute* $f_+'(a)$ *and* $f_-'(a)$ *at the given point a.*
- ***c.*** *Is f continuous at a? Is f differentiable at a?*

71. $f(x) = |x-2|;\ a = 2$

72. $f(x) = \begin{cases} 4-x^2 & \text{if } x \le 1 \\ 2x+1 & \text{if } x > 1 \end{cases};\ a = 1$

73–76. Vertical tangent lines *If a function f is continuous at a and* $\lim_{x\to a} |f'(x)| = \infty$, *then the curve* $y = f(x)$ *has a vertical tangent line at a and the equation of the tangent line is* $x = a$. *If a is an endpoint of a domain, then the appropriate one-sided derivative (Exercises 71–72) is used. Use this information to answer the following questions.*

T **73.** Graph the following functions and determine the location of the vertical tangent lines.

- **a.** $f(x) = (x-2)^{1/3}$
- **b.** $f(x) = (x+1)^{2/3}$
- **c.** $f(x) = \sqrt{|x-4|}$
- **d.** $f(x) = x^{5/3} - 2x^{1/3}$

74. The preceding definition of a vertical tangent line includes four cases: $\lim_{x\to a^+} f'(x) = \pm\infty$ combined with $\lim_{x\to a^-} f'(x) = \pm\infty$ (for example, one case is $\lim_{x\to a^+} f'(x) = -\infty$ and $\lim_{x\to a^-} f'(x) = \infty$). Sketch a continuous function that has a vertical tangent line at a in each of the four cases.

75. Verify that $f(x) = x^{1/3}$ has a vertical tangent line at $x = 0$.

76. Graph the following curves and determine the location of any vertical tangent lines.

- **a.** $x^2 + y^2 = 9$
- **b.** $x^2 + y^2 + 2x = 0$

77.

- **a.** Graph the function $f(x) = \begin{cases} x & \text{if } x \le 0 \\ x+1 & \text{if } x > 0. \end{cases}$
- **b.** For $x < 0$, what is $f'(x)$?
- **c.** For $x > 0$, what is $f'(x)$?
- **d.** Graph f' on its domain.
- **e.** Is f differentiable at 0? Explain.

78. Is $f(x) = \frac{x^2 - 5x + 6}{x - 2}$ differentiable at $x = 2$? Justify your answer.