3 Derivatives

Chapter Preview Now that you are familiar with limits, the door to calculus stands open. The first task is to introduce the fundamental concept of the *derivative*. Suppose a function f represents a quantity of interest, say the variable cost of manufacturing an item, the population of a country, or the position of an orbiting satellite. The derivative of f is another function, denoted f', which gives the changing slope of the curve $y = f(x)$. Equivalently, the derivative of f gives the *instantaneous rate of change* of f at points in the domain. We use limits not only to define the derivative, but also to develop efficient rules for finding derivatives. The applications of the derivative—which we introduce along the way—are endless because almost everything around us is in a state of change, and derivatives describe change.

3.1 Introducing the Derivative

In this section we return to the problem of finding the slope of a line tangent to a curve, introduced at the beginning of Chapter 2. This concept is important for several reasons.

Tangent Lines »

- We identify the slope of the tangent line with the *instantaneous rate of change* of a function (**Figure 3.1**).

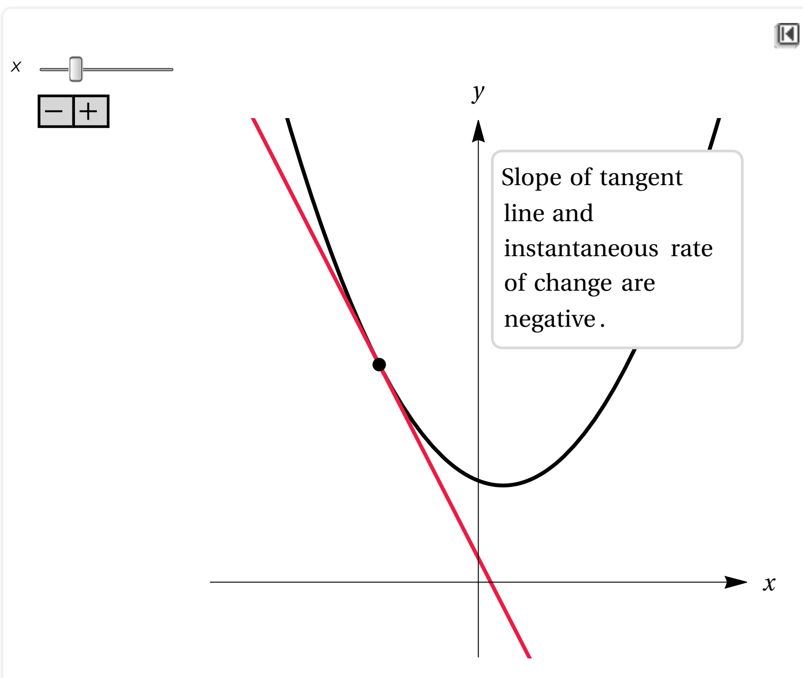


Figure 3.1

- The slopes of the tangent lines as they change along a curve are the values of a new function called the *derivative*.
- Looking farther ahead, if a curve represents the trajectory of a moving object, the tangent line at a point on the curve indicates the direction of motion at that point (**Figure 3.2**).

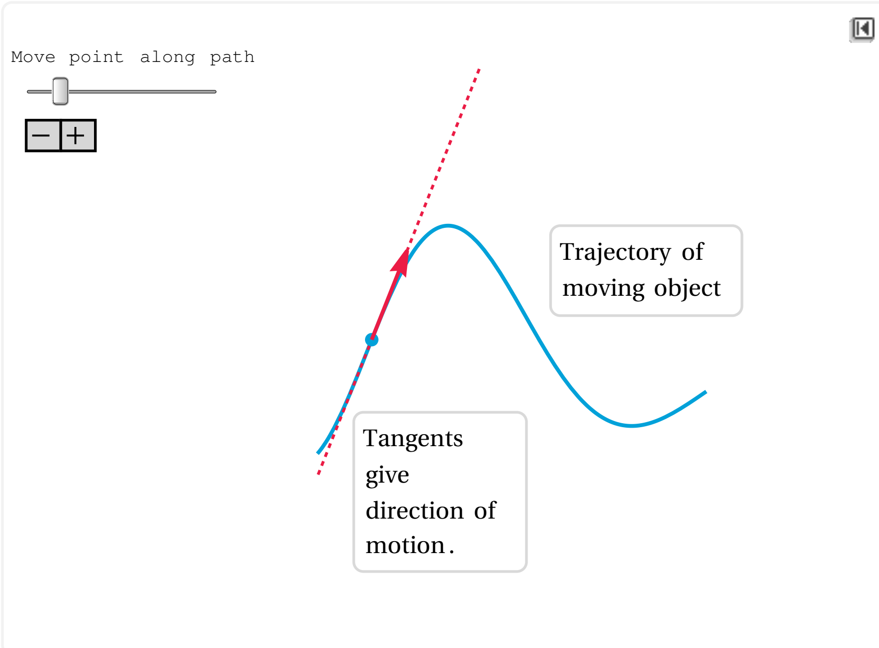


Figure 3.2

In Section 2.1 we used a limit to define the instantaneous velocity of an object that moves along a line. Recall that if $s(t)$ is the position of the object at time t, then the average velocity of the object over the time interval $[a, t]$ is

$$v_{\text{av}} = \frac{s(t) - s(a)}{t - a}.$$

The instantaneous velocity at time $t = a$ is the limit of the average velocity as $t \to a$:

$$v_{\text{inst}} = \lim_{t \to a} \frac{s(t) - s(a)}{t - a}.$$

We also learned that these quantities have important geometric interpretations. The average velocity is the slope of the secant line through the points $(a, s(a))$ and $(t, s(t))$ on the graph of s, and the instantaneous velocity is the slope of the tangent line through the point $(a, s(a))$ (**Figure 3.3**).

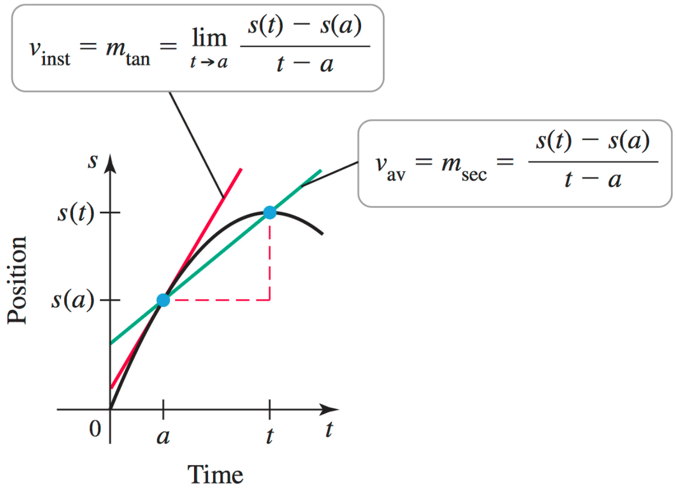


Figure 3.3

EXAMPLE 1 Instantaneous velocity and tangent lines

A rock is launched vertically upward from the ground with an initial speed of 96 ft/s. The position of the rock in feet above the ground after t seconds is given by the function $s(t) = -16\,t^2 + 96\,t$. Consider the point $P(1,\ 80)$ on the curve $y = s(t)$.

a. Find the instantaneous velocity of the rock 1 second after launch and find the slope of the line tangent to the graph of s at P.

b. Find an equation of the tangent line in part (a).

SOLUTION »

a. In Example 2 of Section 2.1, we used numerical evidence to estimate that the instantaneous velocity at $t = 1$ is 64 ft/s. Using limit techniques developed in Chapter 2, we can verify this conjectured value:

$$
\begin{aligned}
v_{\text{inst}} &= \lim_{t \to 1} \frac{s(t) - s(1)}{t - 1} && \text{Definition of instantaneous velocity} \\
&= \lim_{t \to 1} \frac{-16\,t^2 + 96\,t - 80}{t - 1} && s(t) = -16\,t^2 + 96\,t;\ s(1) = 80 \\
&= \lim_{t \to 1} \frac{-16\,(t - 5)\,(t - 1)}{t - 1} && \text{Factor the numerator.} \\
&= -16 \underbrace{\lim_{t \to 1}\,(t - 5)}_{-4} = 64. && \text{Cancel factors } (t \neq 1) \text{ and evaluate the limit.}
\end{aligned}
$$

We see that the instantaneous velocity at $t = 1$ is 64 ft/s, which also equals the slope of the line tangent to the graph of s at the point $P(1,\ 80)$.

b. An equation of the line passing through $(1,\ 80)$ with slope 64 is $y - 80 = 64\,(t - 1)$ or $y = 64\,t + 16$ (**Figure 3.4**).

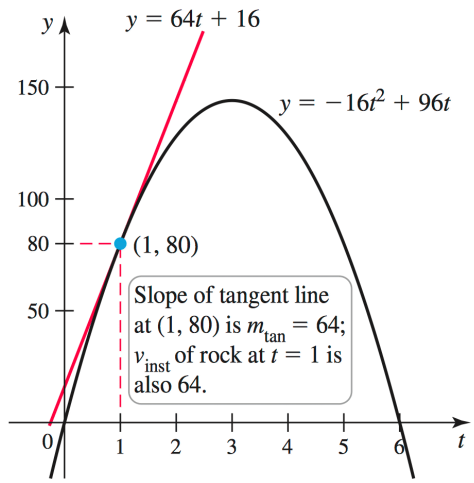


Figure 3.4

Related Exercises 13–14 ◆

Quick Check 1 In Example 1, is the slope of the tangent line at (2, 128) greater than or less than the slope at (1, 80)? ◆

Answer »

The slope is less at $x = 2$.

The connection between the instantaneous rate of change of an object's position and the slope of a tangent line on the graph of the position function can be extended far beyond a discussion of velocity. In fact, the slope of a tangent line is one of the central concepts in calculus because it measures the instantaneous rate of change of a function. Whether a given function describes the position of an object, the population of a city, the concentration of a reactant in a chemical reaction, or the weight of a growing child, the slopes of tangent lines associated with these functions measure rates at which the quantities change.

Tangent Lines and Rates of Change »

Consider the curve $y = f(x)$ and a secant line intersecting the curve at the points $P(a, f(a))$ and $Q(x, f(x))$ (**Figure 3.5**). The difference $f(x) - f(a)$ is the change in the value of f on the interval $[a, x]$, while $x - a$ is the change in x. As discussed in Chapter 2, the slope of the secant line $\overleftrightarrow{PQ}$ is

$$m_{\text{sec}} = \frac{f(x) - f(a)}{x - a},$$

and it gives the *average rate of change* of f on the interval $[a, x]$.

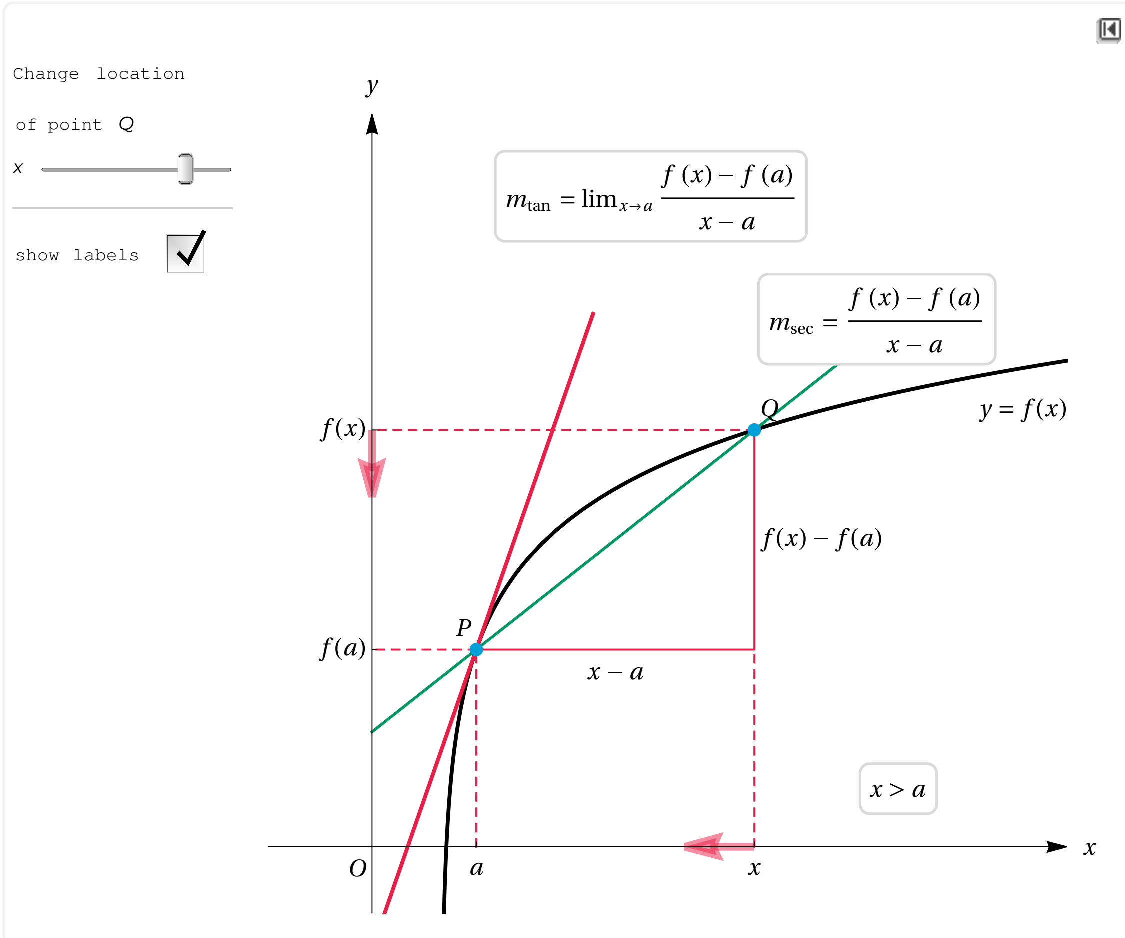


Figure 3.5

Figure 3.5 also shows what happens as the variable point x approaches the fixed point a. If the curve is smooth at $P(a, f(a))$—it has no kinks or corners—the secant lines approach a *unique* line that intersects the curve at P; this line is the *tangent line* at P As x approaches a, the slopes $m_{\sec}$ of the secant lines approach a unique number $m_{\tan}$ that we call the *slope of the tangent line*; that is,

$$m_{\tan} = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}.$$

The slope of the tangent line is also referred to as the *instantaneous rate of change* of f at a because it measures how quickly f changes at a.

The tangent line has another geometric interpretation. As discussed in Section 2.1, if the curve $y = f(x)$ is smooth at a point $P(a, f(a))$, then the curve looks more like a line as we zoom in on P. The line that is approached as we zoom in on P is also the tangent line (**Figure 3.6**). A smooth curve has the property of *local linearity*, which means that if we look at a point on the curve locally (by zooming in), then the curve appears linear.

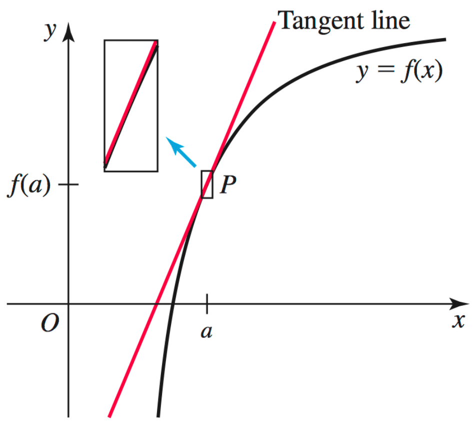


Figure 3.6

Note »

The definition of m_{sec} involves a *difference quotient*, introduced in Section 1.1.

DEFINITION **Rate of Change and the Tangent Line**

The **average rate of change** in f on the interval $[a, x]$ is the slope of the corresponding secant line:

$$m_{\text{sec}} = \frac{f(x) - f(a)}{x - a}.$$

The **instantaneous rate of change** in f at a is

$$m_{\tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}, \qquad (1)$$

which is also the **slope of the tangent line** at a, provided this limit exists. The **tangent line** at $x = a$ is the unique line through $(a, f(a))$ with slope $m_{\tan}$. Its equation is

$$y - f(a) = m_{\tan}(x - a).$$

Note »

If x and y have physical units, then the average and instantaneous rates of change have units of $\dfrac{(\text{units of } y)}{(\text{units of } x)}$. For example, if y has units of meters and x has units of seconds, the units of the rates of change are meters/second (m/s).

Quick Check 2 Sketch the graph of a function f near a point a. As in Figure 3.5, draw a secant line that passes through $(a, f(a))$ and a neighboring point $(x, f(x))$ with $x < a$. Use arrows to show how the secant lines approach the tangent line as x approaches a. ◆

EXAMPLE 2 Equation of a tangent line

Find an equation of the line tangent to the graph of $f(x) = \dfrac{3}{x}$ at $\left(2, \dfrac{3}{2}\right)$.

SOLUTION »

We use the definition of the slope of the tangent line with $a = 2$:

$$\begin{aligned}
m_{\tan} &= \lim_{x\to 2} \frac{f(x) - f(2)}{x - 2} && \text{Definition of slope of tangent line}\\
&= \lim_{x\to 2} \frac{\frac{3}{x} - \frac{3}{2}}{x - 2} && f(x) = \frac{3}{x};\ f(2) = \frac{3}{2}\\
&= \lim_{x\to 2} \frac{\frac{6-3x}{2x}}{x - 2} && \text{Combine fractions with common denominator.}\\
&= \lim_{x\to 2} \frac{-3(x-2)}{2x(x-2)} && \text{Simplify.}\\
&= \lim_{x\to 2} \left(-\frac{3}{2x}\right) = -\frac{3}{4}. && \text{Cancel factors } (x \neq 2) \text{ and evaluate the limit.}
\end{aligned}$$

The tangent line has slope $m_{\tan} = -\frac{3}{4}$ and passes through the point $\left(2, \frac{3}{2}\right)$ (**Figure 3.7**). Its equation is $y - \frac{3}{2} = -\frac{3}{4}(x - 2)$ or $y = -\frac{3}{4}x + 3$. We could also say that the instantaneous rate of change in f at $x = 2$ is $-3/4$.

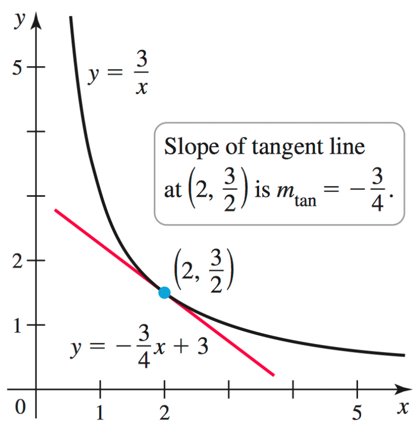


Figure 3.7

Related Exercises 17–18 ◆

An alternative formula for the slope of the tangent line is helpful for future work. Consider again the curve $y = f(x)$ and the secant line intersecting the curve at the points P and Q. We now let $(a, f(a))$ and $(a + h, f(a + h))$ be the coordinates of P and Q, respectively (**Figure 3.8**). The difference in the x-coordinates of P and Q is $(a + h) - a = h$. Note that Q is located to the right of P if $h > 0$ and to the left of P if $h < 0$.

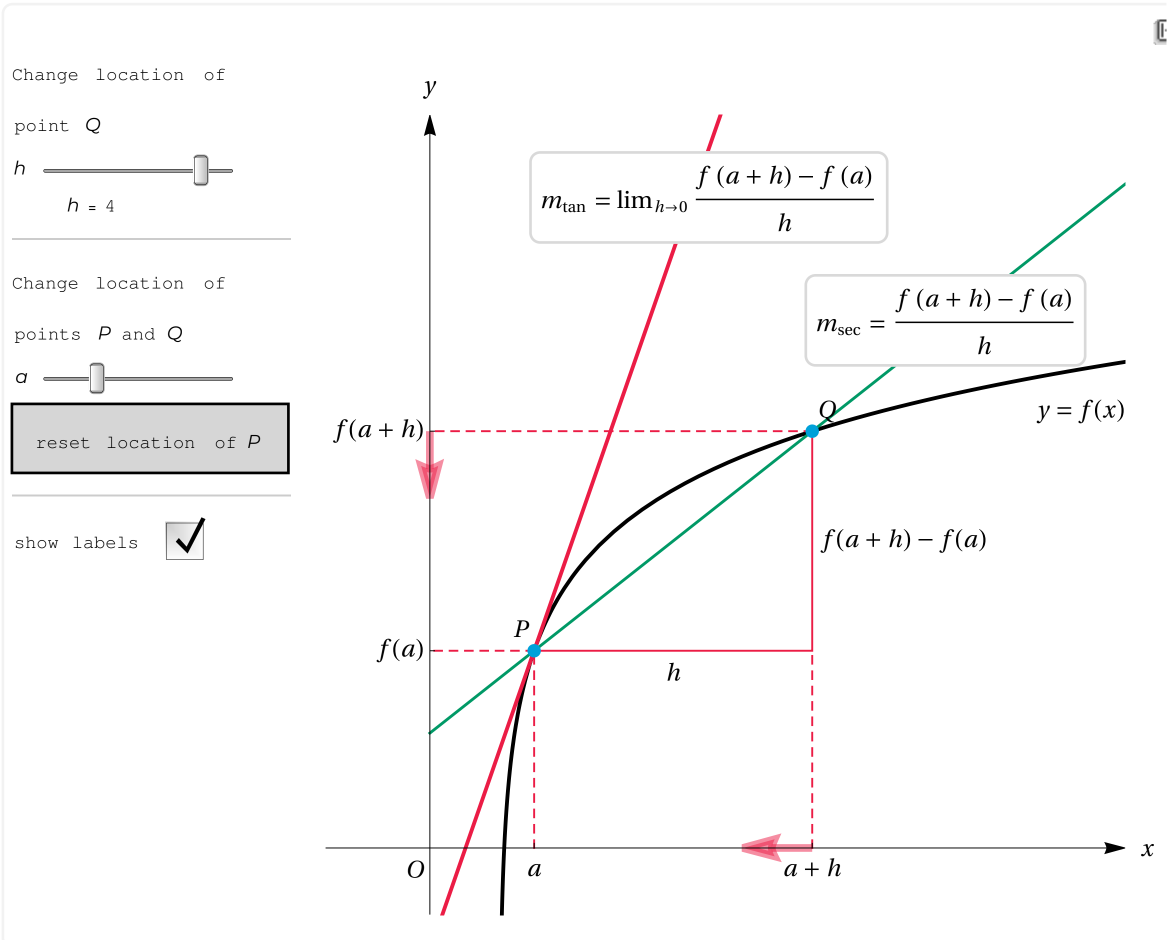


Figure 3.8

The slope of the secant line $\overleftrightarrow{PQ}$ using the new notation is $m_{\sec} = \dfrac{f(a+h) - f(a)}{h}$. As h approaches 0, the variable point Q approaches P and the slopes of the secant lines approach the slope of the tangent line. Therefore, the slope of the tangent line at $(a, f(a))$, which is also the instantaneous rate of change in f at a, is

$$m_{\tan} = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

ALTERNATIVE DEFINITION **Rate of Change and the Slope of the Tangent Line**

The **average rate of change** in f on the interval $[a, a+h]$ is the slope of the corresponding secant line:

$$m_{\text{sec}} = \frac{f(a+h) - f(a)}{h}.$$

The **instantaneous rate of change** in f at a is

$$m_{\tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}, \qquad (2)$$

which is also the **slope of the tangent line** at $(a, f(a))$, provided this limit exists.

EXAMPLE 3 Equation of a tangent line

Find an equation of the line tangent to the graph of $f(x) = x^3 + 4x$ at $(1, 5)$.

SOLUTION »

We let $a = 1$ in definition (2) and first find $f(1+h)$. After expanding and collecting terms, we have

$$f(1+h) = (1+h)^3 + 4(1+h) = h^3 + 3h^2 + 7h + 5.$$

Substituting $f(1+h)$ and $f(1) = 5$, the slope of the tangent line is

$$\begin{aligned}
m_{\tan} &= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} && \text{Definition of } m_{\tan} \\
&= \lim_{h \to 0} \frac{\left(h^3 + 3h^2 + 7h + 5\right) - 5}{h} && \text{Substitute } f(1+h) \text{ and } f(1) = 5. \\
&= \lim_{h \to 0} \frac{h\left(h^2 + 3h + 7\right)}{h} && \text{Simplify.} \\
&= \lim_{h \to 0} \left(h^2 + 3h + 7\right) && \text{Cancel } h, \text{ noting } h \neq 0. \\
&= 7. && \text{Evaluate the limit.}
\end{aligned}$$

Note »

The tangent line has slope $m_{\tan} = 7$ and passes through the point $(1, 5)$ (**Figure 3.9**); its equation is $y - 5 = 7(x - 1)$ or $y = 7x - 2$. We could also say that the instantaneous rate of change in f at $x = 1$ is 7.

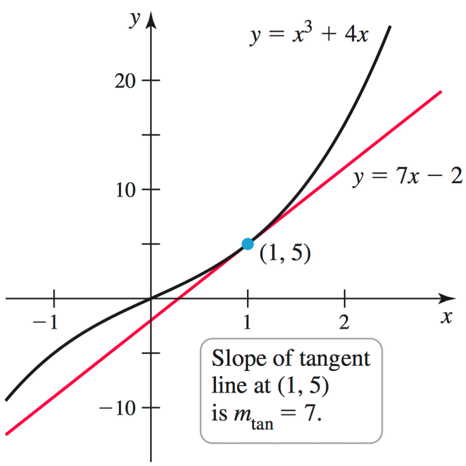


Figure 3.9

Related Exercises 23, 27 ◆

Quick Check 3 Set up the calculation in Example 2 using definition (1) for the slope of the tangent line rather than definition (2). Does the calculation appear more difficult using definition (1)? ◆

Answer »

> Definition (1) requires factoring the numerator or long division in order to cancel $(x - 1)$.

The Derivative »

Computing the slope of the line tangent to the graph of a function f at a given point a gives us the instantaneous rate of change in f at a. This information about the behavior of a function is so important that it has its own name and notation.

DEFINITION **The Derivative of a Function at a Point**

The **derivative of f at a**, denoted $f'(a)$, is given by either of the two following limits, provided the limits exist and a is in the domain of f:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad (1) \qquad \text{or} \qquad f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}. \quad (2)$$

If $f'(a)$ exists, we say that f is **differentiable** at a.

Note »

> The derivative notation $f'(a)$ is read f *prime of* a. A minor modification in the notation for the derivative is necessary when the name of the function changes. For example, given the function $y = g(x)$, its derivative at the point a is $g'(a)$.

The limits that define the derivative of a function at a point are exactly the same limits used to compute the slope of a tangent line and the instantaneous rate of change of a function at a point. When you compute a derivative, remember that you are also finding a rate of change and the slope of a tangent line.

EXAMPLE 4 **Derivatives and tangent lines**

Let $f(x) = \sqrt{2x} + 1$. Compute $f'(2)$, the derivative of f at $x = 2$, and use the result to find an equation of the line tangent to the graph of f at (2, 3).

SOLUTION »

Definition (1) for $f'(a)$ is used here, with $a = 2$.

$$\begin{aligned} f'(2) &= \lim_{x\to 2} \frac{f(x)-f(2)}{x-2} && \text{Definition of the derivative at a point.} \\ &= \lim_{x\to 2} \frac{\sqrt{2x+1}-3}{x-2} && \text{Substitute: } f(x)=\sqrt{2x+1} \text{ and } f(2)=3. \\ &= \lim_{x\to 2} \frac{\sqrt{2x}-2}{x-2}\cdot\frac{\sqrt{2x}+2}{\sqrt{2x}+2} && \text{Multiply numerator and denominator by } \sqrt{2x}+2 \\ &= \lim_{x\to 2} \frac{2x-4}{(x-2)\left(\sqrt{2x}+2\right)} && \text{Simplify.} \\ &= \lim_{x\to 2} \frac{2}{\sqrt{2x}+2} = \frac{1}{2} && \text{Cancel factors } (x \neq 2) \text{ and evaluate the limt.} \end{aligned}$$

Note »

> Definition (2) could also be used to find $f'(2)$ in Example 4.

Because $f'(2) = 1/2$ is the slope of the line tangent to the graph of f at (2, 3), an equation of the tangent line (**Figure 3.10**) is

$$y - 3 = \frac{1}{2}(x-2) \quad \text{or} \quad y = \frac{1}{2}x + 2.$$

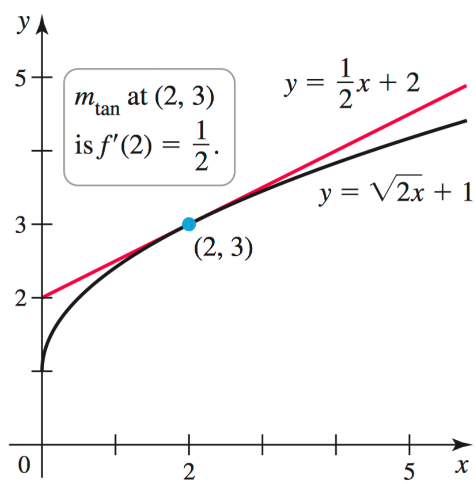


Figure 3.10

Related Exercises 39–40 ◆

Quick Check 4 Verify that the derivative of the function f in Example 4 at the point $(8, 5)$ is $f'(8) = 1/4$. Then find an equation of the line tangent to the graph of f at the point $(8, 5)$. ◆

Answer »

$$y - 5 = \frac{1}{4}(x - 8)$$

Interpreting the Derivative »

The derivative of a function f at a point a measures the instantaneous rate of change in f at a. When the variables associated with a function represent physical quantities, the derivative takes on special meaning. For instance, suppose $T = g(d)$ describes the water temperature T at depth d in the ocean. Then $g'(d)$ measures the instantaneous rate of change in the water temperature at depth d. The following examples illustrate this idea.

EXAMPLE 5 Instantaneous rate of change

Sound intensity I, measured in watts per square meter $(\mathrm{W}/\mathrm{m}^2)$ at a point x meters from a sound source with acoustic power P, is given by $I(x) = \dfrac{P}{4\pi x^2}$. A typical sound system at a rock concert produces an acoustic power of about $P = 3$ W. Compute $I'(3)$ and interpret the result.

SOLUTION »

With $P = 3$, the sound intensity function is $I(x) = \dfrac{3}{4\pi x^2}$. A useful trick is to write I as $I(x) = \dfrac{3}{4\pi} \cdot \dfrac{1}{x^2}$. Using definition (2) of the derivative at a point, with $a = 3$, we have

$$\begin{aligned}
I'(3) &= \lim_{h\to 0} \frac{I(3+h) - I(3)}{h} && \text{Definition of } I'(3). \\
&&& \text{Numerator units: } \mathrm{W}/\mathrm{m}^2\text{; denominator units: m} \\
&= \lim_{h\to 0} \frac{\frac{3}{4\pi}\cdot\frac{1}{(3+h)^2} - \frac{3}{4\pi}\cdot\frac{1}{9}}{h} && \text{Evaluate } I(3+h) \text{ and } I(3). \\
&= \frac{3}{4\pi}\lim_{h\to 0} \frac{\frac{9-(3+h)^2}{9(3+h)^2}}{h} && \text{Factor out } \frac{3}{4\pi} \text{ and simplify remaining fraction.} \\
&= \frac{3}{4\pi}\lim_{h\to 0} \frac{-h(6+h)}{9(3+h)^2}\cdot\frac{1}{h} && \text{Simplify.} \\
&= -\frac{3}{4\pi}\lim_{h\to 0} \frac{6+h}{9(3+h)^2} && \text{Cancel } h;\ h \neq 0. \\
&= -\frac{3}{4\pi}\cdot\frac{6}{81} = -\frac{1}{18\pi}. && \text{Evaluate limit.}
\end{aligned}$$

The units associated with the numerator and denominator help to interpret this result. The change in sound intensity I is measured in W/m^2 while the distance from the sound source is measured in meters (**Figure 3.11**). Therefore, the result $I'(3) = -1/(18\pi)$ means that the sound intensity decreases with an instantaneous rate of change of $1/(18\pi)\ \mathrm{W}/\mathrm{m}^2$ *per meter* at a point 3 m from the sound source. Another useful interpretation is that when the distance from the source increases from 3 m to 4 m, the sound intensity decreases by about $1/(18\pi)\ \mathrm{W/m^2}$.

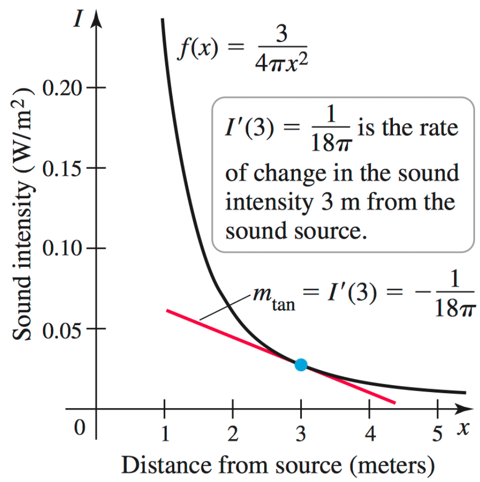


Figure 3.11

Related Exercises 49–50 ◆

EXAMPLE 6 Growth rates of Indian spotted owlets

The Indian spotted owlet is a small owl that is indigenous to Southeast Asia. The body mass $m(t)$ (in grams) of an owl at an age of t weeks is modeled by the graph in **Figure 3.12**. Estimate $m'(2)$ and state the physical meaning of this quantity. (*Source: ZooKeys*, 132, 2011)

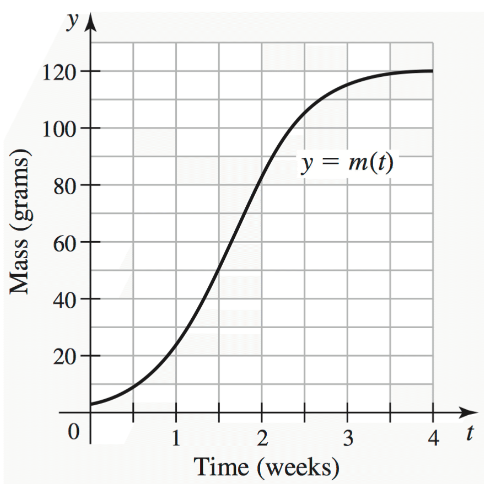


Figure 3.12

SOLUTION »

Recall that the derivative $m'(2)$ equals the slope of the line tangent to the graph of $y = m(t)$ at $t = 2$. One method for estimating $m'(2)$ is to sketch the line tangent to the curve at $t = 2$ and then estimate its slope (**Figure 3.13**). Searching for two convenient points on this line, we see that the tangent line passes through—or close to—the points $(1, 30)$ and $(2.5, 110)$. Therefore,

$$m'(2) \approx \frac{110-30}{2.5-1} = \frac{80\text{ g}}{1.5\text{ weeks}} \approx 53.3\text{ g/week},$$

which means that the owl is growing at 53.3 g/week two weeks after birth.

Note »

> The graphical method used to estimate $m'(2)$ provides only a rough approximation to the value of the derivative: The answer depends on the accuracy with which the tangent line is sketched.

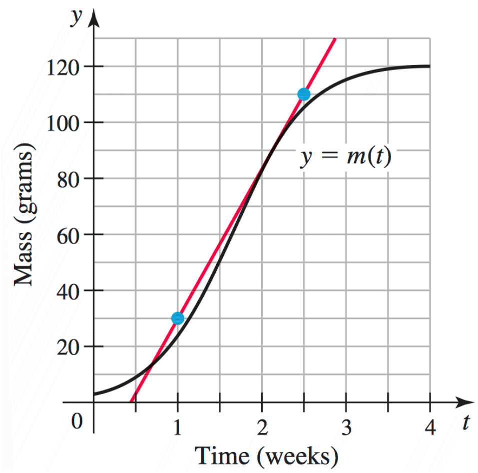


Figure 3.13

Related Exercises 53–54 ◆

Exercises »

Getting Started »

Practice Exercises »

13–14. Velocity functions *A projectile is fired vertically upward into the air; its position (in feet) above the ground after t seconds is given by the function $s(t)$. For the following functions, use limits to determine the instantaneous velocity of the projectile at $t = a$ seconds for the given value of a.*

13. $s(t) = -16t^2 + 100t;\ a = 1$

14. $s(t) = -16t^2 + 128t + 192;\ a = 2$

15–20. Equations of tangent lines by definition (1)

- ***a.*** *Use definition (1) to find the slope of the line tangent to the graph of f at P.*
- ***b.*** *Determine an equation of the tangent line at P.*
- ***c.*** *Plot the graph of f and the tangent line at P.*

15. $f(x) = x^2 - 5;\ P(3, 4)$

T **16.** $f(x) = -3x^2 - 5x + 1;\ P(1, -7)$

17. $f(x) = \dfrac{1}{x};\ P(-1, -1)$

T **18.** $f(x) = \dfrac{4}{x^2}$; $P(-1, 4)$

T **19.** $f(x) = \sqrt{3x+3}$; $P(2, 3)$

T **20.** $f(x) = \dfrac{2}{\sqrt{x}}$; $P(4, 1)$

21–32. Equations of tangent lines by definition (2)

a. *Use definition (2) to find the slope of the line tangent to the graph of f at P.*

b. *Determine an equation of the tangent line at P.*

21. $f(x) = 2x + 1$; $P(0, 1)$

22. $f(x) = -7x$; $P(-1, 7)$

23. $f(x) = 3x^2 - 4x$; $P(1, -1)$

24. $f(x) = 8 - 2x^2$; $P(0, 8)$

25. $f(x) = x^2 - 4$; $P(2, 0)$

26. $f(x) = \dfrac{1}{x}$; $P(1, 1)$

27. $f(x) = x^3$; $P(1, 1)$

28. $f(x) = \dfrac{1}{2x+1}$; $P(0, 1)$

29. $f(x) = \dfrac{1}{3-2x}$; $P\left(-1, \dfrac{1}{5}\right)$

30. $f(x) = \sqrt{x-1}$; $P(2, 1)$

31. $f(x) = \sqrt{x+3}$; $P(1, 2)$

32. $f(x) = \dfrac{x}{x+1}$; $P(-2, 2)$

33–42. Derivatives and tangent lines

a. *For the following functions and values of a, find $f'(a)$.*

b. *Determine an equation of the line tangent to the graph of f at the point $(a, f(a))$ for the given value of a.*

33. $f(x) = 8x$; $a = -3$

34. $f(x) = x^2$; $a = 3$

35. $f(x) = 4x^2 + 2x$; $a = -2$

36. $f(x) = 2\,x^3$; $a = 10$

37. $f(x) = \dfrac{1}{\sqrt{x}}$; $a = \dfrac{1}{4}$

38. $f(x) = \dfrac{1}{x^2}$; $a = 1$

39. $f(x) = \sqrt{2\,x + 1}$; $a = 4$

40. $f(x) = \sqrt{3\,x}$; $a = 12$

41. $f(x) = \dfrac{1}{x + 5}$; $a = 5$

42. $f(x) = \dfrac{1}{3\,x - 1}$; $a = 2$

43–46. Derivative calculations *Evaluate the derivative of the following functions at the given point.*

43. $f(t) = \dfrac{1}{t + 1}$; $a = 1$

44. $f(t) = t - t^2$; $a = 2$

45. $f(s) = 2\sqrt{s} - 1$; $a = 25$

46. $f(r) = \pi\,r^2$; $a = 3$

47. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. For linear functions, the slope of any secant line always equals the slope of any tangent line.

b. The slope of the secant line passing through the points P and Q is less than the slope of the tangent line at P.

c. Consider the graph of the parabola $f(x) = x^2$. For $a > 0$ and $h > 0$, the secant line through $(a,\ f(a))$ and $(a + h,\ f(a + h))$ always has a greater slope than the tangent line at $(a,\ f(a))$.

48–51. *Interpreting the derivative* *Find the derivative of each function at the given point and interpret the physical meaning of this quantity. Include units in your answer.*

48. When a faucet is turned on to fill a bathtub, the volume of water in gallons in the tub after t minutes is $V(t) = 3\,t$. Find $V'(12)$.

49. An object dropped from rest falls $d(t) = 16\,t^2$ feet in t seconds. Find $d'(4)$.

50. The gravitational force of attraction between two masses separated by a distance of x meters is inversely proportional to the square of the distance between them, which implies that the force is described by the function $F(x) = k/x^2$, for some constant k, where $F(x)$ is measured in newtons. Find $F'(10)$, expressing your answer in terms of k.

51. Suppose the speed of a car approaching a stop sign is given by $v(t) = (t - 5)^2$, for $0 \le t \le 5$, where t is measured in seconds and $v(t)$ is measured in meters per second. Find $v'(3)$.

T 52. **Population of Las Vegas** Let $p(t)$ represent the population of the Las Vegas metropolitan area t years after 1970, as shown in the table and figure.

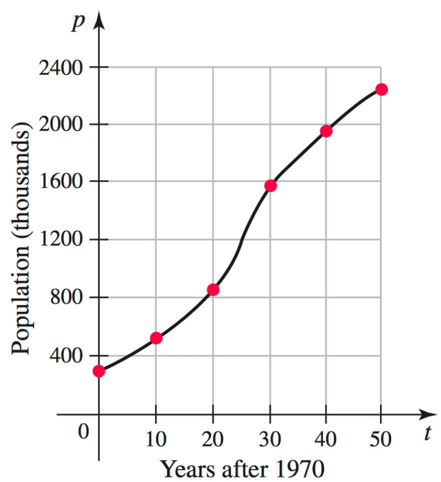
a. Compute the average rate of growth of Las Vegas from 1970 to 1980.

b. Explain why the average rate of growth calculated in part (a) is a good estimate of the instantaneous rate of growth of Las Vegas in 1975.

c. Compute the average rate of growth of Las Vegas from 1970 to 2000. Is the average rate of growth an overestimate or underestimate of the instantaneous rate of growth of Las Vegas in 2000? Approximate the growth rate in 2000.

Year	1970	1980	1990	2000	2010	2020
t	0	10	20	30	40	50
$p(t)$	305,000	528,000	853,000	1,563,000	1,951,000	2,223,000

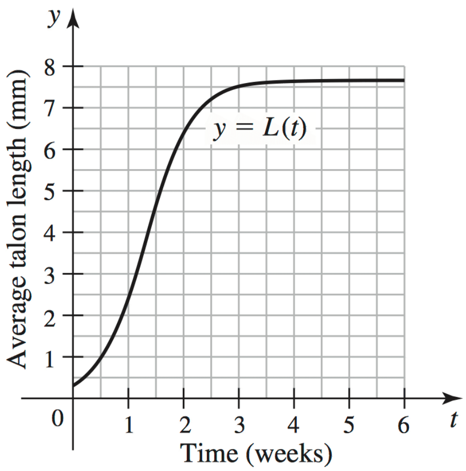
Source: U.S. Bureau of Census



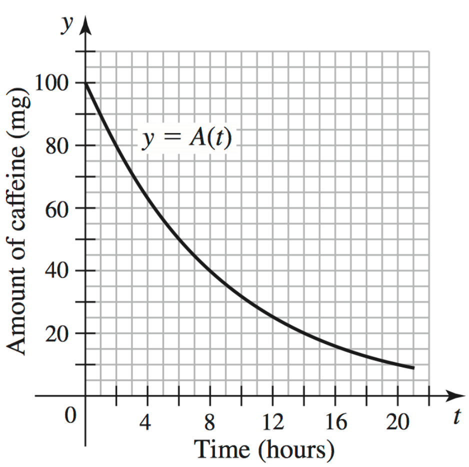
53. **Owlet talons** Let $L(t)$ equal the average length (in mm) of the middle talon on an Indian spotted owlet that is t weeks old, as shown in the figure.

a. Estimate $L'(1.5)$ and state the physical meaning of this quantity.

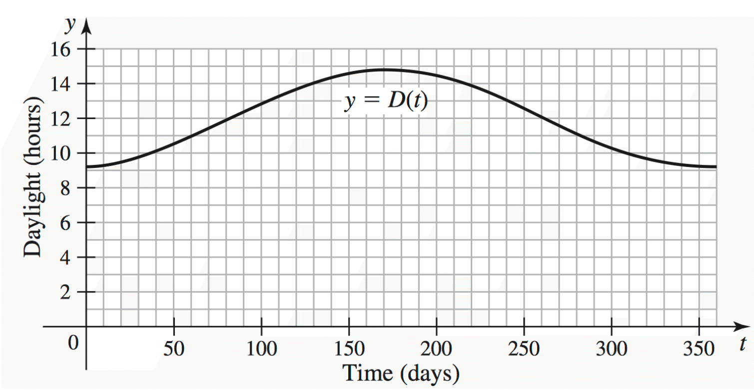
b. Estimate the value of $L'(a)$, for $a \ge 4$. What does this tell you about the talon lengths on these birds? (*Source*: *ZooKeys* 132, 2011)



54. **Caffeine levels** Let $A(t)$ be the amount of caffeine (in mg) in the bloodstream t hours after a cup of coffee has been consumed (see figure). Estimate the values of $A'(7)$ and $A'(15)$, rounding answers to the nearest whole number. Include units in your answers and interpret the physical meaning of these values.



55. Let $D(t)$ equal the number of daylight hours at a latitude of $40°$ N, t days after January 1. Assuming $D(t)$ is approximated by a continuous function (see figure), estimate the values of $D'(60)$ and $D'(170)$. Include units in your answers and interpret your results.



Explorations and Challenges »

56–61. Find the function *The following limits represent the slope of a curve $y = f(x)$ at the point $(a, f(a))$. Determine a possible function f and number a; then calculate the limit.*

56. $\lim_{x\to 1} \dfrac{3x^2 + 4x - 7}{x - 1}$

57. $\lim_{x\to 2} \dfrac{5x^2 - 20}{x - 2}$

58. $\lim_{x\to 2} \dfrac{\frac{1}{x+1} - \frac{1}{3}}{x - 2}$

59. $\lim_{h\to 0} \dfrac{(2+h)^4 - 16}{h}$

60. $\lim_{h\to 0} \dfrac{\sqrt{2+h} - \sqrt{2}}{h}$

61. $\lim_{h\to 0} \dfrac{|-1+h| - 1}{h}$

62–65. Approximating derivatives *Assuming the limit exists, the definition of the derivative* $f'(a) = \lim_{h\to 0} \dfrac{f(a+h) - f(a)}{h}$ *implies that if h is small, then an approximation to $f'(a)$ is given by*

$$f'(a) \approx \frac{f(a+h) - f(a)}{h}.$$

If $h > 0$, then this approximation is called a ***forward difference quotient****; if $h < 0$, it is a* ***backward difference quotient****. As shown in the following exercises, these formulas are used to approximate f' at a point when f is a complicated function or when f is represented by a set of data points.*

62. Let $f(x) = \sqrt{x}$.

a. Find the exact value of $f'(4)$.

b. Show that $f'(4) \approx \dfrac{f(4+h) - f(4)}{h} = \dfrac{\sqrt{4+h} - 2}{h}$.

c. Complete columns 2 and 5 of the following table and describe how $\dfrac{\sqrt{4+h} - 2}{h}$ behaves as h approaches 0.

h	$\frac{\sqrt{4+h}-2}{h}$	Error	h	$\frac{\sqrt{4+h}-2}{h}$	Error
0.1			−0.1		
0.01			−0.01		
0.001			−0.001		
0.0001			−0.0001		

d. The accuracy of an approximation is measured by

$$\text{error} = |\text{exact value} - \text{approximate value}|.$$

Use the exact value of $f'(4)$ in part (a) to complete columns 3 and 6 in the table. Describe the behavior of the errors as h approaches 0.

63. Another way to approximate derivatives is to use the **centered difference quotient:**

$$f'(a) \approx \frac{f(a+h) - f(a-h)}{2h}.$$

Again, consider $f(x) = \sqrt{x}$.

a. Graph f near the point (4, 2) and let $h = 1/2$ in the centered difference quotient. Draw the line whose slope is computed by the centered difference quotient and explain why the centered difference quotient approximates $f'(4)$.

b. Use the centered difference quotient to approximate $f'(4)$ by completing the table.

h	Approximation	Error
0.1		
0.01		
0.001		

c. Explain why it is not necessary to use negative values of h in the table of part (b).

d. Compare the accuracy of the derivative estimates in part (b) with those found in Exercise 62.

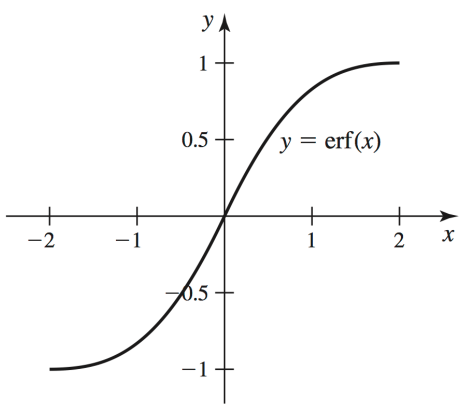
64. The following table gives the distance $f(t)$ fallen by a smoke jumper t seconds after she opens her chute.

a. Use the forward difference quotient with $h = 0.5$ to estimate the velocity of the smoke jumper at $t = 2$ seconds.

b. Repeat part (a) using the centered difference quotient.

t (seconds)	**$f(t)$ (feet)**
0	0
0.5	4
1.0	15
1.5	33
2.0	55
2.5	81
3.0	109
3.5	138
4.0	169

65. The *error function* (denoted $\mathrm{erf}(x)$) is an important function in statistics because it is related to the normal distribution. Its graph is shown in the figure, and values of $\mathrm{erf}(x)$ at several points are shown in the table.



x	**$\mathrm{erf}(x)$**	**x**	**$\mathrm{erf}(x)$**
0.75	0.711156	1.05	0.862436
0.8	0.742101	1.1	0.880205
0.85	0.770668	1.15	0.896124
0.9	0.796908	1.2	0.910314
0.95	0.820891	1.25	0.922900
1.0	0.842701	1.3	0.934008

a. Use forward and centered difference quotients to find approximations to the derivative $\mathrm{erf}'(1)$.

b. Given that $\mathrm{erf}'(1) \approx 0.4151075$, compute the error in the approximations in part (a).