2.7 Precise Definitions of Limits

The limit definitions already encountered in this chapter are adequate for most elementary limits. However some of the terminology used, such as *sufficiently close* and *arbitrarily large*, needs clarification. The goal of this section is to give limits a solid mathematical foundation by transforming the previous limit definitions into precise mathematical statements.

Moving Toward a Precise Definition »

Note »

Assume the function f is defined for all x near a, except possibly at a. Recall that $\lim_{x \to a} f(x) = L$ means that $f(x)$ is arbitrarily close to L for all x sufficiently close (but not equal) to a. This limit definition is made precise by observing that the distance between $f(x)$ and L is $|f(x) - L|$ and that the distance between x and a is $|x - a|$.

Note »

Therefore, we write $\lim_{x \to a} f(x) = L$ if we can make $|f(x) - L|$ arbitrarily small for any x, distinct from a, with $|x - a|$ sufficiently small. For instance, if we want $|f(x) - L|$ to be less than 0.1, then we must find a number $\delta > 0$ such that

$$|f(x) - L| < 0.1 \quad \text{whenever} \quad |x - a| < \delta \quad \text{and} \quad x \neq a.$$

If, instead, we want $|f(x) - L|$ to be less than 0.001, then we must find *another* number $\delta > 0$ such that

$$|f(x) - L| < 0.001 \quad \text{whenever} \quad 0 < |x - a| < \delta.$$

Note »

> The two conditions $|x - a| < \delta$ and $x \neq a$ are written concisely as $0 < |x - a| < \delta$.

For the limit to exist, it must be true that for *any* $\varepsilon > 0$, we can always find a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - a| < \delta.$$

Note »

> The Greek letters δ (delta) and ε (epsilon) represent small positive numbers when discussing limits.

EXAMPLE 1 **Determining values of δ from a graph**

Figure 2.54 shows the graph of a linear function f with $\lim_{x \to 3} f(x) = 5$. For each value of $\varepsilon > 0$, determine a value of $\delta > 0$ satisfying the statement

$$|f(x) - 5| < \varepsilon \quad \text{whenever} \quad 0 < |x - 3| < \delta.$$

a. $\varepsilon = 1$

b. $\varepsilon = \frac{1}{2}$

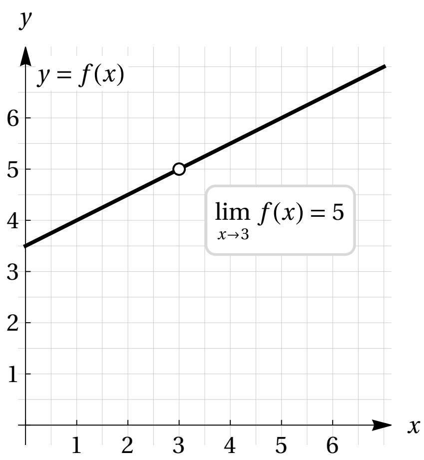


Figure 2.54

SOLUTION »

a. With $\varepsilon = 1$, we want $f(x)$ to be less than 1 unit from 5, which means $f(x)$ is between 4 and 6. To determine a corresponding value of δ, draw the horizontal lines $y = 4$ and $y = 6$ (**Figure 2.55**). Then sketch vertical lines passing through the points where the horizontal lines and the graph of f intersect. We see that the vertical lines intersect the x-axis at $x = 1$ and $x = 5$. Note that $f(x)$ is less than 1 unit from 5 on the y-axis if x is within 2 units of 3 on the x-axis. So, for $\varepsilon = 1$, we let $\delta = 2$ or any smaller positive value.

Note »

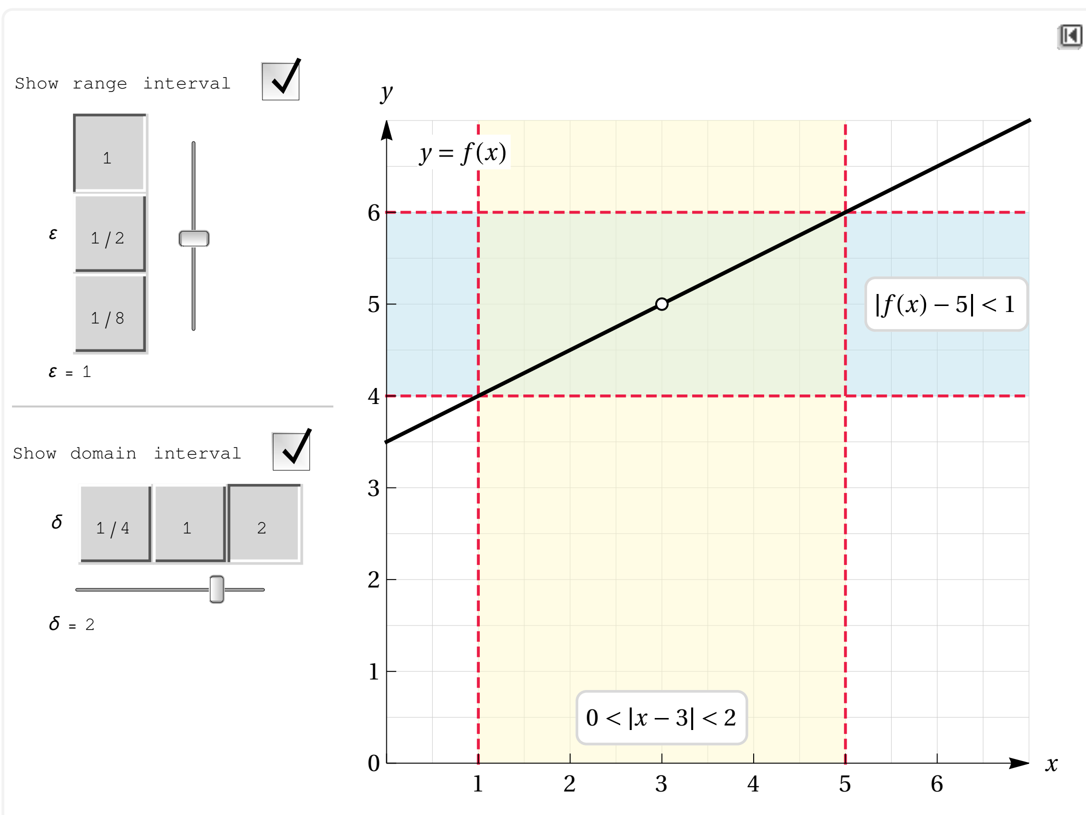


Figure 2.55

b. With $\varepsilon = \frac{1}{2}$, we want $f(x)$ to lie within a half-unit of 5 or, equivalently, $f(x)$ must lie between 4.5 and 5.5. Proceeding as in part (a), we see that $f(x)$ is within a half-unit of 5 on the y-axis if x is less than 1 unit from 3 (**Figure 2.56**). So for $\varepsilon = \frac{1}{2}$, we let $\delta = 1$ or any smaller positive number.

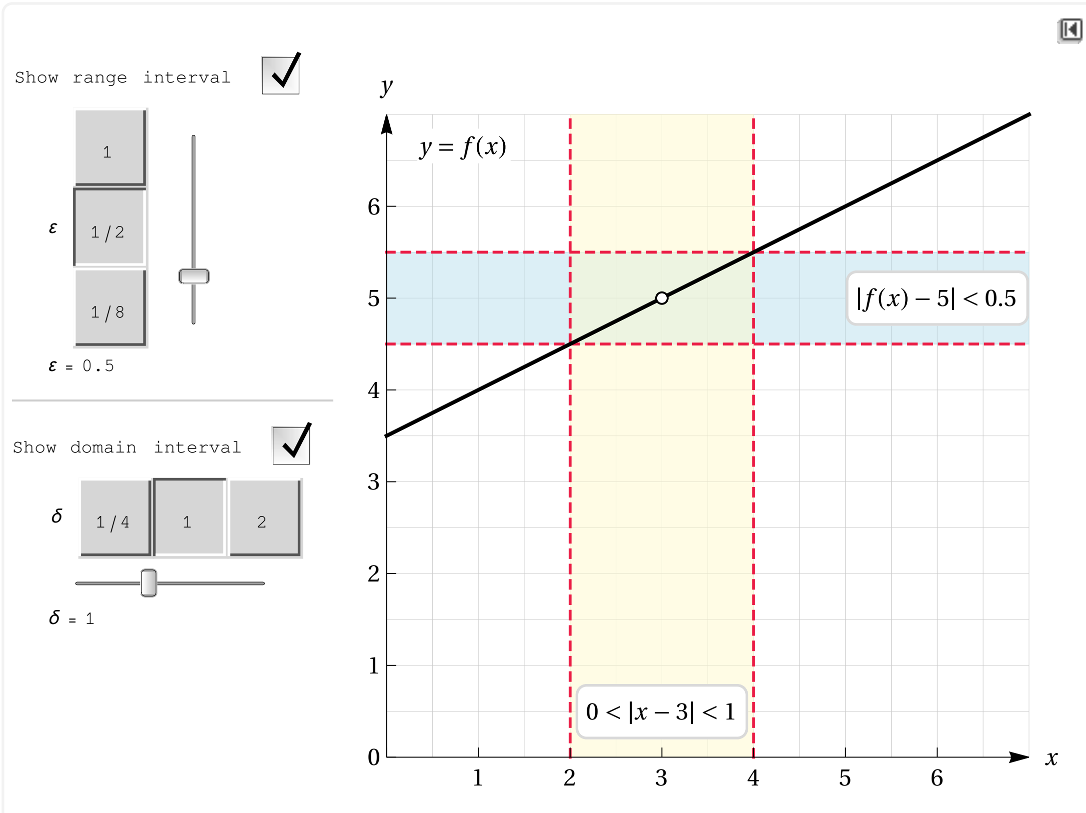


Figure 2.56

Related Exercises 9–10 ◆

The idea of a limit, as illustrated in Example 1, may be described in terms of a contest between two people named Epp and Del. First, Epp picks a particular number $\varepsilon > 0$; then, he challenges Del to find a corresponding value of $\delta > 0$ such that

$$|f(x) - 5| < \varepsilon \quad \text{whenever} \quad 0 < |x - 3| < \delta. \qquad (1)$$

To illustrate, suppose Epp chooses $\varepsilon = 1$. From Example 1, we know that Del will satisfy (1) by choosing $0 < \delta \le 2$. If Epp chooses $\varepsilon = \frac{1}{2}$, then (by Example 1) Del responds by letting $0 < \delta \le 1$. If Epp lets $\varepsilon = \frac{1}{8}$, then Del chooses $0 < \delta \le \frac{1}{4}$ (**Figure 2.57**). In fact, there is a pattern: For *any* $\varepsilon > 0$ that Epp chooses, no matter how small, Del with satisfy (1) by choosing a positive value of δ satisfying $0 < \delta \le 2\varepsilon$. Del has discovered a mathematical relationship: If $0 < \delta \le 2\varepsilon$ and $0 < |x - 3| < \delta$, then $|f(x) - 5| < \varepsilon$, for *any* $\varepsilon > 0$. This conversation illustrates the general procedure for proving that $\lim_{x \to a} f(x) = L$.

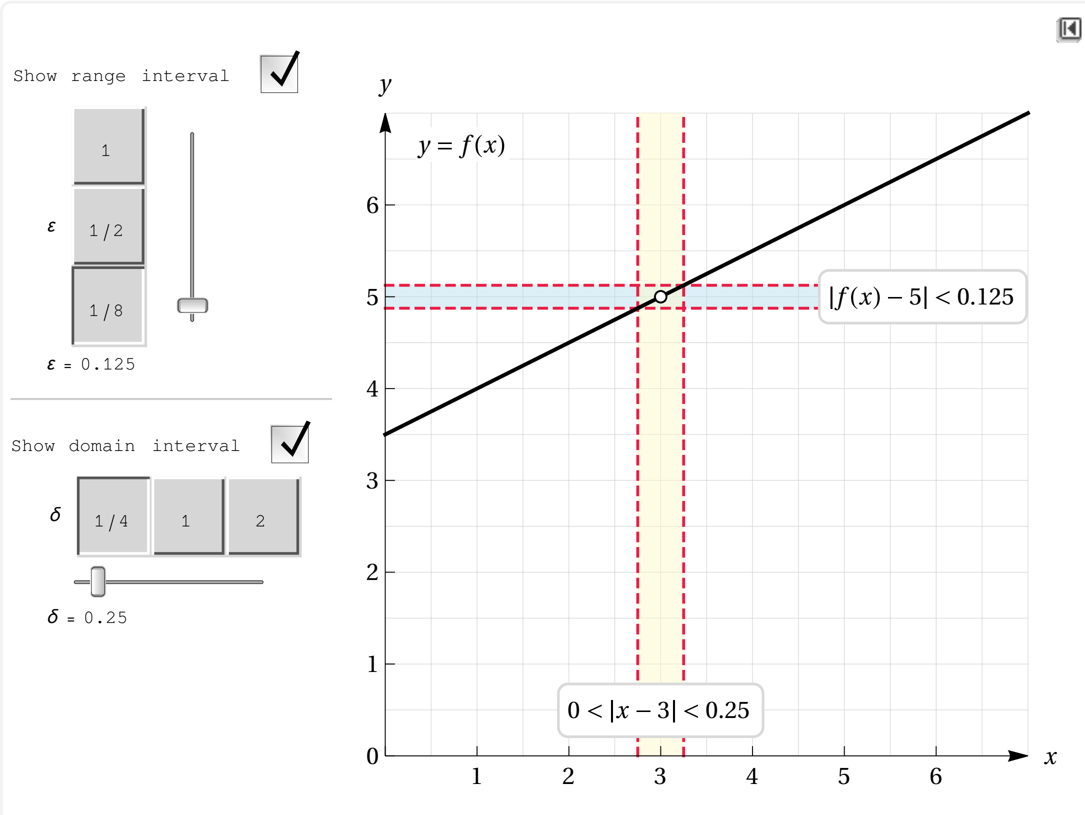


Figure 2.57

Quick Check 1 In Example 1, find a positive number δ satisfying the statement $|f(x) - 5| < \frac{1}{100}$ whenever $0 < |x - 3| < \delta$. ◆

Answer »

$\delta \leq \frac{1}{50}$

A Precise Definition »

Example 1 dealt with a linear function, but it points the way to a precise definition of a limit for any function. As shown in **Figure 2.58**, $\lim_{x \to a} f(x) = L$ means that for *any* positive number ε, there is another positive number δ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - a| < \delta.$$

In all limit proofs, the goal is to find a relationship between ε and δ that gives an admissible value of δ, in terms of ε only. This relationship must work for any positive value of ε.

Note »

The value of δ in the precise definition of a limit depends only on ε.

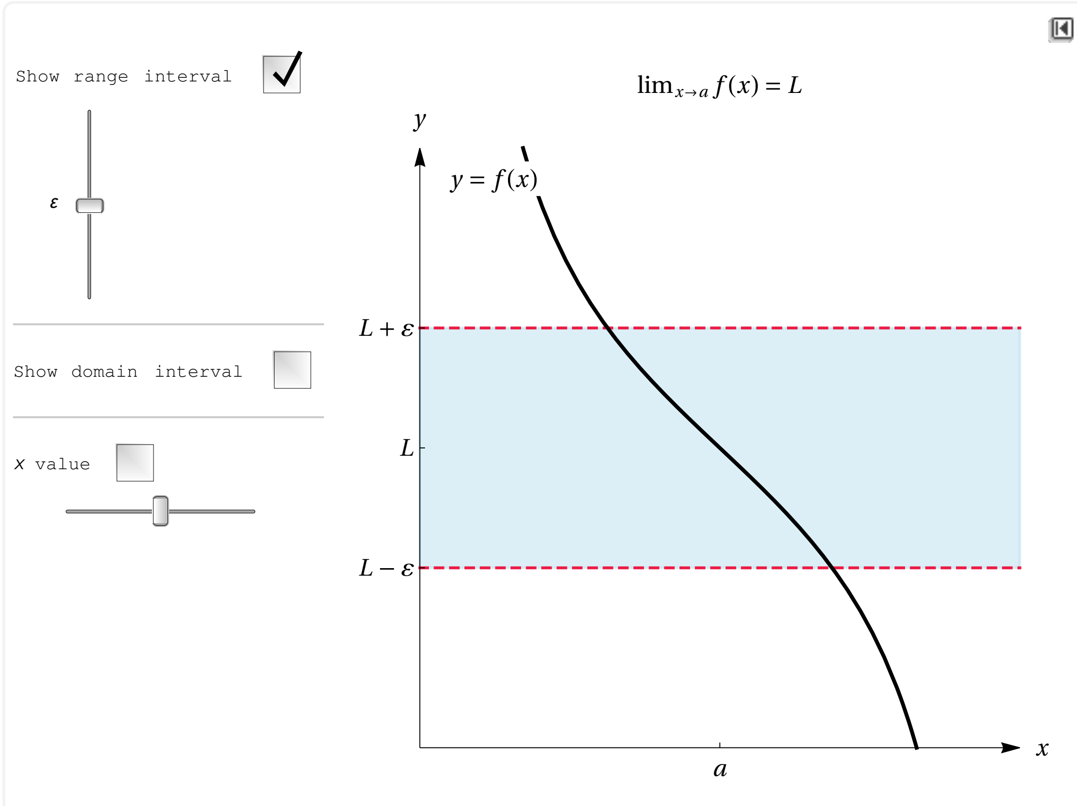


Figure 2.58

DEFINITION **Limit of a Function**

Assume that $f(x)$ is defined for all x in some open interval containing a, except possibly at a. We say that the **limit of $f(x)$ as x approaches a is L**, written

$$\lim_{x\to a} f(x) = L,$$

if for *any* number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - a| < \delta.$$

Note »

EXAMPLE 2 Finding δ for a given ε using a graphing utility

Let $f(x) = x^3 - 6x^2 + 12x - 5$ and demonstrate that $\lim_{x\to 2} f(x) = 3$ as follows. For the given values of ε, use a graphing utility to find a value of $\delta > 0$ such that

$$|f(x) - 3| < \varepsilon \quad \text{whenever} \quad 0 < |x - 2| < \delta.$$

a. $\varepsilon = 1$

b. $\varepsilon = \frac{1}{2}$

SOLUTION »

a. The condition $|f(x) - 3| < \varepsilon = 1$ implies that $f(x)$ lies between 2 and 4. Using a graphing utility, we graph f and the lines $y = 2$ and $y = 4$ (**Figure 2.59**). These lines intersect the graph of f at $x = 1$ and at $x = 3$. We now sketch the vertical lines $x = 1$ and $x = 3$ and observe that $f(x)$ is within 1 unit of 3 whenever x is within 1 unit of 2 on the x-axis (Figure 2.59). Therefore, with $\varepsilon = 1$, we choose δ such that $0 < \delta \le 1$.

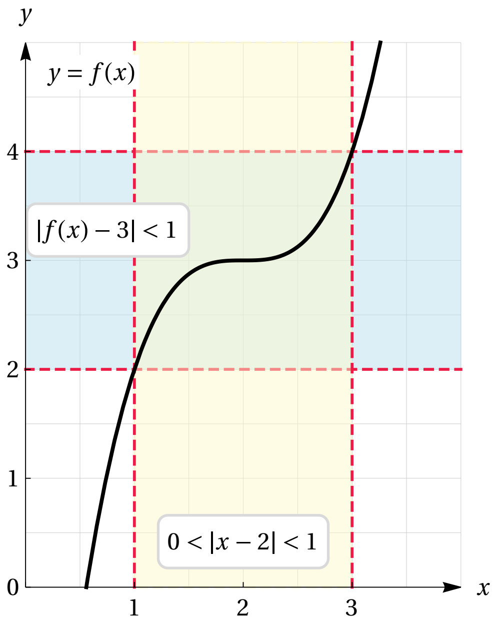


Figure 2.59

b. The condition $|f(x) - 3| < \varepsilon = \frac{1}{2}$ implies that $f(x)$ lies between 2.5 and 3.5 on the y-axis. We now find that the lines $y = 2.5$ and $y = 3.5$ intersect the graph of f at $x \approx 1.21$ and $x \approx 2.79$ (**Figure 2.60**). Observe that if x is less than 0.79 units from 2 on the x-axis, then $f(x)$ is less than a half-unit from 3 on the y-axis. Therefore with $\varepsilon = \frac{1}{2}$ we let $0 < \delta \le 0.79$.

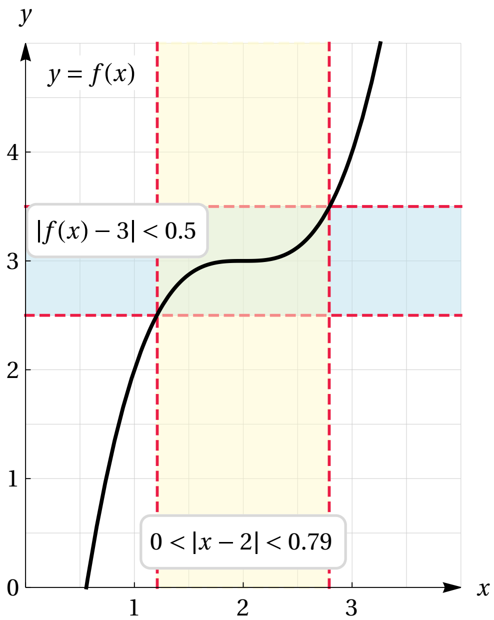


Figure 2.60

This procedure could be repeated for smaller and smaller values of $\varepsilon > 0$. For each value of ε, there exists a corresponding value of δ, proving that the limit exists.

Related Exercise 13 ◆

Quick Check 2 For the function f given in Example 2, estimate a value of $\delta > 0$ satisfying $|f(x) - 3| < 0.25$ whenever $0 < |x - 2| < \delta$. ◆

Answer »

$\delta \le 0.62$

The inequality $0 < |x - a| < \delta$ means that x lies between $a - \delta$ and $a + \delta$ with $x \ne a$. We say that the interval $(a - \delta, a + \delta)$ is **symmetric about *a*** because a is the midpoint of the interval. Symmetric intervals are convenient, but Example 3 demonstrates that we don't always get symmetric intervals without a bit of extra work.

EXAMPLE 3 Finding a symmetric interval

Figure 2.61 shows the graph of g with $\lim_{x \to 2} g(x) = 3$. For each value of ε, find the corresponding values of $\delta > 0$ that satisfy the condition

$$|g(x) - 3| < \varepsilon \quad \text{whenever} \quad 0 < |x - 2| < \delta.$$

a. $\varepsilon = 2$

b. $\varepsilon = 1$

c. For any given value of ε, make a conjecture for the corresponding values of δ that satisfy the limit condition.

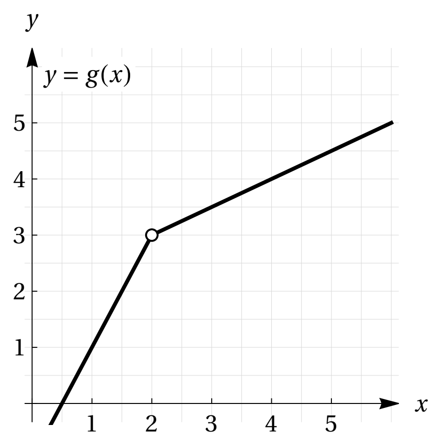


Figure 2.61

SOLUTION »

a. With $\varepsilon = 2$, we need a value of $\delta > 0$ such that $g(x)$ is within 2 units of 3, which means $g(x)$ is between 1 and 5, whenever x is less than δ units from 2. The horizontal lines $y = 1$ and $y = 5$ intersect the graph of g at $x = 1$ and $x = 6$. Therefore, $|g(x) - 3| < 2$ if x lies in the interval $(1, 6)$ with $x \ne 2$ (**Figure 2.62**). However, we want x to lie in an interval that is *symmetric* about 2. We can guarantee that $|g(x) - 3| < 2$ in an interval symmetric about 2 only if x is less than 1 unit away from 2, on either side of 2. Therefore, with $\varepsilon = 2$ we take $\delta = 1$ or any smaller positive number.

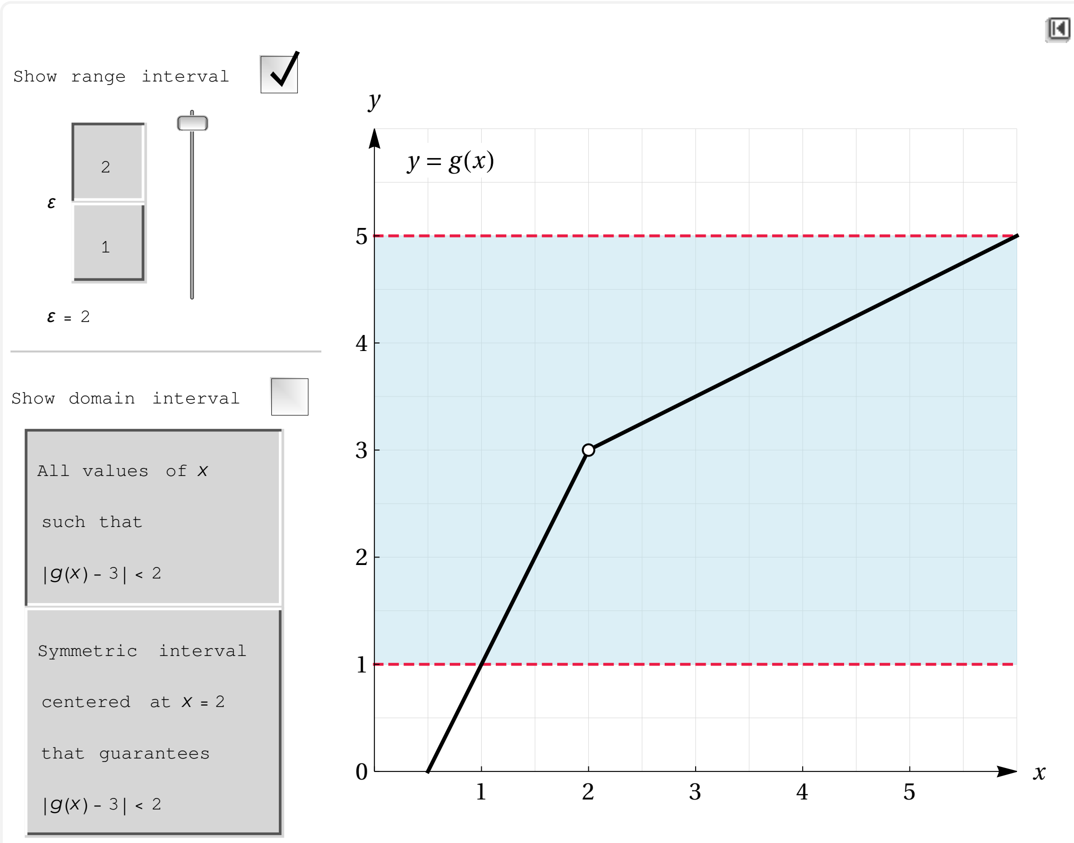


Figure 2.62

b. With $\varepsilon = 1$, $g(x)$ must lie between 2 and 4 (**Figure 2.63**). This implies that x must be within a half-unit to the left of 2 and within 2 units to the right of 2. Therefore $|g(x) - 3| < 1$ provided x lies in the interval $(1.5, 4)$. To obtain a symmetric interval about 2, we take $\delta = \frac{1}{2}$ or any smaller positive number. Then we are still guaranteed that $|g(x) - 3| < 1$ when $0 < |x - 2| < \frac{1}{2}$.

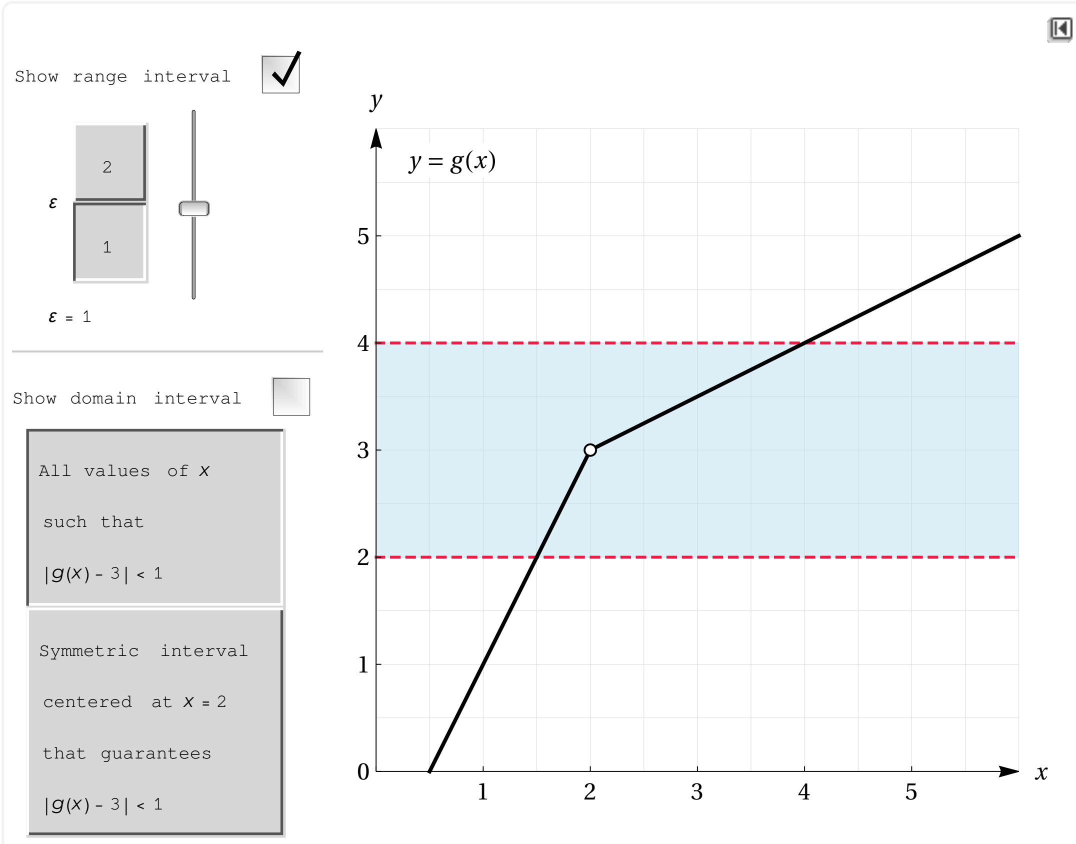


Figure 2.63

c. From parts (a) and (b), it appears that if we choose $\delta \le \frac{\varepsilon}{2}$, the limit condition is satisfied for any $\varepsilon > 0$.

Related Exercises 15–16 ◆

In Examples 2 and 3, we showed that a limit exists by discovering a relationship between ε and δ that satisfies the limit condition. We now generalize this procedure.

Limit Proofs »

We use the following two-step process to prove that $\lim_{x \to a} f(x) = L$.

Steps for proving that $\lim_{x \to a} f(x) = L$

1. **Find δ.** Let ε be an arbitrary positive number. Use the inequality $|f(x) - L| < \varepsilon$ to find a condition of the form $|x - a| < \delta$, where δ depends only on the value of ε.

2. **Write a proof.** For any $\varepsilon > 0$, assume $0 < |x - a| < \delta$ and use the relationship between ε and δ found in Step 1 to prove that $|f(x) - L| < \varepsilon$.

Note »

The first step of the limit-proving process is the preliminary work of finding a candidate for δ. The second step verifies that the δ found in the first step actually works.

EXAMPLE 4 Limit of a linear function

Prove that $\lim_{x \to 4} (4x - 15) = 1$ using the precise definition of a limit.

SOLUTION »

EXAMPLE 5 Limit of a quadratic function

Prove that $\lim_{x \to 5} x^2 = 25$ using the precise definition of a limit.

SOLUTION »

Step 1: *Find* δ. Given $\varepsilon > 0$, our task is to find an expression for $\delta > 0$ that depends only on ε, such that $\left|x^2 - 25\right| < \varepsilon$ whenever $0 < |x - 5| < \delta$. We begin by factoring $\left|x^2 - 25\right|$:

$$\begin{aligned} \left|x^2 - 25\right| &= |(x+5)(x-5)| && \text{Factor.} \\ &= |x+5|\,|x-5|. && |ab| = |a|\,|b| \end{aligned}$$

Because the value of $\delta > 0$ in the inequality $0 < |x - 5| < \delta$ typically represents a small positive number, let's assume $\delta \le 1$ so that $|x - 5| < 1$, which implies that $-1 < x - 5 < 1$ or $4 < x < 6$. It follows that x is positive, $|x + 5| < 11$, and

$$\left|x^2 - 25\right| = |x+5|\,|x-5| < 11\,|x-5|.$$

Using this inequality, we have $\left|x^2 - 25\right| < \varepsilon$, provided $11\,|x - 5| < \varepsilon$ or $|x - 5| < \varepsilon/11$. Note that two restrictions have been placed on $|x - 5|$:

$$|x-5| < 1 \quad \text{and} \quad |x-5| < \frac{\varepsilon}{11}.$$

To ensure that both these inequalities are satisfied, let $\delta = \min\{1, \varepsilon/11\}$ so that δ equals the smaller of 1 and $\varepsilon/11$.

Note »

> The minimum value of a and b is denoted $\min\{a, b\}$. If $x = \min\{a, b\}$, then x is the smaller of a and b. If $a = b$, then x equals the common value of a and b. In either case, $x \le a$ and $x \le b$.

Step 2: *Write a proof*. Let $\varepsilon > 0$ be given and assume $0 < |x - 5| < \delta$, where $\delta = \min\{1, \varepsilon/11\}$. By factoring $x^2 - 25$, we have

$$\left|x^2 - 25\right| = |x+5|\,|x-5|.$$

Because $0 < |x - 5| < \delta$ and $\delta \le \varepsilon/11$, we have $|x - 5| < \varepsilon/11$. It is also the case that $|x - 5| < 1$ because $\delta \le 1$, which implies that $-1 < x - 5 < 1$ or $4 < x < 6$. Therefore, $|x + 5| < 11$ and

$$\left|x^2 - 25\right| = |x+5|\,|x-5| < 11\left(\frac{\varepsilon}{11}\right) = \varepsilon.$$

We have shown that for any $\varepsilon > 0$, $\left|x^2 - 25\right| < \varepsilon$ whenever $0 < |x - 5| < \delta$, provided $0 < \delta = \min\{1, \varepsilon/11\}$. Therefore, $\lim_{x \to 5} x^2 = 25$.

Related Exercises 27, 29 ◆

Justifying Limit Laws »

The precise definition of a limit is used to prove the limit laws in Theorem 2.3. Essential in several of these proofs is the **triangle inequality**, which states that

$$|x + y| \leq |x| + |y|, \text{ for all real numbers } x \text{ and } y.$$

EXAMPLE 6 Proof of Limit Law 1

Prove that if $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist, then

$$\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x).$$

SOLUTION »

Assume that $\varepsilon > 0$ is given. Let $\lim_{x \to a} f(x) = L$, which implies there exists a $\delta_1 > 0$ such that

$$|f(x) - L| < \frac{\varepsilon}{2} \quad \text{whenever} \quad 0 < |x - a| < \delta_1.$$

Note »

> Because $\lim_{x \to a} f(x)$ exists, if there exists a $\delta > 0$ for any given $\varepsilon > 0$, then there also exists a $\delta > 0$ for any given $\frac{\varepsilon}{2}$.

Similarly, let $\lim_{x \to a} g(x) = M$, which implies there exists a $\delta_2 > 0$ such that

$$|g(x) - M| < \frac{\varepsilon}{2} \quad \text{whenever} \quad 0 < |x - a| < \delta_2.$$

Let $\delta = \min\{\delta_1, \delta_2\}$ and suppose $0 < |x - a| < \delta$. Because $\delta \leq \delta_1$, it follows that $0 < |x - a| < \delta_1$ and $|f(x) - L| < \varepsilon/2$. Similarly, because $\delta \leq \delta_2$, it follows that $0 < |x - a| < \delta_2$ and $|g(x) - M| < \varepsilon/2$. Therefore,

$$\begin{aligned} |(f(x) + g(x)) - (L + M)| &= |(f(x) - L) + (g(x) - M)| && \text{Rearrange terms.} \\ &\leq |f(x) - L| + |g(x) - M| && \text{Triangle inequality.} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

We have shown that given any $\varepsilon > 0$, if $0 < |x - a| < \delta$ then $|(f(x) + g(x)) - (L + M)| < \varepsilon$, which implies that $\lim_{x \to a} [f(x) + g(x)] = L + M = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$.

Note »

> Proofs of other limit laws are outlined in Exercises 43–44.

Related Exercises 43–44 ◆

Infinite Limits »

In Section 2.4, we stated that $\lim_{x \to a} f(x) = \infty$ if $f(x)$ grows *arbitrarily large* as x approaches a. More precisely, this means that for any positive number N (no matter how large), $f(x)$ is larger than N if x is sufficiently close to a

but not equal to a.

Note »

Notice that for infinite limits, N plays the role that ε plays for ordinary limits. It sets a tolerance or bound for the function values $f(x)$.

DEFINITION **Two-Sided Infinite Limit**

The **infinite limit** $\lim_{x \to a} f(x) = \infty$ means that for any positive number N there exists a corresponding $\delta > 0$ such that

$$f(x) > N \quad \text{whenever} \quad 0 < |x - a| < \delta.$$

Note »

Precise definitions for $\lim_{x \to a} f(x) = -\infty$, $\lim_{x \to a^+} f(x) = -\infty$, $\lim_{x \to a^+} f(x) = \infty$, $\lim_{x \to a^-} f(x) = -\infty$, and $\lim_{x \to a^-} f(x) = \infty$ are given in Exercises 57–63.

As shown in **Figure 2.64**, to prove that $\lim_{x \to a} f(x) = \infty$, we let N represent *any* positive number. Then we find a value of $\delta > 0$, depending only on N, such that

$$f(x) > N \quad \text{whenever} \quad 0 < |x - a| < \delta\,.$$

This process is similar to the two-step process for finite limits.

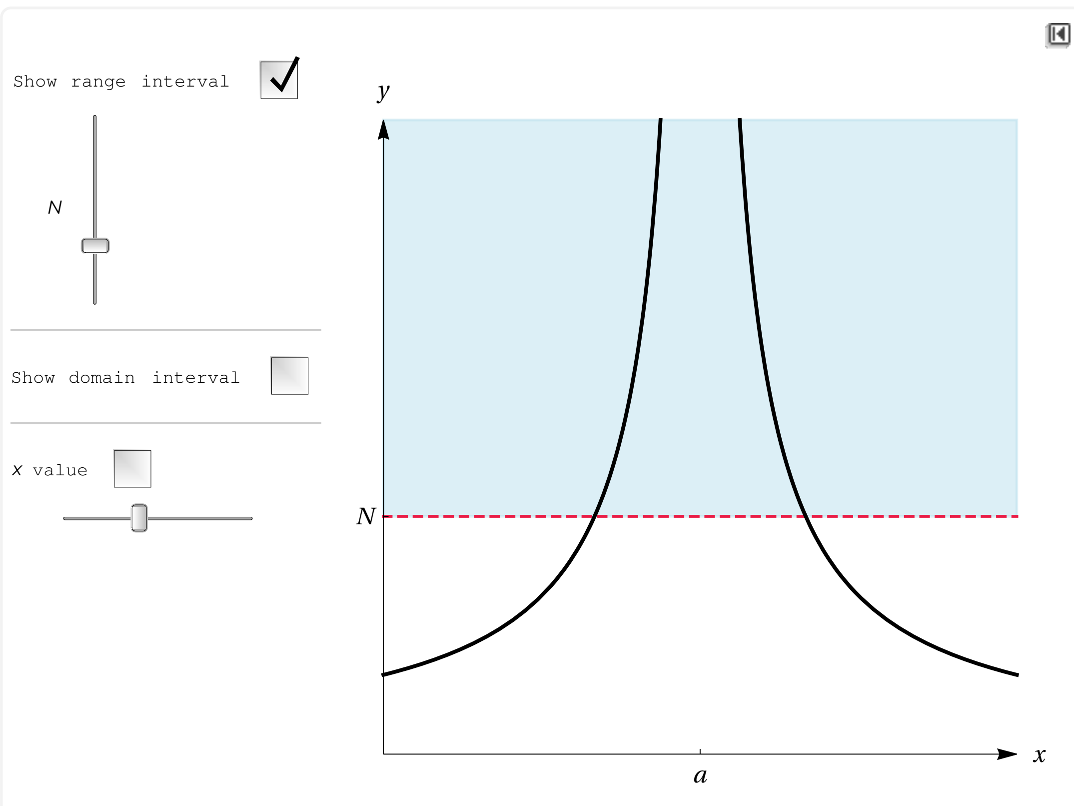


Figure 2.64

Steps for proving that $\lim_{x \to a} f(x) = \infty$

1. Find δ. Let N be an arbitrary positive number. Use the statement $f(x) > N$ to find an inequality of the form $|x - a| < \delta$, where δ depends only on N.

2. Write a proof. For any $N > 0$, assume $0 < |x - a| < \delta$ and use the relationship between N and δ found in Step 1 to prove that $f(x) > N$.

EXAMPLE 7 An Infinite Limit Proof

Let $f(x) = \dfrac{1}{(x-2)^2}$. Prove that $\lim_{x \to 2} f(x) = \infty$.

SOLUTION »

Step 1: *Find $\delta > 0$.* Assuming $N > 0$, we use the inequality $\dfrac{1}{(x-2)^2} > N$ to find δ, where δ depends only on N. Taking reciprocals of this inequality, it follows that

$$(x-2)^2 < \frac{1}{N}$$

$$|x - 2| < \frac{1}{\sqrt{N}}. \quad \text{Take the square root of both sides.}$$

Note »

Recall that $\sqrt{x^2} = |x|$.

The inequality $|x - 2| < \dfrac{1}{\sqrt{N}}$ has the form $|x - 2| < \delta$ if we let $\delta = \dfrac{1}{\sqrt{N}}$. We now write a proof based on this relationship between δ and N.

Step 2: *Write a proof.* Suppose $N > 0$ is given. Let $\delta = \dfrac{1}{\sqrt{N}}$ and assume $0 < |x - 2| < \delta = \dfrac{1}{\sqrt{N}}$. Squaring both sides of the inequality $|x - 2| < \dfrac{1}{\sqrt{N}}$ and taking reciprocals, we have

$$(x-2)^2 < \frac{1}{N} \quad \text{Square both sides.}$$

$$\frac{1}{(x-2)^2} > N. \quad \text{Take reciprocals of both sides.}$$

We see that for any positive N, if $0 < |x - 2| < \delta = \dfrac{1}{\sqrt{N}}$, then $f(x) = \dfrac{1}{(x-2)^2} > N$. It follows that $\lim_{x \to 2} \dfrac{1}{(x-2)^2} = \infty$. Note that because $\delta = \dfrac{1}{\sqrt{N}}$, δ decreases as N increases.

Related Exercises 45–46 ◆

Quick Check 3 In Example 7, if N is increased by a factor of 100, how must δ change? ◆

Answer »

δ must decrease by a factor of $\sqrt{100} = 10$ (at least).

Limits at Infinity »

Precise definitions can also be written for the limits at infinity $\lim_{x\to\infty} f(x) = L$ and $\lim_{x\to-\infty} f(x) = L$. For discussion and examples, see Exercises 64-65.

Exercises »

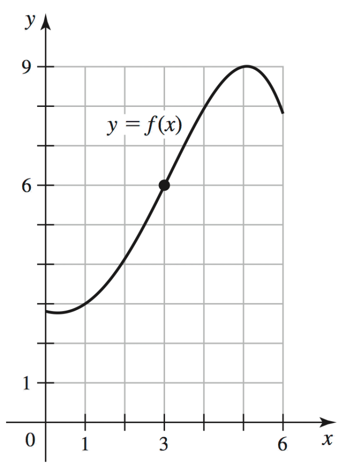
Getting Started »

Practice Exercises »

11. **Determining values of δ from a graph** The function f in the figure satisfies $\lim_{x\to 3} f(x) = 6$. Determine the largest value of $\delta > 0$ satisfying each statement.

a. If $0 < |x - 3| < \delta$, then $|f(x) - 6| < 3$.

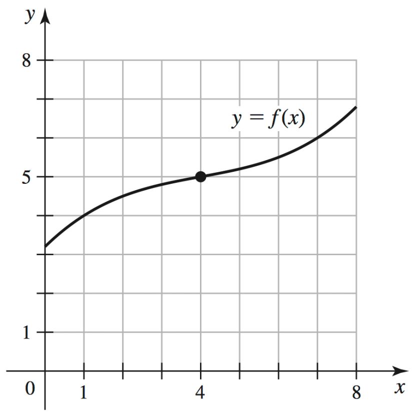
b. If $0 < |x - 3| < \delta$, then $|f(x) - 6| < 1$.



12. **Determining values of δ from a graph** The function f in the figure satisfies $\lim_{x\to 4} f(x) = 5$. Determine the largest value of $\delta > 0$ satisfying each statement.

a. If $0 < |x - 4| < \delta$, then $|f(x) - 5| < 1$.

b. If $0 < |x - 4| < \delta$, then $|f(x) - 5| < 0.5$.



T 13. Finding δ for a given ε using a graph Let $f(x) = x^3 + 3$ and note that $\lim_{x\to 0} f(x) = 3$. For each value of ε, use a graphing utility to find all values of $\delta > 0$ such that $|f(x) - 3| < \varepsilon$ whenever $0 < |x - 0| < \delta$. Sketch graphs illustrating your work.

a. $\varepsilon = 1$

b. $\varepsilon = 0.5$

T 14. Finding δ for a given ε using a graph Let $g(x) = 2x^3 - 12x^2 + 26x + 4$ and note that $\lim_{x\to 2} g(x) = 24$. For each value of ε, use a graphing utility to find all values of $\delta > 0$ such that $|g(x) - 24| < \varepsilon$ whenever $0 < |x - 2| < \delta$. Sketch graphs illustrating your work.

a. $\varepsilon = 1$

b. $\varepsilon = 0.5$

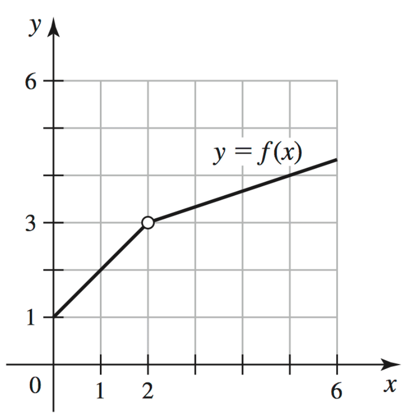
15. Finding a symmetric interval The function f in the figure satisfies $\lim_{x\to 2} f(x) = 3$. For each value of ε, find all values of $\delta > 0$ such that

$$|f(x) - 3| < \varepsilon \quad \text{whenever} \quad 0 < |x - 2| < \delta. \tag{2}$$

a. $\varepsilon = 1$

b. $\varepsilon = \frac{1}{2}$

c. For any $\varepsilon > 0$, make a conjecture about the corresponding values of δ satisfying (2).



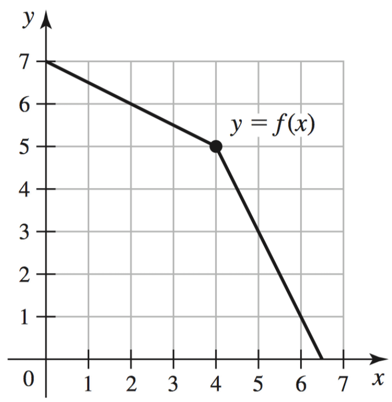
16. Finding a symmetric interval The function f in the figure satisfies $\lim_{x \to 4} f(x) = 5$. For each value of ε, find all values of $\delta > 0$ such that

$$|f(x) - 5| < \varepsilon \quad \text{whenever} \quad 0 < |x - 4| < \delta. \tag{3}$$

a. $\varepsilon = 2$

b. $\varepsilon = 1$

c. For any $\varepsilon > 0$, make a conjecture about the corresponding values of δ satisfying (3).



T **17. Finding a symmetric interval** Let $f(x) = \dfrac{2x^2 - 2}{x - 1}$ and note that $\lim_{x \to 1} f(x) = 4$. For each value of ε, use a graphing utility to find all values of $\delta > 0$ such that $|f(x) - 4| < \varepsilon$ whenever $0 < |x - 1| < \delta$.

a. $\varepsilon = 2$

b. $\varepsilon = 1$

c. For any $\varepsilon > 0$, make a conjecture about the value of δ that satisfies the preceding inequality.

T **18. Finding a symmetric interval** Let $f(x) = \begin{cases} \dfrac{1}{3}x + 1 & \text{if } x \le 3 \\ \dfrac{1}{2}x + \dfrac{1}{2} & \text{if } x > 3 \end{cases}$ and note that $\lim_{x \to 3} f(x) = 2$. For each value of ε, use a graphing utility to find all values of $\delta > 0$ such that $|f(x) - 2| < \varepsilon$ whenever $0 < |x - 3| < \delta$.

a. $\varepsilon = \dfrac{1}{2}$

b. $\varepsilon = \dfrac{1}{4}$

c. For any $\varepsilon > 0$, make a conjecture about the value of δ that satisfies the preceding inequality.

19–42. Limit proofs *Use the precise definition of a limit to prove the following limits. Specify a relationship between ε and δ that guarantees the limit exists.*

19. $\lim_{x \to 1} (8x + 5) = 13$

20. $\lim_{x \to 3} (-2x + 8) = 2$

21. $\lim_{x\to 4} \dfrac{x^2 - 16}{x - 4} = 8$ (*Hint:* Factor and simplify.)

22. $\lim_{x\to 3} \dfrac{x^2 - 7x + 12}{x - 3} = -1$

23. $\lim_{x\to 0} |x| = 0$

24. $\lim_{x\to 0} |5x| = 0$

25. $\lim_{x\to 7} f(x) = 9$, where $f(x) = \begin{cases} 3x - 12 & \text{if } x \le 7 \\ x + 2 & \text{if } x > 7 \end{cases}$

26. $\lim_{x\to 5} f(x) = 4$, where $f(x) = \begin{cases} 2x - 6 & \text{if } x \le 5 \\ -4x + 24 & \text{if } x > 5 \end{cases}$

27. $\lim_{x\to 0} x^2 = 0$ (*Hint:* Use the identity $\sqrt{x^2} = |x|$.)

28. $\lim_{x\to 3} (x-3)^2 = 0$ (*Hint:* Use the identity $\sqrt{x^2} = |x|$.)

29. $\lim_{x\to 2} \left(x^2 + 3x\right) = 10$

30. $\lim_{x\to 4} \left(2x^2 - 4x + 1\right) = 17$

31. $\lim_{x\to -3} |2x| = 6$ (*Hint:* Use the inequality $||a| - |b|| \le |a - b|$, which holds for all constants a and b (see Exercise 74).)

32. $\lim_{x\to 25} \sqrt{x} = 5$ (*Hint*: The factorization $x - 25 = \left(\sqrt{x} - 5\right)\left(\sqrt{x} + 5\right)$ implies that $\sqrt{x} - 5 = \dfrac{x - 25}{\sqrt{x} + 5}$.)

33. $\lim_{x\to 3} \dfrac{1}{x} = \dfrac{1}{3}$ (*Hint:* As $x \to 3$, eventually the distance between x and 3 is less than 1. Start by assuming $|x - 3| < 1$ and show $\dfrac{1}{|x|} < \dfrac{1}{2}$.)

34. $\lim_{x\to 4} \dfrac{x - 4}{\sqrt{x} - 2} = 4$ (*Hint:* Multiply the numerator and denominator by $\sqrt{x} + 2$.)

35. $\lim_{x\to 1/10} \dfrac{1}{x} = 10$ (*Hint:* To find δ, you need to bound x away from 0. So let $\left|x - \dfrac{1}{10}\right| < \dfrac{1}{20}$.)

36. $\lim_{x\to 0} x \sin \dfrac{1}{x} = 0$

37. $\lim_{x\to 0} \left(x^2 + x^4\right) = 0$ (*Hint*: You may use the fact that if $|x| < c$, then $x^2 < c^2$.)

38. $\lim_{x\to a} b = b$ for any constants a and b

39. $\lim_{x\to a} (m\,x + b) = m\,a + b$ for any constants a, b, and m

40. $\lim_{x\to 3} x^3 = 27$

41. $\lim_{x\to 1} x^4 = 1$

42. $\lim_{x\to 5} \frac{1}{x^2} = \frac{1}{25}$

43. **Proof of Limit Law 2** Suppose $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. Prove that $\lim_{x\to a} (f(x) - g(x)) = L - M$.

44. **Proof of Limit Law 3** Suppose $\lim_{x\to a} f(x) = L$. Prove that $\lim_{x\to a} (c\,f(x)) = c\,L$, where c is a constant.

45–48. Limit proofs for infinite limits *Use the precise definition of infinite limits to prove the following limits.*

45. $\lim_{x\to 4} \frac{1}{(x-4)^2} = \infty$

46. $\lim_{x\to -1} \frac{1}{(x+1)^4} = \infty$

47. $\lim_{x\to 0} \left(\frac{1}{x^2} + 1\right) = \infty$

48. $\lim_{x\to 0} \left(\frac{1}{x^4} - \sin x\right) = \infty$

49. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample. Assume a and L are finite numbers and assume $\lim_{x\to a} f(x) = L$.

- **a.** For a given $\varepsilon > 0$, there is one value of $\delta > 0$ for which $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$.
- **b.** The limit $\lim_{x\to a} f(x) = L$ means that given an arbitrary $\delta > 0$, we can always find an $\varepsilon > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$.
- **c.** The limit $\lim_{x\to a} f(x) = L$ means that for any arbitrary $\varepsilon > 0$, we can always find a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$.
- **d.** If $|x - a| < \delta$, then $a - \delta < x < a + \delta$.

50. **Finding δ algebraically** Let $f(x) = x^2 - 2\,x + 3$.

- **a** For $\varepsilon = 0.25$, find the largest value of $\delta > 0$ satisfying the statement

$$|f(x) - 2| < \varepsilon \quad \text{whenever} \quad 0 < |x - 1| < \delta.$$

- **b.** Verify that $\lim_{x\to 1} f(x) = 2$ as follows. For any $\varepsilon > 0$, find the largest value of $\delta > 0$ satisfying the statement

$$|f(x) - 2| < \varepsilon \quad \text{whenever} \quad 0 < |x - 1| < \delta.$$

51–55. Precise definitions for left- and right-sided limits *Use the following definitions.*

Assume f exists for all x near a with $x > a$. We say that ***the limit of $f(x)$ as x approaches a from the right of a is L*** *and write $\lim_{x\to a^+} f(x) = L$, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$|f(x) - L| < \varepsilon \quad \textit{whenever} \quad 0 < x - a < \delta.$$

Assume f exists for all x near a with $x < a$. We say that ***the limit of $f(x)$ as x approaches a from the left of a is L*** *and write $\lim_{x\to a^-} f(x) = L$, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$|f(x) - L| < \varepsilon \quad \textit{whenever} \quad 0 < a - x < \delta.$$

51. Comparing definitions Why is the last inequality in the definition of $\lim_{x\to a} f(x) = L$, namely, $0 < |x - a| < \delta$, replaced with $0 < x - a < \delta$ in the definition of $\lim_{x\to a^+} f(x) = L$?

52. Comparing definitions Why is the last inequality in the definition of $\lim_{x\to a} f(x) = L$, namely, $0 < |x - a| < \delta$, replaced with $0 < a - x < \delta$ in the definition of $\lim_{x\to a^-} f(x) = L$?

53. One-sided limit proofs Prove the following limits for

$$f(x) = \begin{cases} 3x - 4 & \text{if } x < 0 \\ 2x - 4 & \text{if } x \ge 0. \end{cases}$$

a. $\lim_{x\to 0^+} f(x) = -4$

b. $\lim_{x\to 0^-} f(x) = -4$

c. $\lim_{x\to 0} f(x) = -4$

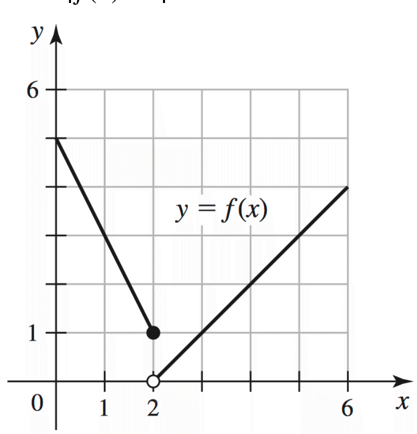
54. Determining values of δ from a graph The function f in the figure satisfies $\lim_{x\to 2^+} f(x) = 0$ and $\lim_{x\to 2^-} f(x) = 1$. Determine all values of $\delta > 0$ that satisfy each statement.

a. $|f(x) - 0| < 2$ whenever $0 < x - 2 < \delta$

b. $|f(x) - 0| < 1$ whenever $0 < x - 2 < \delta$

c. $|f(x) - 1| < 2$ whenever $0 < 2 - x < \delta$

d. $|f(x) - 1| < 1$ whenever $0 < 2 - x < \delta$



55. One-sided limit proof Prove that $\lim_{x\to 0^+} \sqrt{x} = 0$.

Explorations and Challenges »

56. The relationship between one-sided and two-sided limits Prove the following statements to establish the fact that $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^-} f(x) = L$ and $\lim_{x\to a^+} f(x) = L$.

a. If $\lim_{x\to a^-} f(x) = L$ and $\lim_{x\to a^+} f(x) = L$, then $\lim_{x\to a} f(x) = L$.

b. If $\lim_{x\to a} f(x) = L$, then $\lim_{x\to a^-} f(x) = L$ and $\lim_{x\to a^+} f(x) = L$.

57. Definition of one-sided infinite limits We write $\lim_{x\to a^+} f(x) = -\infty$ if for any negative number N, there exists $\delta > 0$ such that

$$f(x) < N \quad \text{whenever} \quad 0 < x - a < \delta.$$

a. Write an analogous formal definition for $\lim_{x\to a^+} f(x) = \infty$.

b. Write an analogous formal definition for $\lim_{x\to a^-} f(x) = -\infty$.

c. Write an analogous formal definition for $\lim_{x\to a^-} f(x) = \infty$.

58–59. One-sided infinite limits *Use the definitions given in Exercise 57 to prove the following infinite limits.*

58. $\lim_{x\to 1^+} \dfrac{1}{1-x} = -\infty$

59. $\lim_{x\to 1^-} \dfrac{1}{1-x} = \infty$

60–61. Definition of an infinite limit *We write* $\lim_{x\to a} f(x) = -\infty$ *if for any negative number* M, *there exists a* $\delta > 0$ *such that*

$$f(x) < M \quad \textit{whenever} \quad 0 < |x - a| < \delta.$$

Use this definition to prove the following statements.

60. $\lim_{x\to 1} \dfrac{-2}{(x-1)^2} = -\infty$

61. $\lim_{x\to -2} \dfrac{-10}{(x+2)^4} = -\infty$

62. Suppose $\lim_{x\to a} f(x) = \infty$. Prove that $\lim_{x\to a} (f(x) + c) = \infty$ for any constant c.

63. Suppose $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$. Prove that $\lim_{x\to a} (f(x) + g(x)) = \infty$.

64–65. Definition of a limit at infinity *The limit at infinity* $\lim_{x\to \infty} f(x) = L$ *means that for any* $\varepsilon > 0$, *there exists* $N > 0$ *such that*

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad x > N.$$

Use this definition to prove the following statements.

64. $\lim\limits_{x\to\infty} \dfrac{10}{x} = 0$

65. $\lim\limits_{x\to\infty} \dfrac{2x+1}{x} = 2$

66–67. Definition of infinite limits at infinity *We write* $\lim\limits_{x\to\infty} f(x) = \infty$ *if for any positive number* M, *there is a corresponding* $N > 0$ *such that*

$$f(x) > M \quad \text{whenever} \quad x > N.$$

Use this definition to prove the following statements.

66. $\lim\limits_{x\to\infty} \dfrac{x}{100} = \infty$

67. $\lim\limits_{x\to\infty} \dfrac{x^2+x}{x} = \infty$

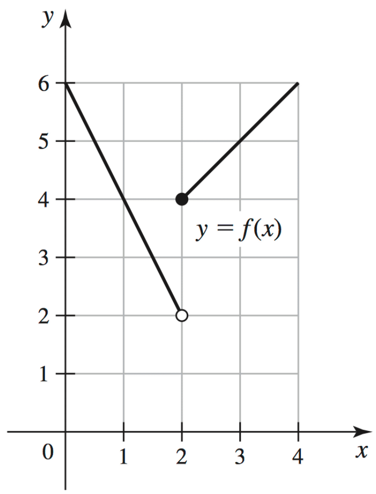
68. Proof of the Squeeze Theorem Assume the functions f, g, and h satisfy the inequality $f(x) \le g(x) \le h(x)$ for all values of x near a, except possibly at a. Prove that if $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} h(x) = L$, then $\lim\limits_{x\to a} g(x) = L$.

69. Limit proof Suppose f is defined for all x near a, except possibly at a. Assume that for any integer $N > 0$ there is another integer $M > 0$ such that $|f(x) - L| < 1/N$ whenever $|x - a| < 1/M$. Prove that $\lim\limits_{x\to a} f(x) = L$ using the precise definition of a limit.

70–72. Proving that $\lim\limits_{x\to a} f(x) \ne L$ *Use the following definition for the* nonexistence *of a limit. Assume f is defined for all x near a, except possibly at a. We write* $\lim\limits_{x\to a} f(x) \ne L$ *if for some* $\varepsilon > 0$, *there is no value of* $\delta > 0$ *satisfying the condition*

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - a| < \delta.$$

70. For the following function, note that $\lim\limits_{x\to 2} f(x) \ne 3$. Find all values of $\varepsilon > 0$ for which the preceding condition for nonexistence is satisfied.



71. Prove that $\lim_{x\to 0} \frac{|x|}{x}$ does not exist.

72. Let

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational}. \end{cases}$$

Prove that $\lim_{x\to a} f(x)$ does not exist for any value of a. (*Hint:* Assume $\lim_{x\to a} f(x) = L$ for some values of a and L, and let $\varepsilon = \frac{1}{2}$.)

73. **A continuity proof** Suppose f is continuous at a and defined for all x near a. If $f(a) > 0$, show that there is a positive number $\delta > 0$ for which $f(x) > 0$ for all x in $(a - \delta, a + \delta)$. (In other words, f is positive for all x in some interval containing a.)

74. Show that $||a| - |b|| \le |a - b|$ for all constants a and b. (*Hint*: Write $|a| = |(a - b) + b|$ and apply the triangle inequality to $|(a - b) + b|$.)