2.6 Continuity

The graphs of many functions encountered in this text contain no holes, jumps, or breaks. For example, if $L = f(t)$ represents the length of a fish t years after it is hatched, then the length of the fish changes gradually as t increases. Consequently, the graph of $L = f(t)$ contains no breaks (**Figure 2.42a**). Some functions, however, do contain abrupt changes in their values. Consider a parking meter that accepts only quarters and each quarter buys 15 minutes of parking. Letting $c(t)$ be the cost (in dollars) of parking for t minutes, the graph of c has breaks at integer multiples of 15 minutes (**Figure 2.42b**).

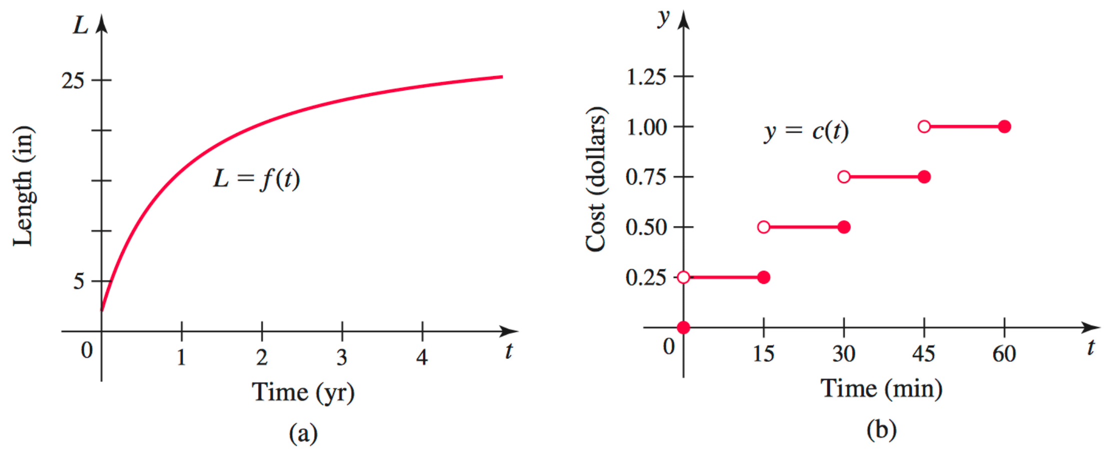


Figure 2.42

Informally, we say that a function f is *continuous* at a if the graph of f does not have a hole or break at a (that is, if the graph near a can be drawn without lifting the pencil).

Continuity at a Point »

This informal description of continuity is sufficient for determining the continuity of simple functions, but it is not precise enough to deal with more complicated functions such as

$$h(x) = \begin{cases} x \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

It is difficult to determine whether the graph of h has a break at $x = 0$ because it oscillates rapidly as x approaches 0 (**Figure 2.43**). We need a better definition.

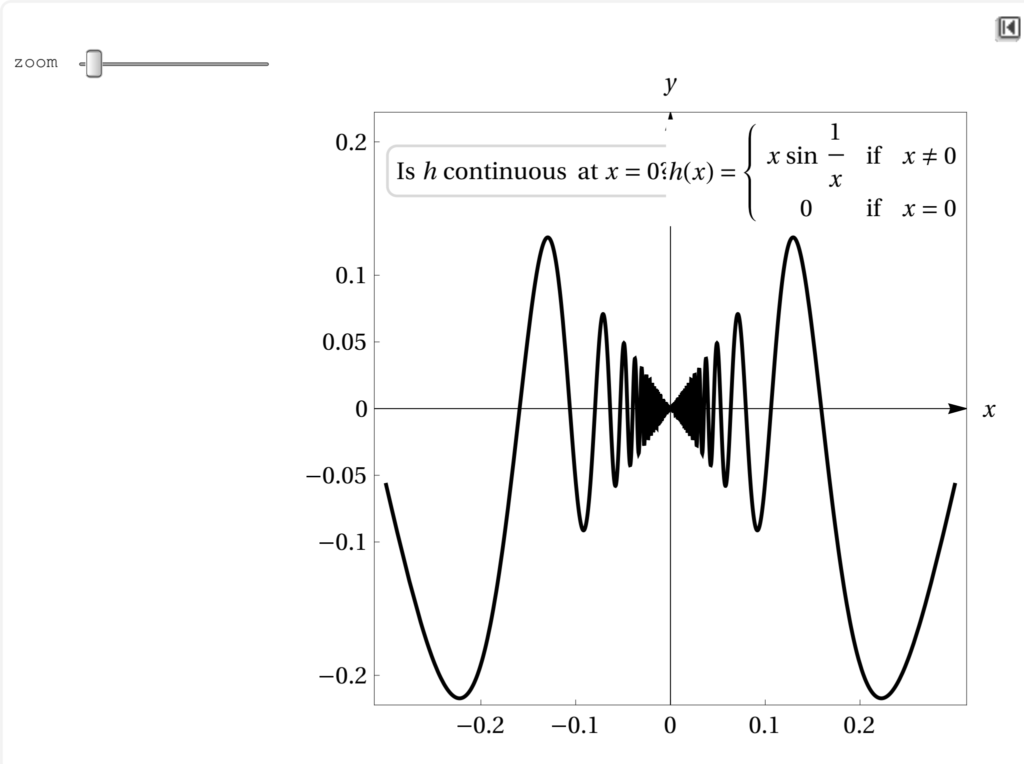


Figure 2.43

DEFINITION **Continuity at a Point**

A function f is **continuous** at a if $\lim_{x \to a} f(x) = f(a)$.

There is more to this definition than first appears. If $\lim_{x \to a} f(x) = f(a)$, then $f(a)$ and $\lim_{x \to a} f(x)$ must both exist, and they must be equal. The following checklist is helpful in determining whether a function is continuous at a.

Continuity Checklist

In order for f to be continuous at a, the following three conditions must hold.

1. $f(a)$ is defined (a is in the domain of f).

2. $\lim_{x \to a} f(x)$ exists.

3. $\lim_{x \to a} f(x) = f(a)$ (the value of f equals the limit of f at a).

If *any* item in the continuity checklist fails to hold, the function fails to be continuous at a. From this definition, we see that continuity has an important practical consequence:

If f is continuous at a, then $\lim_{x \to a} f(x) = f(a)$, and direct substitution may be used to evaluate $\lim_{x \to a} f(x)$.

Note that when f is defined on an open interval containing a (except possibly at a), we say that f has a **discontinuity** at a (or that a is a **point of discontinuity**) if f is not continuous at a.

EXAMPLE 1 Points of discontinuity

Use the graph of f in **Figure 2.44** to identify values of x on the interval $(0, 7)$ at which f is not continuous.

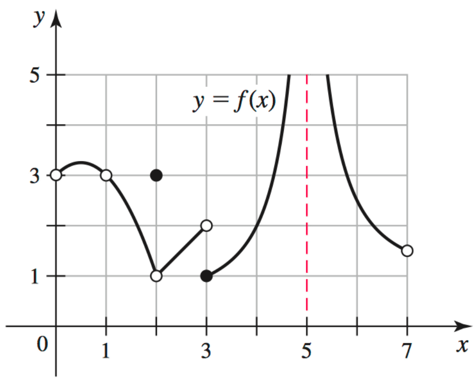


Figure 2.44

SOLUTION »

Quick Check 1 For what values of t in $(0, 60)$ does the graph of $y = c(t)$ in Figure 2.42b have discontinuities? ◆

Answer »

$t = 15, 30, 45$

EXAMPLE 2 Continuity at a point

Determine whether the following functions are continuous at a. Justify each answer using the continuity checklist.

a. $f(x) = \dfrac{3x^2 + 2x + 1}{x - 1}$; $a = 1$

b. $g(x) = \dfrac{3x^2 + 2x + 1}{x - 1}$; $a = 2$

c. $h(x) = \begin{cases} x \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$; $a = 0$

SOLUTION »

The following theorems make it easier to test various combinations of functions for continuity at a point.

THEOREM 2.8 **Continuity Rules**

If f and g are continuous at a, then the following functions are also continuous at a. Assume c is a constant and $n > 0$ is an integer.

a. $f + g$

b. $f - g$

c. cf

d. fg

e. $\frac{f}{g}$, provided $g(a) \neq 0$

f. $(f(x))^n$

To prove the first result, note that if f and g are continuous at a, then $\lim_{x \to a} f(x) = f(a)$ and $\lim_{x \to a} g(x) = g(a)$. From the limit laws of Theorem 2.3, it follows that

$$\lim_{x \to a} (f(x) + g(x)) = f(a) + g(a).$$

Therefore, $f + g$ is continuous at a. Similar arguments lead to the continuity of differences, products, quotients, and powers of continuous functions. The next theorem is a direct consequence of Theorem 2.8.

THEOREM 2.9 **Polynomials and Rational Functions**

a. A polynomial function is continuous for all x.

b. A rational function (a function of the form $\frac{p}{q}$, where p and q are polynomials) is continuous for all x for which $q(x) \neq 0$.

EXAMPLE 3 Applying the continuity theorems

For what values of x is the function $f(x) = \frac{x}{x^2 - 7x + 12}$ continuous?

SOLUTION »

Because f is rational, Theorem 2.9b implies it is continuous for all x at which the denominator is nonzero. The denominator factors as $(x - 3)(x - 4)$, so it is zero at $x = 3$ and $x = 4$. Therefore, f is continuous for all x except $x = 3$ and $x = 4$ (**Figure 2.45**).

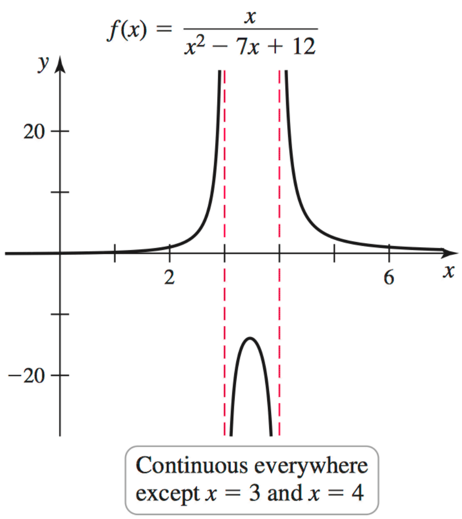


Figure 2.45

Related Exercises 26–27 ◆

The following theorem allows us to determine when a composition of two functions is continuous at a point. Its proof is informative and is outlined in Exercise 96.

> **THEOREM 2.10** **Continuity of Composite Functions at a Point**
>
> If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ is continuous at a.

Theorem 2.10 is useful because it allows us to conclude that the composition of two continuous functions is continuous at a point. For example, the composite function $\left(\frac{x}{x-1}\right)^3$ is continuous for all $x \neq 1$. Furthermore, under the stated conditions on f and g, the limit of their composition is evaluated by direct substitution; that is,

$$\lim_{x\to a} f(g(x)) = f(g(a)).$$

EXAMPLE 4 Limit of a composition

Evaluate $\lim_{x\to 0}\left(\frac{x^4 - 2x + 2}{x^6 + 2x^4 + 1}\right)^{10}$.

SOLUTION »

The rational function $\frac{x^4 - 2x + 2}{x^6 + 2x^4 + 1}$ is continuous for all x because its denominator is always positive (Theorem 2.9b). Therefore $\left(\frac{x^4 - 2x + 2}{x^6 + 2x^4 + 1}\right)^{10}$, which is the composition of the continuous function $f(x) = x^{10}$ and a continuous rational function, is continuous for all x by Theorem 2.10. By direct substitution,

$$\lim_{x\to 0}\left(\frac{x^4 - 2x + 2}{x^6 + 2x^4 + 1}\right)^{10} = \left(\frac{0^4 - 2\cdot 0 + 2}{0^6 + 2\cdot 0^4 + 1}\right)^{10} = 2^{10} = 1024.$$

Related Exercises 31–32 ◆

Closely related to Theorem 2.10 are two results dealing with limits of composite functions; they are used frequently in upcoming chapters. We present these two results—one a more general version of the other—in a single theorem.

THEOREM 2.11 **Limits of Composite Functions**

1. If g is continuous at a and f is continuous at $g(a)$, then

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right).$$

2. If $\lim_{x\to a} g(x) = L$ and f is continuous at L, then

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right).$$

Proof: The first statement follows directly from Theorem 2.10, which states that $\lim_{x\to a} f(g(x)) = f(g(a))$. If g is continuous at a, then $\lim_{x\to a} g(x) = g(a)$, and it follows that

$$\lim_{x\to a} f(g(x)) = f(\underbrace{g(a)}_{\lim_{x\to a} g(x)}) = f\left(\lim_{x\to a} g(x)\right).$$

The proof of the second statement (see Appendix A) relies on the formal definition of a limit, which is discussed in Section 2.7. ◆

Both statements of Theorem 2.11 justify interchanging the order of a limit and a function evaluation. By the second statement, the inner function of the composition needn't be continuous at the point of interest, but it must have a limit at that point. Note also that $\lim_{x\to a}$ can be replaced with $\lim_{x\to a^+}$ or $\lim_{x\to a^-}$ in statement (1) of Theorem 2.11, provided g is *right-* or *left-continuous* at a (see the definition of *Continuity at Endpoints* in the next subsection), respectively, in statement (1). In statement (2), $\lim_{x\to a}$ can be replaced with $\lim_{x\to\infty}$ or $\lim_{x\to-\infty}$.

EXAMPLE 5 **Limits of composite functions**

Evaluate the following limits.

a. $\lim_{x\to -1} \sqrt{2x^2 - 1}$

b. $\lim_{x\to 2} \cos\left(\dfrac{x^2 - 4}{x - 2}\right)$

SOLUTION »

a. The inner function of the composite function $\sqrt{2x^2 - 1}$ is $g(x) = 2x^2 - 1$; it is continuous and positive at -1, and $g(-1) = 1$. Because $f(x) = \sqrt{x}$ is continuous at $g(-1) = 1$ (a consequence of Law 7, Theorem 2.3), we have, by the first statement of Theorem 2.11,

$$\lim_{x\to -1} \sqrt{2x^2 - 1} = \sqrt{\underbrace{\lim_{x\to -1}\left(2x^2 - 1\right)}_{1}} = \sqrt{1} = 1.$$

b. We show later in this section that cos x is continuous at all points of its domain. The inner function of the composite function $\cos\left(\frac{x^2-4}{x-2}\right)$ is $\frac{x^2-4}{x-2}$, which is not continuous at 2. However,

$$\lim_{x\to 2}\left(\frac{x^2-4}{x-2}\right) = \lim_{x\to 2}\frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2}(x+2) = 4.$$

Therefore, by the second statement of Theorem 2.11,

$$\lim_{x\to 2}\cos\left(\frac{x^2-4}{x-2}\right) = \cos\left(\lim_{x\to 2}\left(\frac{x^2-4}{x-2}\right)\right) = \cos 4 \approx -0.654.$$

Related Exercises 33–34 ◆

Continuity on an Interval »

A function is continuous *on an interval* if it is continuous at every point in that interval. Consider the functions f and g whose graphs are shown in **Figure 2.46**. Both these functions are continuous for all x in (a, b), but what about the endpoints? To answer this question, we introduce the ideas of *left-continuity* and *right-continuity*.

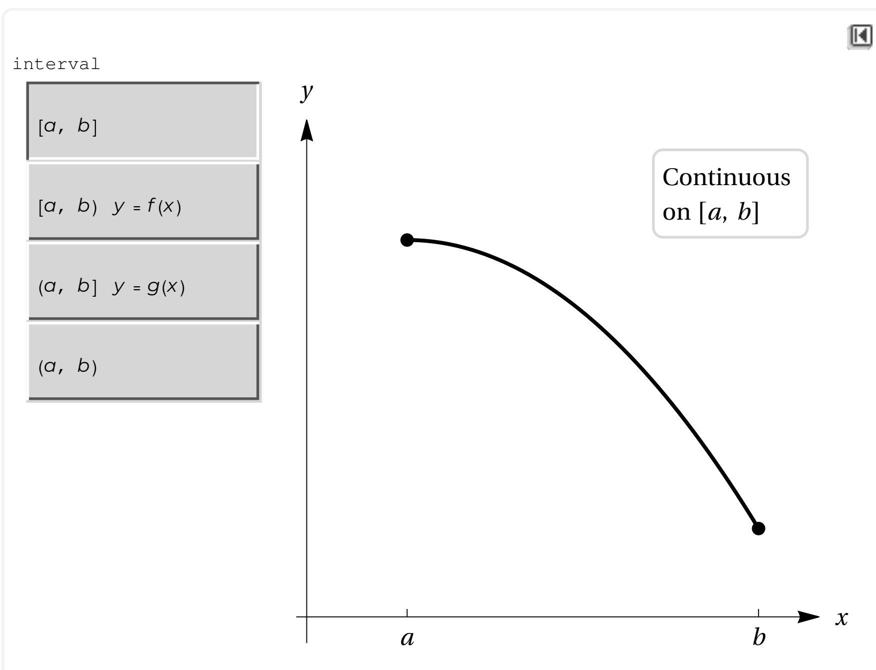


Figure 2.46

DEFINITION **Continuity at Endpoints**

A function f is **continuous from the right** (or **right-continuous**) at a if $\lim_{x \to a^+} f(x) = f(a)$ and f is **continuous from the left** (or **left-continuous**) at b if $\lim_{x \to b^-} f(x) = f(b)$.

Combining the definitions of left-continuous and right-continuous with the definition of continuity at a point, we define what it means for a function to be continuous on an interval.

DEFINITION **Continuity on an Interval**

A function f is **continuous on an interval** $\boldsymbol{I}$ if it is continuous at all points of I. If I contains its endpoints, continuity on I means continuous from the right or left at the endpoints.

To illustrate these definitions, consider again the functions in Figure 2.46. In Figure 2.46, f is continuous from the right at a because $\lim_{x \to a^+} f(x) = f(a)$; but it is not continuous from the left at b because $f(b)$ is not defined. Therefore, f is continuous on the interval $[a, b)$. The behavior of the function g in Figure 2.46 is the opposite: It is continuous from the left at b, but it is not continuous from the right at a. Therefore, g is continuous on $(a, b]$.

Quick Check 2 Modify the graphs of the functions f and g in Figure 2.46 to obtain functions that are continuous on $[a, b]$. ◆

Answer »

Fill in the endpoints.

EXAMPLE 6 Intervals of continuity

Determine the intervals of continuity for

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x \le 0 \\ 3x + 5 & \text{if } x > 0. \end{cases}$$

SOLUTION »

This piecewise function consists of two polynomials that describe a parabola and a line (**Figure 2.47**). By Theorem 2.9, f is continuous for all $x \ne 0$. From its graph, it appears that f is left-continuous at $x = 0$. This observation is verified by noting that

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \left(x^2 + 1\right) = 1,$$

which means that $\lim_{x \to 0^-} f(x) = f(0)$. However, because

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (3x + 5) = 5 \ne f(0),$$

we see that f is not right-continuous at $x = 0$. Therefore, f is continuous on $(-\infty, 0]$ and on $(0, \infty)$.

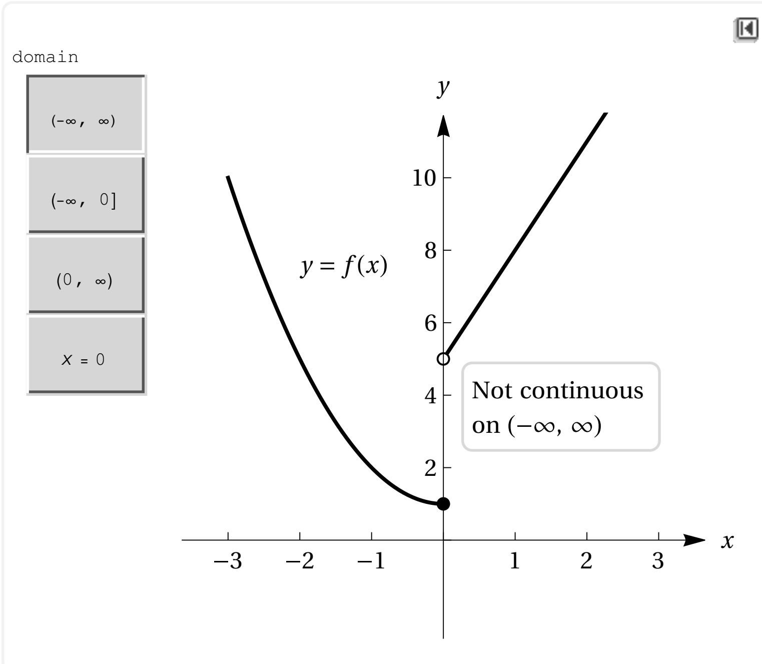


Figure 2.47

Related Exercises 39–40 ◆

Functions Involving Roots »

Recall that Limit Law 7 of Theorem 2.3 states

$$\lim_{x \to a} (f(x))^{1/n} = \left(\lim_{x \to a} f(x)\right)^{1/n},$$

provided $f(x) \geq 0$, for x near a, if n is even. Therefore, if n is odd and f is continuous at a, then $(f(x))^{1/n}$ is continuous at a, because

$$\lim_{x \to a} (f(x))^{1/n} = \left(\lim_{x \to a} f(x)\right)^{1/n} = (f(a))^{1/n}.$$

When n is even, the continuity of $(f(x))^{1/n}$ must be handled more carefully because this function is defined only when $f(x) \geq 0$. Exercise 73 of Section 2.7 establishes an important fact:

If f is continuous at a and $f(a) > 0$, then $f(x) > 0$ for all values of x in some interval containing a.

Combining this fact with Theorem 2.10 (the continuity of composite functions), it follows that $(f(x))^{1/n}$ is continuous at a provided $f(a) > 0$. At points where $f(a) = 0$, the behavior of $(f(x))^{1/n}$ varies. Often we find that $(f(x))^{1/n}$ is left- or right-continuous at that point, or it may be continuous from both sides.

THEOREM 2.12 **Continuity of Functions with Roots**

Assume n is a positive integer. If n is an odd integer, then $(f(x))^{1/n}$ is continuous at all points at which f is continuous. If n is even, then $(f(x))^{1/n}$ is continuous at all points a at which f is continuous and $f(a) > 0$.

EXAMPLE 7 Continuity with roots

For what values of x are the following functions continuous?

a. $g(x) = \sqrt{9 - x^2}$

b. $f(x) = \left(x^2 - 2x + 4\right)^{2/3}$

SOLUTION »

a. The graph of g is the upper half of the circle $x^2 + y^2 = 9$ (which can be verified by solving $x^2 + y^2 = 9$ for y). From **Figure 2.48**, it appears that g is continuous on $[-3, 3]$. To verify this fact, note that g involves an even root ($n = 2$ in Theorem 2.12). If $-3 < x < 3$, then $9 - x^2 > 0$ and by Theorem 2.12, g is continuous for all x on $(-3, 3)$.

At the right endpoint, $\lim_{x \to 3^-} \sqrt{9 - x^2} = 0 = g(3)$ by Limit Law 7 for one-sided limits, which implies that g is left-continuous at 3. Similarly, g is right-continuous at -3 because $\lim_{x \to -3^+} \sqrt{9 - x^2} = 0 = g(-3)$. Therefore, g is continuous on $[-3, 3]$.

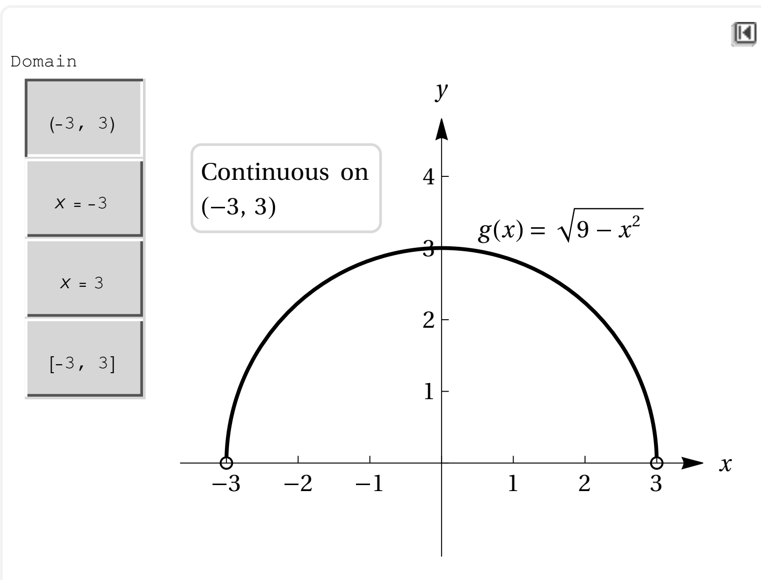


Figure 2.48

b. The polynomial $x^2 - 2x + 4$ is continuous for all x by Theorem 2.9a. Rewriting f as $f(x) = \left(\left(x^2 - 2x + 4\right)^{1/3}\right)^2$, we see that f involves an odd root ($n = 3$ in Theorem 2.12). Therefore, f is continuous for all x.

Related Exercises 44–45 ◆

Quick Check 3 On what interval is $f(x) = x^{1/4}$ continuous? On what interval is $f(x) = x^{2/5}$ continuous? ◆

Answer »

$[0, \infty)$; $(-\infty, \infty)$

Continuity of Trigonometric Functions »

In Section 2.3, we used the Squeeze Theorem to show that $\lim_{x \to 0} \sin x = 0$ and $\lim_{x \to 0} \cos x = 1$. Because $\sin 0 = 0$ and $\cos 0 = 1$, these limits imply that $\sin x$ and $\cos x$ are continuous at 0. The graph of $y = \sin x$ (**Figure 2.49**) suggests that $\lim_{x \to a} \sin x = \sin a$ for any value of a, which means that $\sin x$ is continuous everywhere. The graph of $y = \cos x$ also indicates that $\cos x$ is continuous for all x. Exercise 99 outlines a proof of these results.

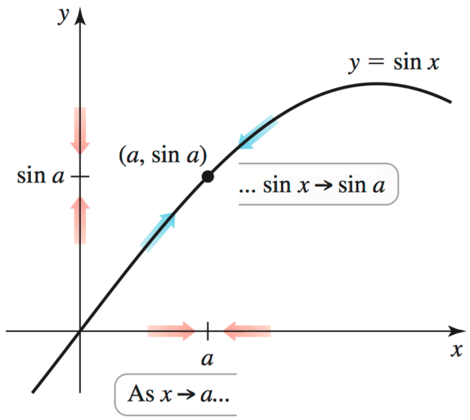


Figure 2.49

With these facts in hand, we appeal to Theorem 2.9e to discover that the remaining trigonometric functions are continuous on their domains. For example, because $\sec x = \dfrac{1}{\cos x}$, the secant function is continuous for all x for which $\cos x \neq 0$ (for all x except odd multiples of $\dfrac{\pi}{2}$; **Figure 2.50**). Likewise, the tangent, cotangent, and cosecant functions are continuous at all points of their domains.

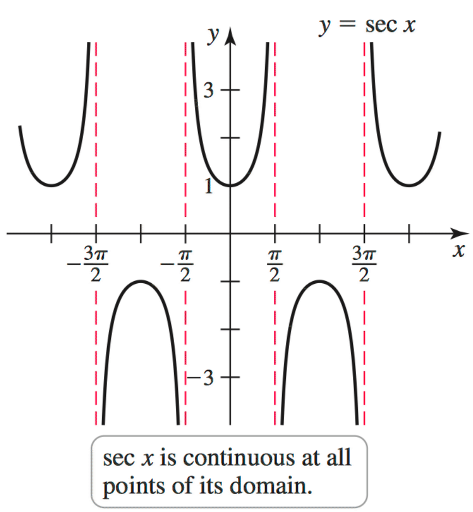


Figure 2.50

THEOREM 2.13 **Continuity of Trigonometric Functions**

The functions $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, and $\csc x$ are continuous at all points of their domains.

For each function listed in Theorem 2.13, we have $\lim_{x \to a} f(x) = f(a)$, provided a is in the domain of the function. This means that limits involving these functions may be evaluated by direct substitution at points in the domain.

EXAMPLE 8 Limits involving trigonometric functions

Evaluate $\lim\limits_{x\to 0} \dfrac{\cos^2 x - 1}{\cos x - 1}$.

SOLUTION »

Both $\cos^2 x - 1$ and $\cos x - 1$ are continuous for all x by Theorems 2.8 and 2.13. However, the ratio of these functions is not continuous when $\cos x - 1 = 0$, which corresponds to all integer multiples of 2π. Note that both the numerator and denominator of $\dfrac{\cos^2 x - 1}{\cos x - 1}$ approach 0 as $x \to 0$. To evaluate the limit, we factor and simplify:

$$\lim_{x\to 0} \frac{\cos^2 x - 1}{\cos x - 1} = \lim_{x\to 0} \frac{(\cos x - 1)(\cos x + 1)}{\cos x - 1} = \lim_{x\to 0} (\cos x + 1)$$

(where $\cos x - 1$ may be canceled because it is nonzero as x approaches 0). The limit on the right is now evaluated using direct substitution:

$$\lim_{x\to 0} (\cos x + 1) = \cos 0 + 1 = 2.$$

Note »

> Limits like the one in Example 8 are denoted 0/0 and are known as *indeterminate forms*, to be studied further in Section 4.7.

Related Exercises 62–63 ◆

We close this section with an important theorem that has both practical and theoretical uses.

Intermediate Value Theorem »

A common problem in mathematics is finding solutions to equations of the form $f(x) = L$. Before attempting to find values of x satisfying this equation, it is worthwhile to determine whether a solution exists.

The existence of solutions is often established using a result known as the *Intermediate Value Theorem*. Given a function f and a constant L, we assume L lies between $f(a)$ and $f(b)$. The Intermediate Value Theorem says that if f is continuous on $[a, b]$, then the graph of f must cross the horizontal line $y = L$ at least once (**Figure 2.51**). Although this theorem is easily illustrated, its proof goes beyond the scope of this text.

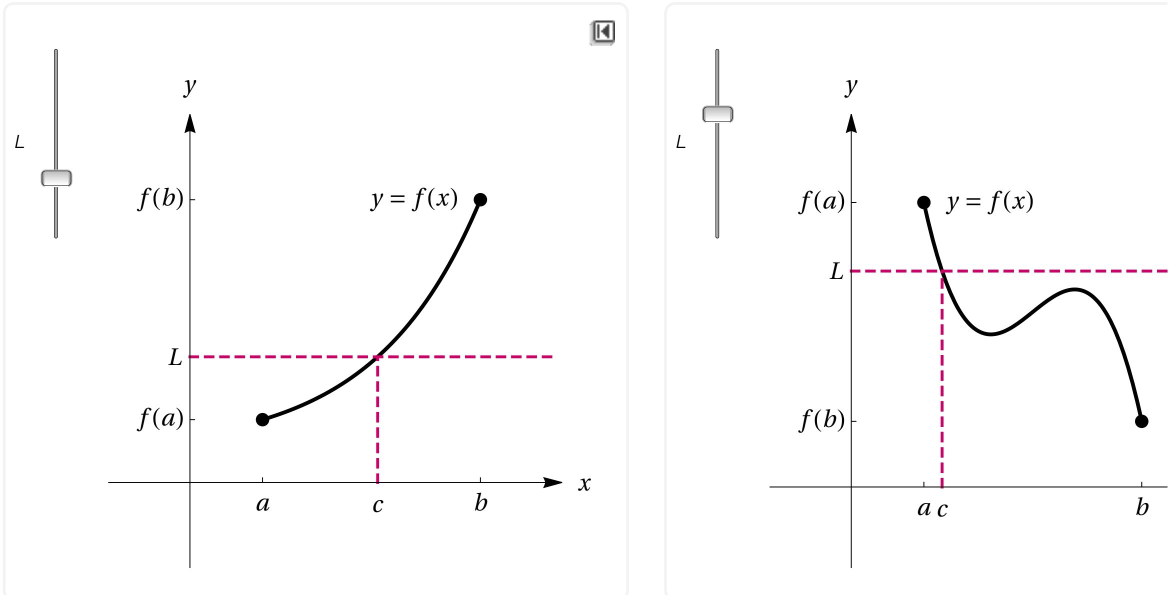


Figure 2.51

> **THEOREM 2.14** **Intermediate Value Theorem**
>
> Suppose f is continuous on the interval $[a, b]$ and L is a number strictly between $f(a)$ and $f(b)$. Then there exists at least one number c in (a, b) satisfying $f(c) = L$.

The importance of continuity in Theorem 2.14 is illustrated in **Figure 2.52**, where we see a function f that is not continuous on $[a, b]$. For the value of L shown in the figure, there is no value of c in (a, b) satisfying $f(c) = L$. The next example illustrates a practical application of the Intermediate Value Theorem.

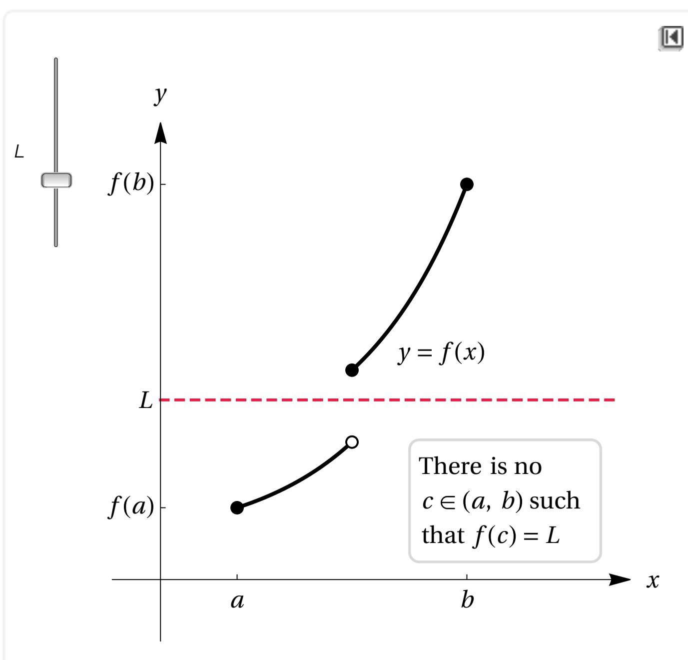


Figure 2.52

Quick Check 4 Does the equation $f(x) = x^3 + x + 1 = 0$ have a solution on the interval $[-1, 1]$? Explain. ◆

Answer »

The equation has a solution on the interval $[-1, 1]$ because f is continuous on $[-1, 1]$ and $f(-1) < 0 < f(1)$.

EXAMPLE 9 **Finding an interest rate**

Suppose you invest $1000 in a special 5-year savings account with a fixed annual interest rate r and with monthly compounding. The amount of money A in the account after 5 years (60 months) is $A(r) = 1000\left(1 + \frac{r}{12}\right)^{60}$. Your goal is to have $1400 in the account after 5 years.

a. Use the Intermediate Value Theorem to show there is a value of r in $(0, 0.08)$—that is, an interest rate between 0% and 8%—for which $A(r) = 1400$.

b. Use a graphing utility to illustrate your explanation in part (a), and then estimate the interest rate required to reach your goal.

SOLUTION »

a. As a polynomial in r (of degree 60), $A(r) = 1000\left(1 + \frac{r}{12}\right)^{60}$ is continuous for all r. Evaluating $A(r)$ at the endpoints of the interval $[0, 0.08]$, we have $A(0) = 1000$ and $A(0.08) = 1489.85$. Therefore,

$$A(0) < 1400 < A(0.08),$$

and it follows, by the Intermediate Value Theorem, that there is a value of r in $(0, 0.08)$ for which $A(r) = 1400$.

b. The graphs of $y = A(r)$ and the horizontal line $y = 1400$ are shown in **Figure 2.53**; it is evident that they intersect between $r = 0$ and $r = 0.08$. Solving $A(r) = 1400$ algebraically or using a root finder reveals that the curve and line intersect at $r \approx 0.0675$. Therefore, an interest rate of approximately 6.75% is required for the investment to be worth $1400 after 5 years.

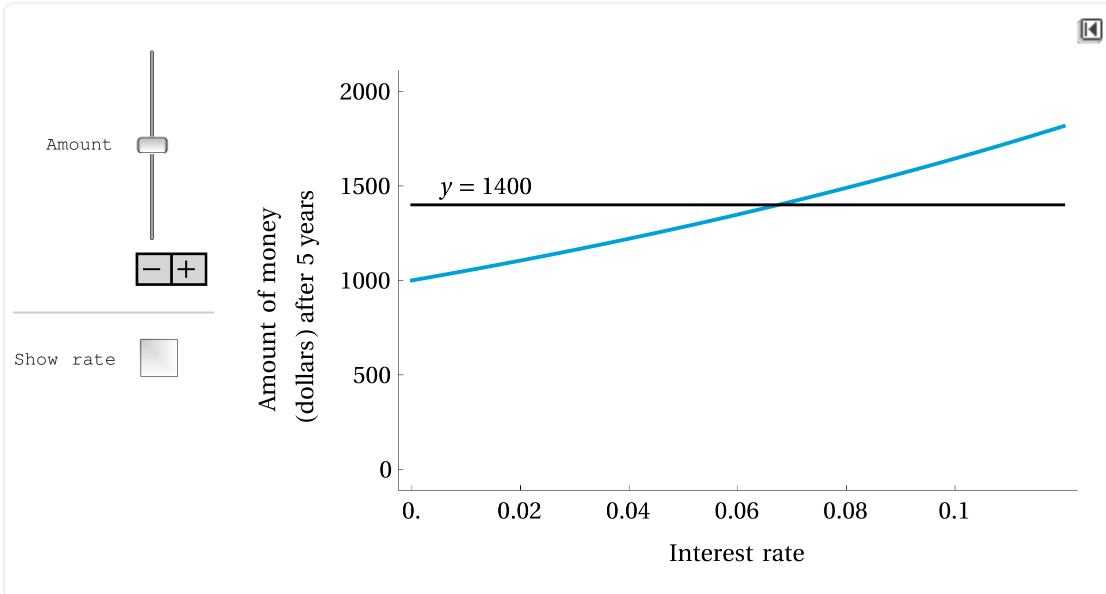


Figure 2.53

Related Exercises 65, 71 ◆

Exercises »

Getting Started »

Practice Exercises »

17–24. Continuity at a point *Determine whether the following functions are continuous at a. Use the continuity checklist to justify your answer.*

17. $f(x) = \dfrac{2x^2 + 3x + 1}{x^2 + 5x}$; $a = -5$

18. $f(x) = \dfrac{2x^2 + 3x + 1}{x^2 + 5x}$; $a = 5$

19. $f(x) = \sqrt{x - 2}$; $a = 1$

20. $g(x) = \dfrac{1}{x - 3}$; $a = 3$

21. $f(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1} & \text{if } x \neq 1 \\ 3 & \text{if } x = 1 \end{cases}$; $a = 1$

22. $f(x) = \begin{cases} \dfrac{x^2 - 4x + 3}{x - 3} & \text{if } x \neq 3 \\ 2 & \text{if } x = 3 \end{cases}$; $a = 3$

23. $f(x) = \dfrac{5x - 2}{x^2 - 9x + 20}$; $a = 4$

24. $f(x) = \begin{cases} \dfrac{x^2 + x}{x + 1} & \text{if } x \neq -1 \\ 2 & \text{if } x = -1 \end{cases}$; $a = -1$

25–30. Continuity *Determine the interval(s) on which the following functions are continuous.*

25. $p(x) = 4x^5 - 3x^2 + 1$

26. $g(x) = \dfrac{3x^2 - 6x + 7}{x^2 + x + 1}$

27. $f(x) = \dfrac{x^5 + 6x + 17}{x^2 - 9}$

28. $s(x) = \dfrac{x^2 - 4x + 3}{x^2 - 1}$

29. $f(x) = \dfrac{1}{x^2 - 4}$

30. $f(t) = \dfrac{t + 2}{t^2 - 4}$

31–38. Limits *Evaluate each limit and justify your answer.*

31. $\lim\limits_{x \to 0} \left(x^8 - 3x^6 - 1\right)^{40}$

32. $\lim\limits_{x \to 2} \left(\dfrac{3}{2x^5 - 4x^2 - 50}\right)^4$

33. $\lim\limits_{x \to 4} \sqrt{\dfrac{x^3 - 2x^2 - 8x}{x - 4}}$

34. $\lim\limits_{t \to 4} \dfrac{t - 4}{\sqrt{t} - 2}$

35. $\lim\limits_{x \to 1} \left(\dfrac{x + 5}{x + 2}\right)^4$

36. $\lim\limits_{x \to \infty} \left(\dfrac{2x + 1}{x}\right)^3$

37. $\lim\limits_{x \to 5} \left(\dfrac{\sqrt{x^2 - 16} - 3}{5x - 25}\right)^4$

38. $\lim_{x\to 0} \left(\frac{x}{\sqrt{16x+1} - 1}\right)^{1/3}$

39–40. Intervals of continuity *Complete the following steps for each function.*

a. *Use the continuity checklist to show that f is not continuous at the given value of a.*

b. *Determine whether f is continuous from the left or right at a.*

c. *State the interval(s) of continuity.*

39. $f(x) = \begin{cases} 2x & \text{if } x < 1 \\ x^2 + 3x & \text{if } x \geq 1 \end{cases}; a = 1$

40. $f(x) = \begin{cases} x^3 + 4x + 1 & \text{if } x \leq 0 \\ 2x^3 & \text{if } x > 0 \end{cases}; a = 0$

41–48. Functions with roots *Determine the interval(s) on which the following functions are continuous. At which finite endpoints of the intervals of continuity is f continuous from the left or continuous from the right?*

41. $f(x) = \sqrt{5 - x}$

42. $f(x) = \sqrt{25 - x^2}$

43. $f(x) = \sqrt{2x^2 - 16}$

44. $f(x) = \sqrt{x^2 - 3x + 2}$

45. $f(x) = \sqrt[3]{x^2 - 2x - 3}$

46. $f(t) = \left(t^2 - 1\right)^{3/2}$

47. $f(x) = (2x - 3)^{2/3}$

48. $f(z) = (z - 1)^{3/4}$

49–60. *Evaluate each limit.*

49. $\lim_{x\to 2} \sqrt{\frac{4x + 10}{2x - 2}}$

50. $\lim_{x\to -1} \left(x^2 - 4 + \sqrt[3]{x^2 - 9}\right)$

51. $\lim_{x\to \pi} \frac{\cos^2 x + 3\cos x + 2}{\cos x + 1}$

52. $\lim_{x\to 3\pi/2} \frac{\sin^2 x + 6\sin x + 5}{\sin^2 x - 1}$

53. $\lim_{x \to 3} \sqrt{x^2 + 7}$

54. $\lim_{t \to 2} \dfrac{t^2 + 5}{1 + \sqrt{t^2 + 5}}$

55. $\lim_{x \to \pi/2} \dfrac{\sin x - 1}{\sqrt{\sin x} - 1}$

56. $\lim_{\theta \to 0} \dfrac{\frac{1}{2+\sin\theta} - \frac{1}{2}}{\sin\theta}$

57. $\lim_{x \to 0} \dfrac{\cos x - 1}{\sin^2 x}$

58. $\lim_{x \to 0^+} \dfrac{1 - \cos^2 x}{\sin x}$

59. $\lim_{t \to \infty} \dfrac{\sin t}{t^2}$

60. $\lim_{x \to 0^+} \dfrac{\cos x}{x}$

61–64. Continuity and limits with trigonometric functions *Determine the interval(s) on which the following functions are continuous; then analyze the given limits.*

61. $f(x) = \csc x$; $\lim_{x \to \pi/4} f(x)$; $\lim_{x \to 2\pi^-} f(x)$

62. $f(x) = \sqrt{\sin x}$; $\lim_{x \to \pi/2} f(x)$; $\lim_{x \to 0^+} f(x)$

63. $f(x) = \dfrac{1 + \sin x}{\cos x}$; $\lim_{x \to \pi/2^-} f(x)$; $\lim_{x \to 4\pi/3} f(x)$

64. $f(x) = \dfrac{1}{2\cos x - 1}$; $\lim_{x \to \pi/6} f(x)$

T 65–68. Applying the Intermediate Value Theorem

a. *Use the Intermediate Value Theorem to show that the following equations have a solution on the given interval.*

b. *Use a graphing utility to find all the solutions to the equation on the given interval.*

c. *Illustrate your answers with an appropriate graph.*

65. $2x^3 + x - 2 = 0$; $(-1, 1)$

66. $\sqrt{x^4 + 25x^3 + 10} = 5$; $(0, 1)$

67. $x^3 - 5x^2 + 2x = -1$; $(-1, 5)$

68. $-x^5 - 4x^2 + 2\sqrt{x} + 5 = 0$; $(0, 3)$

69. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. If a function is left-continuous and right-continuous at a, then it is continuous at a.

b. If a function is continuous at a, then it is left-continuous and right-continuous at a.

c. If $a < b$ and $f(a) \le L \le f(b)$, then there is some value of c in (a, b) for which $f(c) = L$.

d. Suppose f is continuous on $[a, b]$. Then there is a point c in (a, b) such that $f(c) = (f(a) + f(b))/2$.

T 70. Mortgage payments You are shopping for a $250,000, 30-year (360-month) loan to buy a house. The monthly payment is given by

$$m(r) = \frac{250,000\,(r/12)}{1 - (1 + r/12)^{-360}},$$

where r is the annual interest rate. Suppose banks are currently offering interest rates between 4% and 5%.

a. Show there is a value of r in $(0.04, 0.05)$—an interest rate between 4% and 5%—that allows you to make monthly payments of $1300 per month.

b. Use a graph to illustrate your explanation to part (a). Then determine the interest rate you need for monthly payments of $1300.

T 71. Interest rates Suppose $5000 is invested in a savings account for 10 years (120 months), with an annual interest rate of r, compounded monthly. The amount of money in the account after 10 years is given by $A(r) = 5000\,(1 + r/12)^{120}$.

a. Show there is a value of r in $(0, 0.08)$—an interest rate between 0% and 8%—that allows you to reach your savings goal of $7000 in 10 years.

b. Use a graph to illustrate your explanation in part (a). Then approximate the interest rate required to reach your goal.

T 72. Investment problem Assume you invest $250 at the end of each year for 10 years at an annual interest rate of r. The amount of money in your account after 10 years is given by $A(r) = \dfrac{250\left((1 + r)^{10} - 1\right)}{r}$. Assume your goal is to have $3500 in your account after 10 years.

a. Show that there is an interest rate r in the interval $(0.01, 0.10)$—between 1% and 10%—that allows you to reach your financial goal.

b. Use a calculator to estimate the interest rate required to reach your financial goal.

T 73. Find an interval containing a solution to the equation $2x = \cos x$. Use a graphing utility to approximate the solution.

Explorations and Challenges »

74. Continuity of the absolute value function Prove that the absolute value function $|x|$ is continuous for all values of x. (*Hint:* Using the definition of the absolute value function, compute $\lim_{x \to 0^-} |x|$ and $\lim_{x \to 0^+} |x|$.)

75–78. Continuity of functions with absolute values *Use the continuity of the absolute value function (Exercise 74) to determine the interval(s) on which the following functions are continuous.*

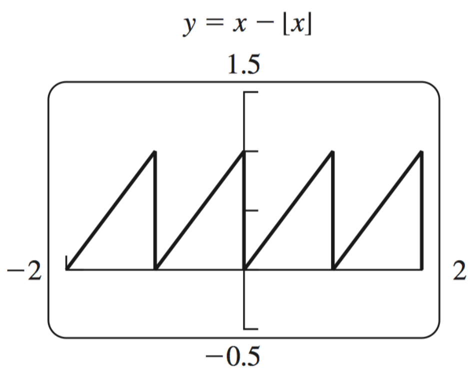
75. $f(x) = |x^2 + 3x - 18|$

76. $g(x) = \left|\frac{x+4}{x^2-4}\right|$

77. $h(x) = \left|\frac{1}{\sqrt{x} - 4}\right|$

78. $h(x) = |x^2 + 2x + 5| + \sqrt{x}$

79. Pitfalls using technology The graph of the *sawtooth function* $y = x - \lfloor x \rfloor$, where $\lfloor x \rfloor$ is the greatest integer function or floor function (Exercise 51, Section 2.2), was obtained using a graphing utility (see figure). Identify any inaccuracies appearing in the graph and then plot an accurate graph by hand.



T 80. Pitfalls using technology Graph the function $f(x) = \frac{\sin x}{x}$ using a graphing window of $[-\pi, \pi] \times [0, 2]$.

a. Sketch a copy of the graph obtained with your graphing device and describe any inaccuracies appearing in the graph.

b. Sketch an accurate graph of the function. Is f continuous at 0?

c. What is the value of $\lim_{x \to 0} \frac{\sin x}{x}$?

81. Sketching functions

a. Sketch the graph of a function that is not continuous at 1, but is defined at 1.

b. Sketch the graph of a function that is not continuous at 1, but has a limit at 1.

82. An unknown constant Determine the value of the constant a for which the function

$$f(x) = \begin{cases} \frac{x^2 + 3x + 2}{x + 1} & \text{if } x \neq -1 \\ a & \text{if } x = -1 \end{cases}$$

is continuous at -1.

83. An unknown constant Let

$$g(x) = \begin{cases} x^2 + x & \text{if } x < 1 \\ a & \text{if } x = 1 \\ 3x + 5 & \text{if } x > 1. \end{cases}$$

a. Determine the value of a for which g is continuous from the left at 1.

b. Determine the value of a for which g is continuous from the right at 1.

c. Is there a value of a for which g is continuous at 1? Explain.

T **84–85. Applying the Intermediate Value Theorem** *Use the Intermediate Value Theorem to verify that the following equations have three solutions on the given interval. Use a graphing utility to find the approximate roots.*

84. $x^3 + 10x^2 - 100x + 50 = 0;\ (-20, 10)$

85. $70x^3 - 87x^2 + 32x - 3 = 0;\ (0, 1)$

86. Walk in the park Suppose you park your car at a trailhead in a national park and begin a 2-hr hike to a lake at 7 A.M. on a Friday morning. On Sunday morning, you leave the lake at 7 A.M. and start the 2-hr hike back to your car. Assume the lake is 3 mi from your car. Let $f(t)$ be your distance from the car t hours after 7 A.M. on Friday morning, and let $g(t)$ be your distance from the car t hours after 7 A.M. on Sunday morning.

a. Evaluate $f(0)$, $f(2)$, $g(0)$, and $g(2)$.

b. Let $h(t) = f(t) - g(t)$. Find $h(0)$ and $h(2)$.

c. Show that there is some point along the trail that you will pass at exactly the same time of morning on both days.

87. The monk and the mountain A monk set out from a monastery in the valley at dawn. He walked all day up a winding path, stopping for lunch and taking a nap along the way. At dusk, he arrived at a temple on the mountaintop. The next day the monk made the return walk to the valley, leaving the temple at dawn, walking the same path for the entire day, and arriving at the monastery in the evening. Must there be one point along the path that the monk occupied at the same time of day on both the ascent and the descent? Explain. (*Hint:* The question can be answered without the Intermediate Value Theorem.) (*Source:* Arthur Koestler, *The Act of Creation*)

88. Does continuity of $|f|$ imply continuity of f? Let

$$g(x) = \begin{cases} 1 & \text{if } x \ge 0 \\ -1 & \text{if } x < 0. \end{cases}$$

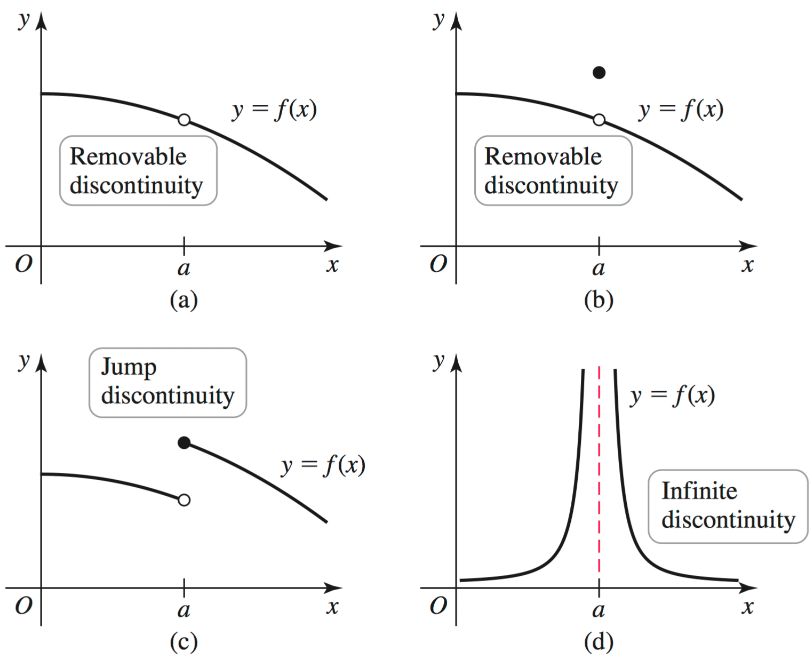
a. Write a formula for $|g(x)|$.

b. Is g continuous at $x = 0$? Explain.

c. Is $|g|$ continuous at $x = 0$? Explain.

d. For any function f, if $|f|$ is continuous at a, does it necessarily follow that f is continuous at a? Explain.

89–90. Classifying discontinuities *The discontinuities in graphs (a) and (b) are* **removable discontinuities** *because they disappear if we define or redefine f at a so that $f(a) = \lim_{x \to a} f(x)$. The function in graph (c) has a* **jump discontinuity** *because left and right limits exist at a but are unequal. The discontinuity in graph (d) is an* **infinite discontinuity** *because the function has a vertical asymptote at a.*



89. Is the discontinuity at a in graph (c) removable? Explain.

90. Is the discontinuity at a in graph (d) removable? Explain.

T 91–94. *Classify the discontinuities in the following functions at the given points.*

91. $f(x) = \dfrac{x^2 - 7x + 10}{x - 2}$; $x = 2$

92. $g(x) = \begin{cases} \dfrac{x^2 - 1}{1 - x} & \text{if } x \neq 1 \\ 3 & \text{if } x = 1 \end{cases}$; $x = 1$

93. $h(x) = \dfrac{x^3 - 4x^2 + 4x}{x(x - 1)}$; $x = 0$ and $x = 1$

94. $f(x) = \dfrac{|x - 2|}{x - 2}$; $x = 2$

95. Do removable discontinuities exist?

a. Does the function $f(x) = x \sin(1/x)$ have a removable discontinuity at $x = 0$? Explain.

b. Does the function $g(x) = \sin(1/x)$ have a removable discontinuity at $x = 0$? Explain.

96. Continuity of composite functions Prove Theorem 2.10: If g is continuous at a and f is continuous at $g(a)$, then the composition $f \circ g$ is continuous at a. (*Hint:* Write the definition of continuity for f and g separately; then combine them to form the definition of continuity for $f \circ g$.)

97. Continuity of compositions

a. Find functions f and g such that each function is continuous at 0 but $f \circ g$ is not continuous at 0.

b. Explain why examples satisfying part (a) do not contradict Theorem 2.10.

98. Violation of the Intermediate Value Theorem? Let $f(x) = \frac{|x|}{x}$. Then $f(-2) = -1$ and $f(2) = 1$. Therefore, $f(-2) < 0 < f(2)$, but there is no value of c between -2 and 2 for which $f(c) = 0$. Does this fact violate the Intermediate Value Theorem? Explain.

99. Continuity of sin *x* and cos *x*

a. Use the identity $\sin(a + h) = \sin a \cos h + \cos a \sin h$ with the fact that $\lim_{x \to 0} \sin x = 0$ to prove that $\lim_{x \to a} \sin x = \sin a$, thereby establishing that $\sin x$ is continuous for all x. (*Hint:* Let $h = x - a$ so that $x = a + h$ and note that $h \to 0$ as $x \to a$.)

b. Use the identity $\cos(a + h) = \cos a \cos h - \sin a \sin h$ with the fact that $\lim_{x \to 0} \cos x = 1$ to prove that $\lim_{x \to a} \cos x = \cos a$.