2.5 Limits at Infinity

Limits at infinity—as opposed to infinite limits—occur when the independent variable becomes large in magnitude. For this reason, limits at infinity determine what is called the *end behavior* of a function. An application of these limits is to determine whether a system (such as an ecosystem or a large oscillating structure) reaches a steady state as time increases.

Limits at Infinity and Horizontal Asymptotes »

Consider the function $f(x) = \dfrac{x}{\sqrt{x^2 + 1}}$ (**Figure 2.29**), whose domain is $(-\infty, \infty)$. As x becomes arbitrarily large (denoted $x \to \infty$), $f(x)$ approaches 1, and as x becomes arbitrarily large in magnitude and negative (denoted $x \to -\infty$), $f(x)$ approaches -1 These limits are expressed as

$$\lim_{x\to\infty} f(x) = 1 \quad \text{and} \quad \lim_{x\to-\infty} f(x) = -1.$$

The graph of f approaches the horizontal line $y = 1$ as $x \to \infty$ and it approaches the horizontal line $y = -1$ as $x \to -\infty$. These lines are called *horizontal asymptotes.*

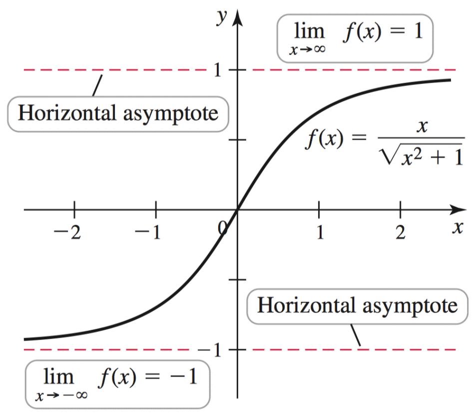


Figure 2.29

> **DEFINITION** **Limits at Infinity and Horizontal Asymptotes**
>
> If $f(x)$ becomes arbitrarily close to a finite number L for all sufficiently large and positive x, then we write
>
> $$\lim_{x\to\infty} f(x) = L.$$
>
> We say the limit of $f(x)$ as x approaches infinity is L. In this case the line $y = L$ is a **horizontal asymptote** of f (**Figure 2.30**). The limit at negative infinity, $\lim_{x\to-\infty} f(x) = M$, is defined analogously and in this case the horizontal asymptote is $y = M$.

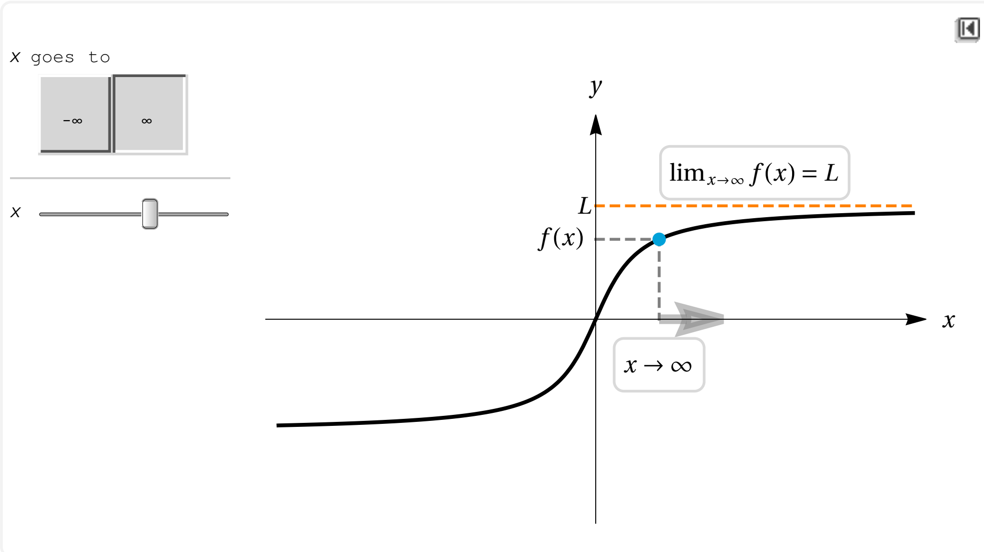


Figure 2.30

Quick Check 1 Evaluate $\frac{x}{x+1}$ for $x = 10,\ 100,$ and 1000. What is $\lim_{x\to\infty} \frac{x}{x+1}$? ◆

Answer »

$10/11,\ 100/101,\ 1000/1001;\ 1$

EXAMPLE 1 Limits at infinity

Evaluate the following limits.

a. $\lim_{x\to-\infty} \left(2 + \frac{10}{x^2}\right)$

b. $\lim_{x\to\infty} \left(5 + \frac{\sin x}{\sqrt{x}}\right)$

SOLUTION »

a. As x becomes large and negative, x^2 becomes large and positive; in turn, $\frac{10}{x^2}$ approaches 0. By the limit laws of Theorem 2.3,

$$\lim_{x\to-\infty} \left(2 + \frac{10}{x^2}\right) = \underbrace{\lim_{x\to-\infty} 2}_{\text{equals } 2} + \underbrace{\lim_{x\to-\infty} \left(\frac{10}{x^2}\right)}_{\text{equals } 0} = 2 + 0 = 2.$$

Note »

The limit laws of Theorem 2.3 and the Squeeze Theorem apply if $x \to a$ is replaced with $x \to \infty$ or $x \to -\infty$.

Therefore, the graph of $y = 2 + \frac{10}{x^2}$ approaches the horizontal asymptote $y = 2$ as $x \to \infty$ and as $x \to -\infty$ (**Figure**

2.31). Notice that $\lim_{x\to\infty}\left(2+\frac{10}{x^2}\right)$ is also equal to 2, which implies that the graph has a single horizontal asymptote.

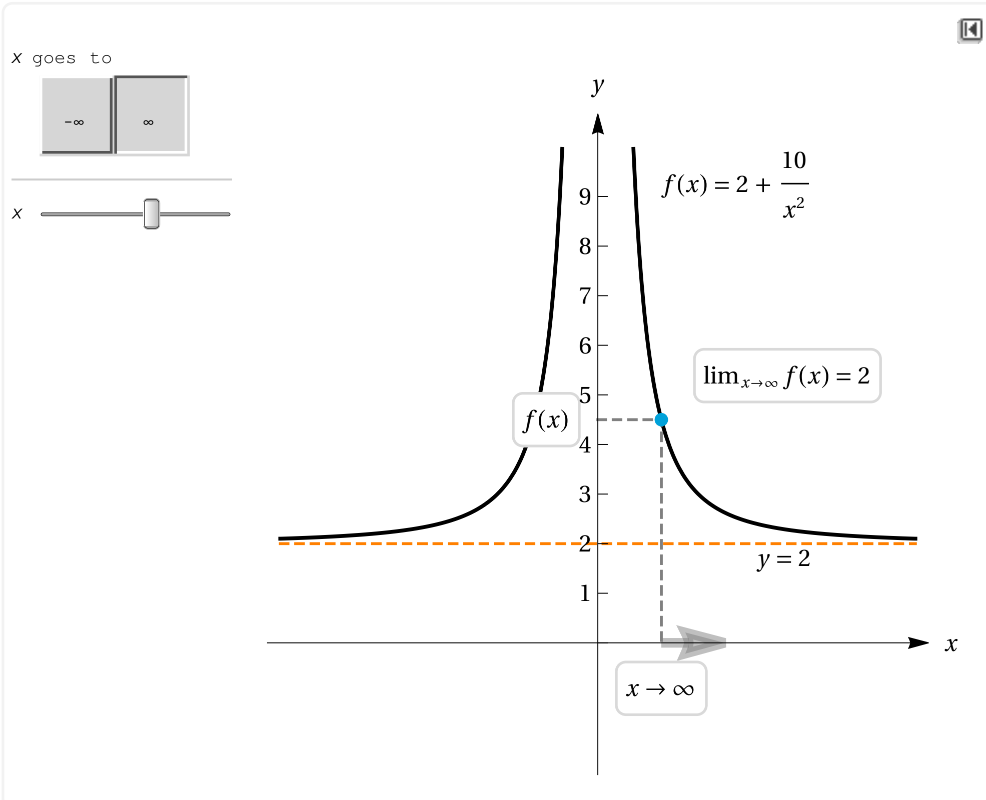


Figure 2.31

b. The numerator of $\frac{\sin x}{\sqrt{x}}$ is bounded between −1 and 1; therefore, for $x > 0$,

$$-\frac{1}{\sqrt{x}} \le \frac{\sin x}{\sqrt{x}} \le \frac{1}{\sqrt{x}}.$$

As $x \to \infty$, $\sqrt{x}$ becomes arbitrarily large, which means that

$$\lim_{x\to\infty}\frac{-1}{\sqrt{x}} = \lim_{x\to\infty}\frac{1}{\sqrt{x}} = 0.$$

It follows by the Squeeze Theorem (Theorem 2.5) that $\lim_{x\to\infty}\frac{\sin x}{\sqrt{x}} = 0$.

Using the limit laws of Theorem 2.3,

$$\lim_{x\to\infty}\left(5+\frac{\sin x}{\sqrt{x}}\right) = \underbrace{\lim_{x\to\infty} 5}_{\text{equals } 5} + \underbrace{\lim_{x\to\infty}\left(\frac{\sin x}{\sqrt{x}}\right)}_{\text{equals } 0} = 5.$$

The graph of $y = 5 + \frac{\sin x}{\sqrt{x}}$ approaches the horizontal asymptote $y = 5$ as x becomes large (**Figure 2.32**). Note that the curve intersects its asymptote infinitely many times.

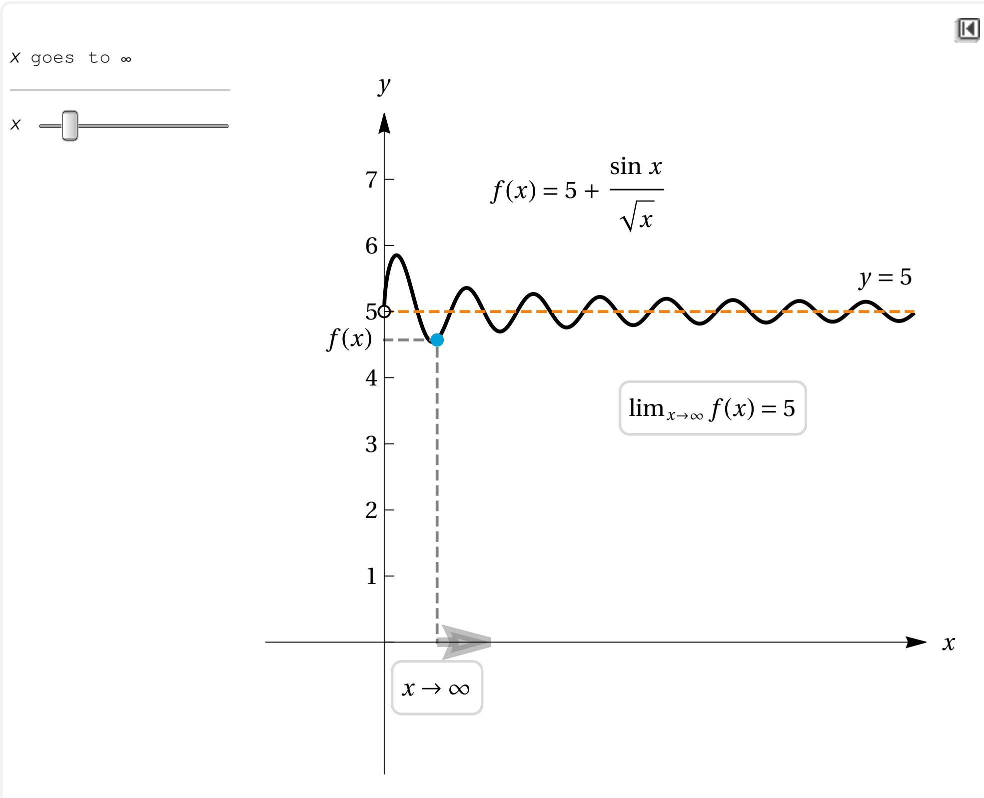


Figure 2.32

Related Exercises 10, 19 ◆

Infinite Limits at Infinity »

It is possible for a limit to be *both* an infinite limit and a limit at infinity. This type of limit occurs if $f(x)$ becomes arbitrarily large in magnitude as x becomes arbitrarily large in magnitude. Such a limit is called an *infinite limit at infinity* and is illustrated by the function $f(x) = x^3$ (**Figure 2.33**).

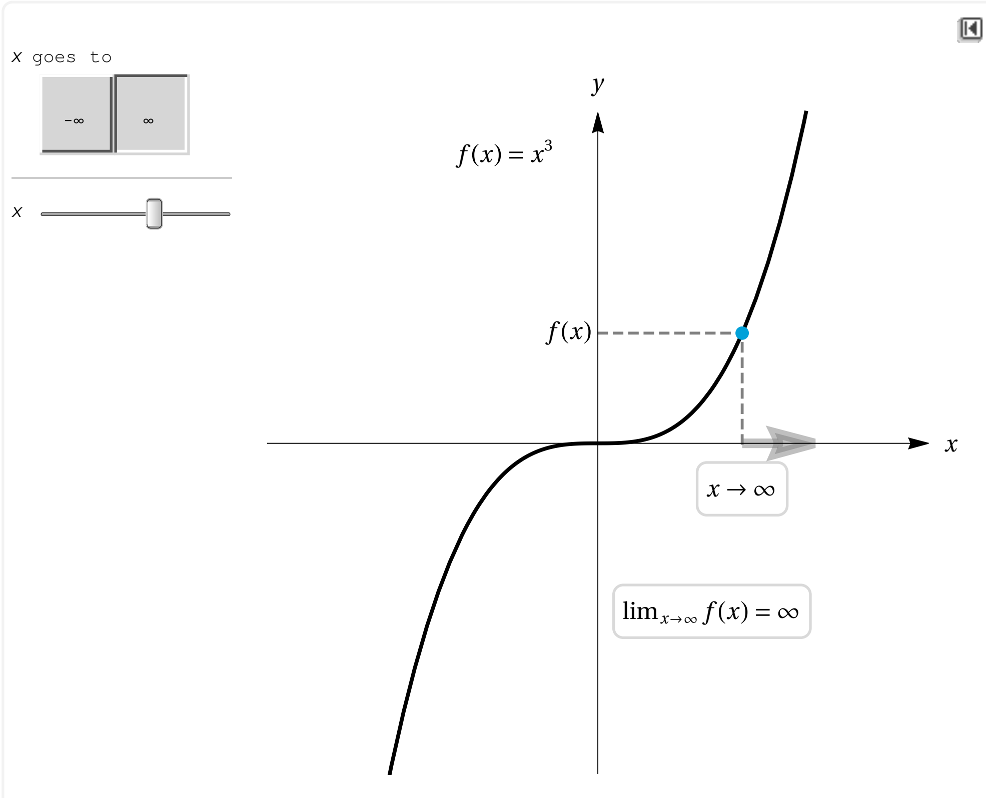


Figure 2.33

DEFINITION **Infinite Limits at Infinity**

If $f(x)$ becomes arbitrarily large as x becomes arbitrarily large, then we write

$$\lim_{x\to\infty} f(x) = \infty.$$

The limits $\lim_{x\to\infty} f(x) = -\infty$, $\lim_{x\to-\infty} f(x) = \infty$, and $\lim_{x\to-\infty} f(x) = -\infty$ are defined similarly.

Infinite limits at infinity tell us about the behavior of polynomials for large-magnitude values of x. First, consider power functions $f(x) = x^n$, where n is a positive integer. **Figure 2.34** shows that when n is even, $\lim_{x\to\pm\infty} x^n = \infty$, and when n is odd, $\lim_{x\to\infty} x^n = \infty$ and $\lim_{x\to-\infty} x^n = -\infty$.

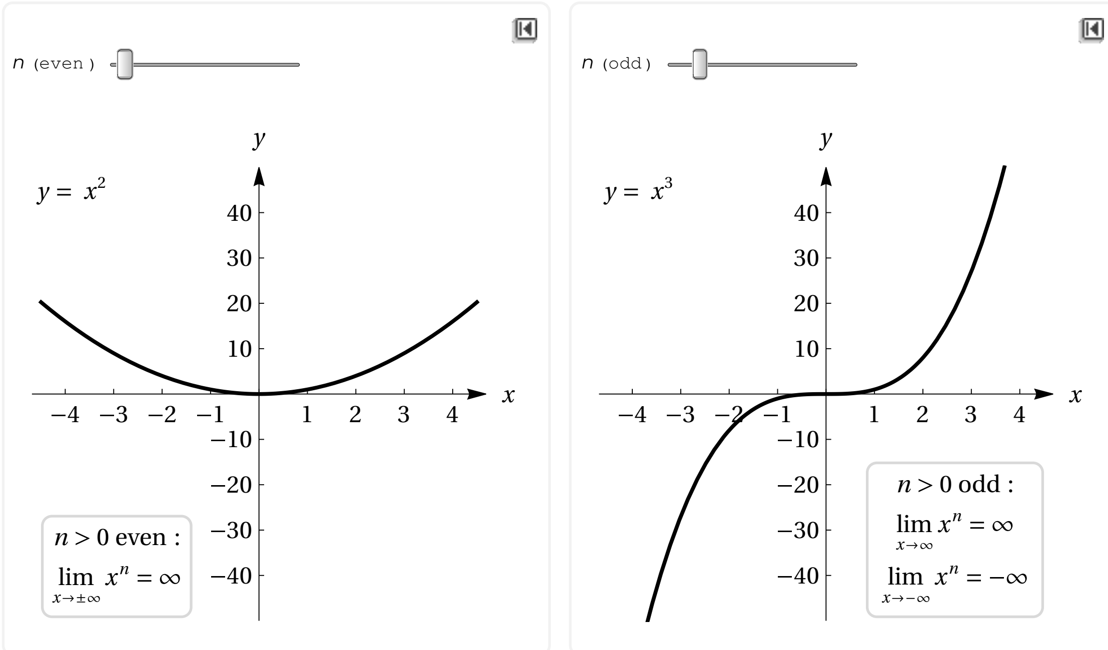


Figure 2.34

It follows that reciprocals of power functions, $f(x) = \dfrac{1}{x^n} = x^{-n}$, where n is a positive integer, behave as follows:

$$\lim_{x\to\infty} \frac{1}{x^n} = \lim_{x\to\infty} x^{-n} = 0 \quad \text{and} \quad \lim_{x\to-\infty} \frac{1}{x^n} = \lim_{x\to-\infty} x^{-n} = 0.$$

From here, it is a short step to finding the behavior of any polynomial as $x \to \pm\infty$. Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$. We now write p in the equivalent form

$$p(x) = x^n \left(a_n + \underbrace{\frac{a_{n-1}}{x}}_{\to 0} + \underbrace{\frac{a_{n-2}}{x^2}}_{\to 0} + \cdots + \underbrace{\frac{a_0}{x^n}}_{\to 0} \right).$$

Notice that as x becomes large in magnitude, all the terms in p except the first term approach zero. Therefore, as $x \to \pm\infty$, we see that $p(x) \approx a_n x^n$. This means that as $x \to \pm\infty$, the behavior of p is determined by the term $a_n x^n$ with the highest power of x.

Quick Check 2 Describe the behavior of $p(x) = -3x^3$ as $x \to \infty$ and as $x \to -\infty$. ◆

Answer »

$p(x) \to -\infty$ as $x \to \infty$ and $p(x) \to \infty$ as $x \to -\infty$

THEOREM 2.6 Limits at Infinity of Powers and Polynomials

Let n be a positive integer and let p be the polynomial

$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$, where $a_n \neq 0$.

1. $\lim_{x \to \pm\infty} x^n = \infty$ when n is even.

2. $\lim_{x \to \infty} x^n = \infty$ and $\lim_{x \to -\infty} x^n = -\infty$ when n is odd.

3. $\lim_{x \to \pm\infty} \frac{1}{x^n} = \lim_{x \to \pm\infty} x^{-n} = 0$.

4. $\lim_{x \to \pm\infty} p(x) = \lim_{x \to \pm\infty} a_n x^n = \pm\infty$, depending on the degree of the polynomial and the sign of the leading coefficient a_n.

EXAMPLE 2 Limits at infinity

Determine the limits as $x \to \pm\infty$ of the following functions.

a. $p(x) = 3x^4 - 6x^2 + x - 10$

b. $q(x) = -2x^3 + 3x^2 - 12$

SOLUTION »

a. We use the fact that the limit is determined by the behavior of the leading term:

$$\lim_{x \to \infty} \left(3x^4 - 6x^2 + x - 10\right) = \lim_{x \to \infty} 3 \underbrace{x^4}_{\to \infty} = \infty.$$

Similarly,

$$\lim_{x \to -\infty} \left(3x^4 - 6x^2 + x - 10\right) = \lim_{x \to -\infty} 3 \underbrace{x^4}_{\to \infty} = \infty.$$

Figure 2.35 illustrates these limits.

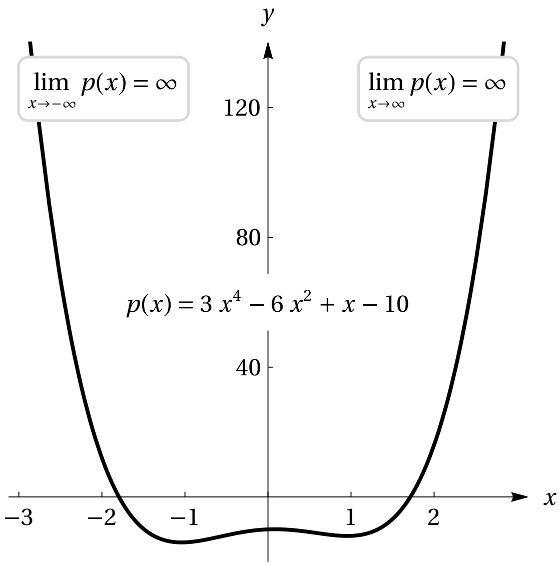


Figure 2.35

b. Noting that the leading coefficient is negative, we have

$$\lim_{x\to\infty} \left(-2x^3 + 3x^2 - 12\right) = \lim_{x\to\infty} \left(-2\underbrace{x^3}_{\to\infty}\right) = -\infty$$

$$\lim_{x\to-\infty} \left(-2x^3 + 3x^2 - 12\right) = \lim_{x\to-\infty} \left(-2\underbrace{x^3}_{\to-\infty}\right) = \infty.$$

The graph of q (**Figure 2.36**) confirms these results.

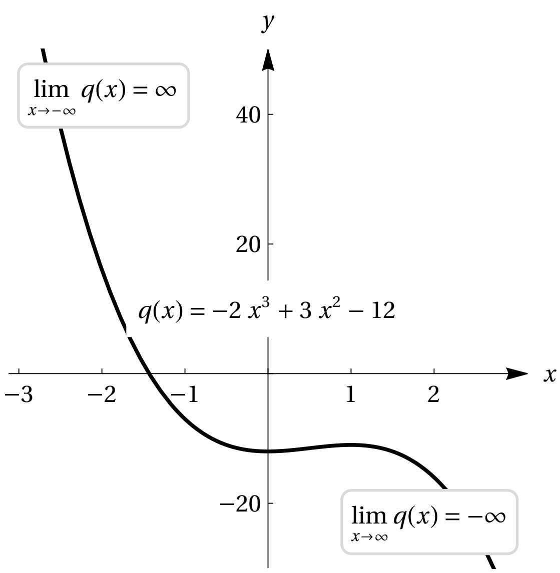


Figure 2.36

Related Exercises 21, 23 ◆

End Behavior »

The behavior of polynomials as $x \to \pm\infty$ is an example of what is often called *end behavior*. Having treated polynomials, we now turn to the end behavior of rational and algebraic functions.

EXAMPLE 3 End behavior of rational functions

Use limits at infinity to determine the end behavior of the following rational functions.

a. $f(x) = \dfrac{3x+2}{x^2-1}$

b. $g(x) = \dfrac{40x^4+4x^2-1}{10x^4+8x^2+1}$

c. $h(x) = \dfrac{x^3-2x+1}{2x+4}$

SOLUTION »

A special case of end behavior arises with rational functions. As shown in the next example, if the graph of a function f approaches a non-horizontal line as $x \to \pm\infty$, then that line is a **slant asymptote**, or **oblique asymptote**, of f.

EXAMPLE 4 Slant asymptotes

Determine the end behavior of the function $f(x) = \dfrac{2x^2+6x-2}{x+1}$.

SOLUTION »

We first divide the numerator and denominator by the largest power of x appearing in the denominator, which is x:

$$\lim_{x\to\infty} \frac{2x^2+6x-2}{x+1} = \lim_{x\to\infty} \frac{\frac{2x^2}{x}+\frac{6x}{x}-\frac{2}{x}}{\frac{x}{x}+\frac{1}{x}} \quad \text{Divide the numerator and denominator by } x.$$

$$= \lim_{x\to\infty} \frac{\overbrace{2x}^{\text{arbitrarily large}} + \overbrace{6}^{\text{constant}} - \overbrace{\frac{2}{x}}^{\text{approaches } 0}}{\underbrace{1}_{\text{constant}} + \underbrace{\frac{1}{x}}_{\text{approaches } 0}} \quad \text{Simplify.}$$

$$= \infty. \quad \text{Take limits.}$$

A similar analysis shows that $\displaystyle\lim_{x\to-\infty} \frac{2x^2+6x-2}{x+1} = -\infty$. Because these limits are not finite, f has no horizontal asymptotes.

However, there is more to be learned about the end behavior of this function. Using long division, the function f is written

$$f(x) = \frac{2x^2 + 6x - 2}{x+1} = \underbrace{2x + 4}_{\ell(x)} - \underbrace{\frac{6}{x+1}}_{\substack{\text{approaches } 0 \\ \text{as } x \to \infty}}.$$

As $x \to \infty$, the term $\frac{6}{x+1}$ approaches 0, and we see that the function f behaves like the linear function $l(x) = 2x + 4$. For this reason, the graphs of f and l approach each other as $x \to \infty$ (**Figure 2.40**). A similar argument shows that the graphs of f and l approach each other as $x \to -\infty$. The line described by l is a slant asymptote; it occurs with rational functions only when the degree of the polynomial in the numerator exceeds the degree of the polynomial in the denominator by exactly 1.

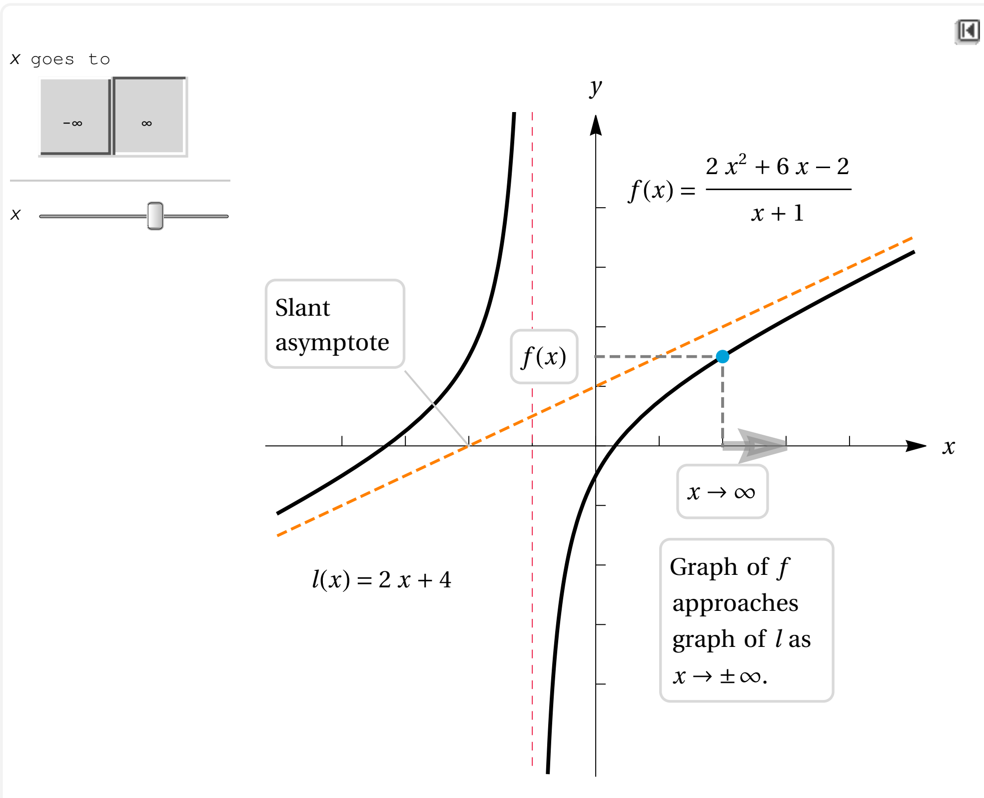


Figure 2.40

Related Exercises 51–52 ◆

The conclusions reached in Examples 3 and 4 can be generalized for all rational functions. These results are summarized in Theorem 2.7; its proof is assigned in Exercise 80.

THEOREM 2.7 End Behavior and Asymptotes of Rational Functions

Suppose $f(x) = \dfrac{p(x)}{q(x)}$ is a rational function, where

$$p(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_2 x^2 + a_1 x + a_0 \text{ and}$$
$$q(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_2 x^2 + b_1 x + b_0$$

and $a_m \neq 0$ and $b_n \neq 0$.

a. If $m < n$, then $\lim_{x\to\pm\infty} f(x) = 0$, and $y = 0$ is a horizontal asymptote of f.

b. If $m = n$, then $\lim_{x\to\pm\infty} f(x) = a_m / b_n$, and $y = a_m / b_n$ is a horizontal asymptote of f.

c. If $m > n$, then $\lim_{x\to\pm\infty} f(x) = \infty$ or $-\infty$, and f has no horizontal asymptote.

d. If $m = n + 1$, then $\lim_{x\to\pm\infty} f(x) = \infty$ or $-\infty$, f has no horizontal asymptote, but f has a slant asymptote.

e. Assuming that $f(x)$ is in reduced form (p and q share no common factors), vertical asymptotes occur at the zeros of q.

Note »

More generally, a non-horizontal line $y = l(x)$ is a slant asymptote of a function f if $\lim_{x\to\infty} (f(x) - l(x)) = 0$ or $\lim_{x\to-\infty} (f(x) - l(x)) = 0$.

Quick Check 3 Use Theorem 2.7 to find the vertical and horizontal asymptotes of $y = \dfrac{10x}{3x - 1}$. ◆

Answer »

Horizontal asymptote is $y = \dfrac{10}{3}$; vertical asymptote is $x = \dfrac{1}{3}$

Although it isn't stated explicitly, Theorem 2.7 implies that a rational function can have at most one horizontal asymptote, and whenever a horizontal asymptote exists, $\lim_{x\to\infty} \dfrac{p(x)}{q(x)} = \lim_{x\to-\infty} \dfrac{p(x)}{q(x)}$. The same cannot be said of other functions, as shown in the next example.

EXAMPLE 5 End behavior of an algebraic function

Use limits at infinity to determine the end behavior of $f(x) = \dfrac{10x^3 - 3x^2 + 8}{\sqrt{25x^6 + x^4 + 2}}$.

SOLUTION »

The square root in the denominator forces us to revise the strategy used with rational functions. First, consider the limit as $x \to \infty$. The highest power of the polynomial in the denominator is 6. However, the polynomial is under a square root, so effectively, the term with the highest power in the denominator is $\sqrt{x^6} = x^3$. Dividing the numerator and denominator by x^3, for $x > 0$, the limit becomes

$$\lim_{x\to\infty} \frac{10x^3 - 3x^2 + 8}{\sqrt{25x^6 + x^4 + 2}} = \lim_{x\to\infty} \frac{\frac{10x^3}{x^3} - \frac{3x^2}{x^3} + \frac{8}{x^3}}{\sqrt{\frac{25x^6}{x^6} + \frac{x^4}{x^6} + \frac{2}{x^6}}} \quad \text{Divide by } \sqrt{x^6} = x^3.$$

$$= \lim_{x\to\infty} \frac{10 - \overbrace{\frac{3}{x}}^{\text{approaches } 0} + \overbrace{\frac{8}{x^3}}^{\text{approaches } 0}}{\sqrt{25 + \underbrace{\frac{1}{x^2}}_{\text{approaches } 0} + \underbrace{\frac{2}{x^6}}_{\text{approaches } 0}}} \quad \text{Simplify.}$$

$$= \frac{10}{\sqrt{25}} = 2. \quad \text{Evaluate limits.}$$

As $x \to -\infty$, x^3 is negative, so we divide numerator and denominator by $\sqrt{x^6} = -x^3$ (which is positive):

Note »

$$\lim_{x\to-\infty} \frac{10x^3 - 3x^2 + 8}{\sqrt{25x^6 + x^4 + 2}} = \lim_{x\to-\infty} \frac{\frac{10x^3}{-x^3} - \frac{3x^2}{-x^3} + \frac{8}{-x^3}}{\sqrt{\frac{25x^6}{x^6} + \frac{x^4}{x^6} + \frac{2}{x^6}}} \quad \text{Divide by } \sqrt{x^6} = -x^3 > 0.$$

$$= \lim_{x\to-\infty} \frac{-10 + \overbrace{\frac{3}{x}}^{\text{approaches } 0} - \overbrace{\frac{8}{x^3}}^{\text{approaches } 0}}{\sqrt{25 + \underbrace{\frac{1}{x^2}}_{\text{approaches } 0} + \underbrace{\frac{2}{x^6}}_{\text{approaches } 0}}} \quad \text{Simplify.}$$

$$= -\frac{10}{\sqrt{25}} = -2. \quad \text{Evaluate limits.}$$

The limits reveal two asymptotes, $y = 2$ and $y = -2$. Observe that the graph crosses both horizontal asymptotes (**Figure 2.41**).

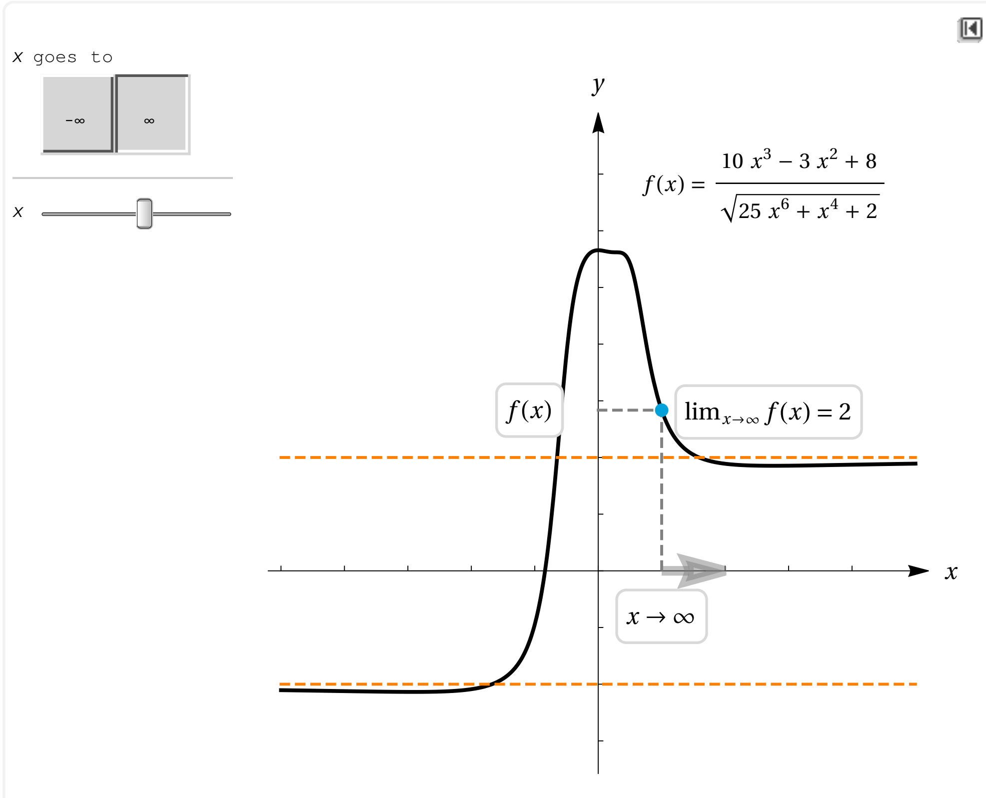


Figure 2.41

Related Exercises 46–47 ◆

End Behavior of sin *x* and cos *x*

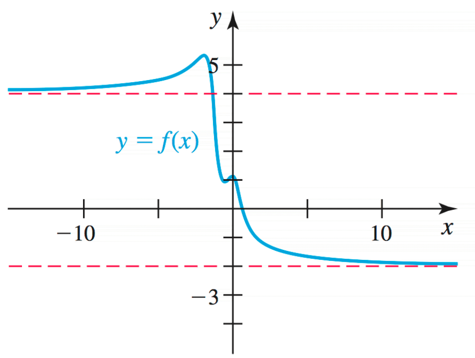
Our future work requires knowing about the end behavior of $\sin x$ and $\cos x$. The values of both functions oscillate between -1 and 1 as x increases in magnitude. Therefore, $\lim_{x\to\pm\infty} \sin x$ and $\lim_{x\to\pm\infty} \cos x$ do not exist. However, both functions are bounded as $x \to \pm\infty$; that is, $|\sin x| \le 1$ and $|\cos x| \le 1$ for all x.

Exercises »

Getting Started »

1. Explain the meaning of $\lim_{x\to-\infty} f(x) = 10$.

2. Evaluate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ using the figure.



3–13. *Determine the following limits at infinity.*

3. $\lim_{x\to\infty} x^{12}$

4. $\lim_{x\to-\infty} 3x^{11}$

5. $\lim_{x\to\infty} x^{-6}$

6. $\lim_{x\to-\infty} x^{-11}$

7. $\lim_{t\to\infty} \left(-12t^{-5}\right)$

8. $\lim_{x\to-\infty} 2x^{-8}$

9. $\lim_{x\to\infty} \left(3+\frac{10}{x^2}\right)$

10. $\lim_{x\to\infty} \left(5+\frac{1}{x}+\frac{10}{x^2}\right)$

11. $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ if $f(x)\to 100{,}000$ and $g(x)\to\infty$ as $x\to\infty$

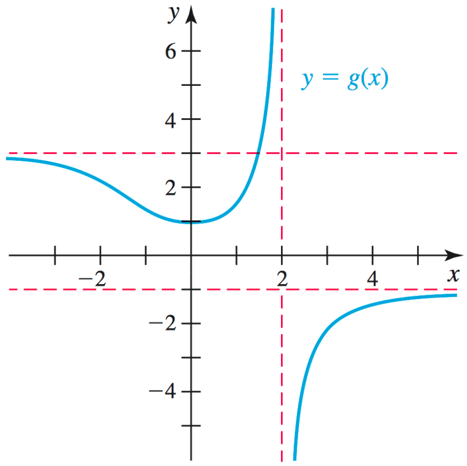
12. $\lim_{x\to\infty} \frac{4x^2+2x+3}{x^2}$

13. $\lim_{x\to\infty} \frac{1-x}{2x}$, $\lim_{x\to-\infty} \frac{1-x}{x^2}$, and $\lim_{x\to\infty} \frac{1-x^2}{2x}$

14. Describe with a sketch the end behavior of $f(x)=\cos x$.

15. Suppose the function g satisfies the inequality $3-\frac{1}{x^2}\le g(x)\le 3+\frac{1}{x^2}$ for all nonzero values of x. Evaluate $\lim_{x\to\infty} g(x)$ and $\lim_{x\to-\infty} g(x)$.

16. The graph of g has a vertical asymptote at $x = 2$ and horizontal asymptotes at $y = -1$ and $y = 3$ (see figure). Determine the following limits: $\lim_{x \to -\infty} g(x)$, $\lim_{x \to \infty} g(x)$, $\lim_{x \to 2^-} g(x)$, and $\lim_{x \to 2^+} g(x)$.



Practice Exercises »

17–36. Limits at infinity *Determine the following limits.*

17. $\lim_{\theta \to \infty} \frac{\cos \theta}{\theta^2}$

18. $\lim_{t \to \infty} \frac{5\,t^2 + t \sin t}{t^2}$

19. $\lim_{x \to \infty} \frac{\cos x^5}{\sqrt{x}}$

20. $\lim_{x \to -\infty} \left(5 + \frac{100}{x} + \frac{\sin^4 x^3}{x^2}\right)$

21. $\lim_{x \to \infty} \left(3\,x^{12} - 9\,x^7\right)$

22. $\lim_{x \to -\infty} \left(3\,x^7 + x^2\right)$

23. $\lim_{x \to -\infty} \left(-3\,x^{16} + 2\right)$

24. $\lim_{x \to -\infty} \left(2\,x^{-8} + 4\,x^3\right)$

25. $\lim_{x \to \infty} \frac{14\,x^3 + 3\,x^2 - 2\,x}{21\,x^3 + x^2 + 2\,x + 1}$

26. $\lim_{x \to \infty} \frac{9\,x^3 + x^2 - 5}{3\,x^4 + 4\,x^2}$

27. $\lim_{x \to -\infty} \frac{3x^2 + 3x}{x + 1}$

28. $\lim_{x \to \infty} \frac{x^4 + 7}{x^5 + x^2 - x}$

29. $\lim_{w \to \infty} \frac{15w^2 + 3w + 1}{\sqrt{9w^4 + w^3}}$

30. $\lim_{x \to -\infty} \frac{40x^4 + x^2 + 5x}{\sqrt{64x^8 + x^6}}$

31. $\lim_{x \to -\infty} \frac{\sqrt{16x^2 + x}}{x}$

32. $\lim_{x \to \infty} \frac{6x^2}{4x^2 + \sqrt{16x^4 + x^2}}$

33. $\lim_{x \to \infty} \left(x^2 - \sqrt{x^4 + 3x^2}\right)$ (*Hint*: Multiply by $\frac{x^2 + \sqrt{x^4 + 3x^2}}{x^2 + \sqrt{x^4 + 3x^2}}$ first.)

34. $\lim_{x \to -\infty} \left(x + \sqrt{x^2 - 5x}\right)$

35. $\lim_{x \to \infty} \frac{\sin x}{x^2 + 1}$

36. $\lim_{x \to -\infty} \frac{x^4 \cos x}{3x^6 + 7x^4}$

37–50. Horizontal asymptotes *Determine* $\lim_{x \to \infty} f(x)$ *and* $\lim_{x \to -\infty} f(x)$ *for the following functions. Then give the horizontal asymptotes of f (if any).*

37. $f(x) = \frac{4x}{20x + 1}$

38. $f(x) = \frac{3x^2 - 7}{x^2 + 5x}$

39. $f(x) = \frac{6x^2 - 9x + 8}{3x^2 + 2}$

40. $f(x) = \frac{12x^8 - 3}{3x^8 - 2x^7}$

41. $f(x) = \dfrac{3x^3 - 7}{x^4 + 5x^2}$

42. $f(x) = \dfrac{2x + 1}{3x^4 - 2}$

43. $f(x) = \dfrac{40x^5 + x^2}{16x^4 - 2x}$

44. $f(x) = \dfrac{6x^2 + 1}{\sqrt{4x^4 + 3x + 1}}$

45. $f(x) = \dfrac{1}{2x^4 - \sqrt{4x^8 - 9x^4}}$

46. $f(x) = \dfrac{\sqrt{x^2 + 1}}{2x + 1}$

47. $f(x) = \dfrac{4x^3 + 1}{2x^3 + \sqrt{16x^6 + 1}}$

48. $f(x) = x - \sqrt{x^2 - 9x}$

49. $f(x) = \dfrac{\sqrt[3]{x^6 + 8}}{4x^2 + \sqrt{3x^4 + 1}}$

50. $f(x) = 4x\left(3x - \sqrt{9x^2 + 1}\right)$

T **51–56. Slant (oblique) asymptotes** *Complete the following steps for the given functions.*

a. *Find the slant asymptote of f.*

b. *Find the vertical asymptotes of f (if any).*

c. *Graph f and all of its asymptotes with a graphing utility. Then sketch a graph of the function by hand, correcting any errors appearing in the computer-generated graph.*

51. $f(x) = \dfrac{x^2 - 3}{x + 6}$

52. $f(x) = \dfrac{x^2 - 1}{x + 2}$

53. $f(x) = \dfrac{x^2 - 2x + 5}{3x - 2}$

54. $f(x) = \dfrac{5x^2 - 4}{5x - 5}$

55. $f(x)=\dfrac{4x^3+4x^2+7x+4}{x^2+1}$

56. $f(x)=\dfrac{3x^2-2x+5}{3x+4}$

57. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.

a. The graph of a function can never cross one of its horizontal asymptotes.

b. A rational function f has both $\lim_{x\to\infty} f(x)=L$ (where L is finite) and $\lim_{x\to-\infty} f(x)=\infty$.

c. The graph of a function can have any number of vertical asymptotes but at most two horizontal asymptotes.

d. $\lim_{x\to\infty}\left(x^3-x\right)=\lim_{x\to\infty} x^3-\lim_{x\to\infty} x=\infty-\infty=0$

58–61. Steady states *If a function f represents a system that varies in time, the existence of* $\lim_{t\to\infty} f(t)$ *means that the system reaches a steady state (or equilibrium). For the following systems, determine whether a steady state exists and give the steady-state value.*

58. The population of a bacteria culture is given by $p(t)=\dfrac{2500}{t+1}$.

59. The population of a culture of tumor cells is given by $p(t)=\dfrac{3500\,t}{t+1}$.

60. The population of a colony of squirrels is given by $p(t)=\dfrac{1500\,t^2}{2t^2+3}$.

61. The amplitude of an oscillator is given by $a(t)=2\left(\dfrac{t+\sin t}{t}\right)$.

62–73. Horizontal and vertical asymptotes

a. *Analyze* $\lim_{x\to\infty} f(x)$ *and* $\lim_{x\to-\infty} f(x)$*, and then identify any horizontal asymptotes.*

b. *Find the vertical asymptotes. For each vertical asymptote* $x=a$*, analyze* $\lim_{x\to a^-} f(x)$ *and* $\lim_{x\to a^+} f(x)$*.*

62. $f(x)=\dfrac{x^2-4x+3}{x-1}$

63. $f(x)=\dfrac{2x^3+10x^2+12x}{x^3+2x^2}$

64. $f(x)=\dfrac{\sqrt{16x^4+64x^2}+x^2}{2x^2-4}$

65. $f(x)=\dfrac{3x^4+3x^3-36x^2}{x^4-25x^2+144}$

66. $f(x) = x^2\left(4x^2 - \sqrt{16x^4 + 1}\right)$

67. $f(x) = \dfrac{x^2 - 9}{x(x - 3)}$

68. $f(x) = \dfrac{x^4 - 1}{x^2 - 1}$

69. $f(x) = \dfrac{\sqrt{x^2 + 2x + 6} - 3}{x - 1}$

70. $f(x) = \dfrac{\left|1 - x^2\right|}{x(x + 1)}$

71. $f(x) = \sqrt{|x|} - \sqrt{|x - 1|}$

72. $f(x) = \dfrac{2x}{\sqrt{x^2 - x - 2}}$

73. $f(x) = \dfrac{\cos x + 2\sqrt{x}}{\sqrt{x}}$

74–75. Sketching graphs *Sketch a possible graph of a function f that satisfies all of the given conditions. Be sure to identify all vertical and horizontal asymptotes.*

74. $f(-1) = -2,\ f(1) = 2,\ f(0) = 0,\ \lim_{x\to\infty} f(x) = 1,\ \lim_{x\to-\infty} f(x) = -1$

75. $\lim_{x\to0^+} f(x) = \infty,\ \lim_{x\to0^-} f(x) = -\infty,\ \lim_{x\to\infty} f(x) = 1,\ \lim_{x\to-\infty} f(x) = -2$

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76–79. Looking ahead to sequences *A sequence is an infinite, ordered list of numbers that is often defined by a function. For example, the sequence $\{2, 4, 6, 8, \ldots\}$ is specified by the function $f(n) = 2n$, where $n = 1, 2, 3, \ldots$. The limit of such a sequence is $\lim_{n\to\infty} f(n)$, provided the limit exists. All the limit laws for limits at infinity may be applied to limits of sequences. Find the limit of the following sequences or state that the limit does not exist.*

76. $\left\{4, 2, \frac{4}{3}, 1, \frac{4}{5}, \frac{2}{3}, \ldots\right\}$, which is defined by $f(n) = \frac{4}{n}$, for $n = 1, 2, 3, \ldots$

77. $\left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\right\}$, which is defined by $f(n) = \frac{n-1}{n}$, for $n = 1, 2, 3, \ldots$

78. $\left\{\frac{1}{2}, \frac{4}{3}, \frac{9}{4}, \frac{16}{5}, \ldots\right\}$, which is defined by $f(n) = \frac{n^2}{n+1}$, for $n = 1, 2, 3, \ldots$

79. $\left\{2, \frac{3}{4}, \frac{4}{9}, \frac{5}{16}, \ldots\right\}$, which is defined by $f(n) = \frac{n+1}{n^2}$, for $n = 1, 2, 3, \ldots$

80. **End behavior of rational functions** Suppose $f(x) = \frac{p(x)}{q(x)}$ is a rational function, where $p(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_2 x^2 + a_1 x + a_0$, $q(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_2 x^2 + b_1 x + b_0$, $a_m \neq 0$, and $b_n \neq 0$.

a. Prove that if $m = n$, then $\lim\limits_{x \to \pm\infty} f(x) = \frac{a_m}{b_n}$.

b. Prove that if $m < n$, then $\lim\limits_{x \to \pm\infty} f(x) = 0$.

81. **Horizontal and slant asymptotes**

a. Is it possible for a rational function to have both slant and horizontal asymptotes? Explain.

b. Is it possible for an algebraic function to have two distinct slant asymptotes? Explain or give an example.