2.3 Techniques for Computing Limits

Graphical and numerical techniques for estimating limits, like those presented in the previous section, provide intuition about limits. These techniques, however, occasionally lead to incorrect results. Therefore, we turn our attention to analytical methods for evaluating limits precisely.

Limits of Linear Functions »

The graph of $f(x) = mx + b$ is a line with slope m and y-intercept b. From **Figure 2.15**, we see that for any value of a, $f(x)$ approaches $f(a)$ as x approaches a. Therefore, if f is a linear function we have $\lim_{x \to a} f(x) = f(a)$. It follows that for linear functions, $\lim_{x \to a} f(x)$ is found by direct substitution of $x = a$ into $f(x)$. This observation leads to the following theorem, which is proved in Exercise 39 of Section 2.7.

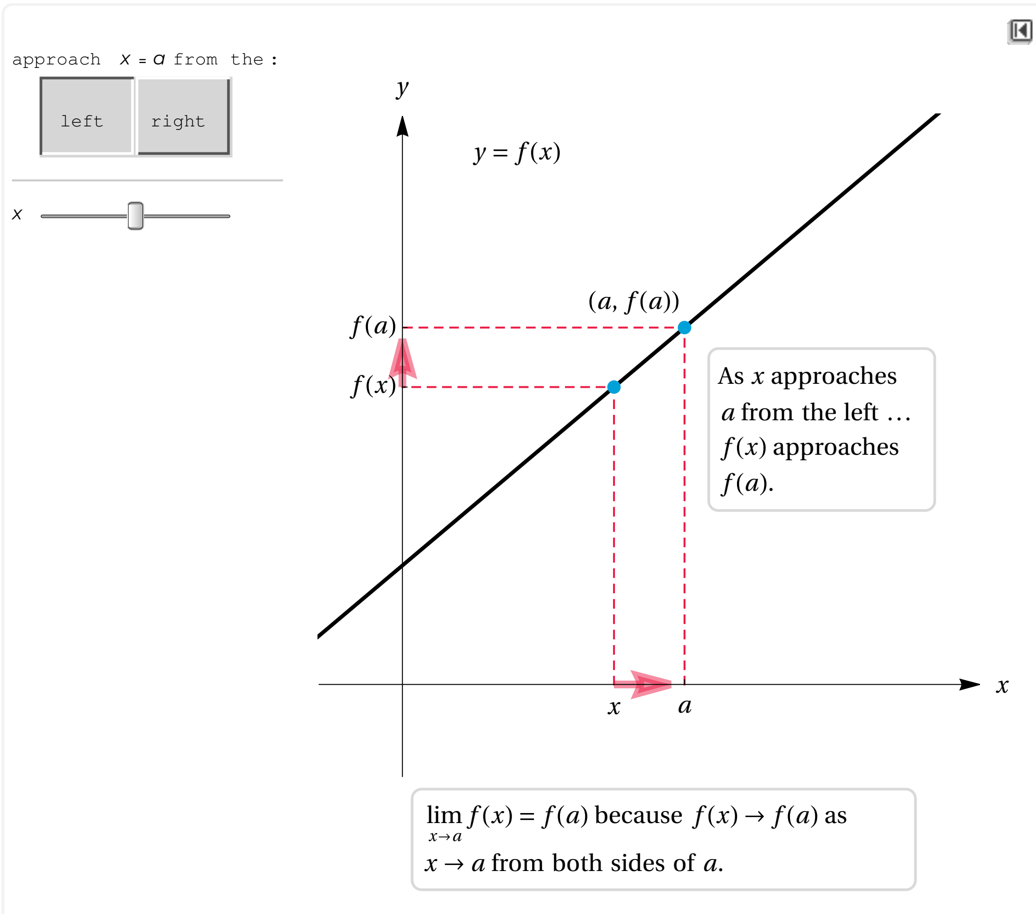


Figure 2.15

THEOREM 2.2 Limits of Linear Functions

Let a, b, and m be real numbers. For linear functions $f(x) = mx + b$,

$$\lim_{x \to a} f(x) = f(a) = ma + b.$$

EXAMPLE 1 **Limits of linear functions**

Evaluate the following limits.

a. $\lim_{x \to 3} f(x)$, where $f(x) = \frac{1}{2}x - 7$

b. $\lim_{x \to 2} g(x)$, where $g(x) = 6$

SOLUTION »

a. $\lim_{x \to 3} f(x) = \lim_{x \to 3} \left(\frac{1}{2}x - 7\right) = f(3) = -\frac{11}{2}$.

b. $\lim_{x \to 2} g(x) = \lim_{x \to 2} 6 = g(2) = 6$.

Related Exercises 19, 22 ◆

Limit Laws »

The following limit laws greatly simplify the evaluation of many limits.

THEOREM 2.3 **Limit Laws**

Assume $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist. The following properties hold, where c is a real number, and $n > 0$ is an integer.

1. **Sum** $\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
2. **Difference** $\lim_{x \to a} (f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$
3. **Constant multiple** $\lim_{x \to a} (c\, f(x)) = c \lim_{x \to a} f(x)$
4. **Product** $\lim_{x \to a} (f(x)\, g(x)) = \left(\lim_{x \to a} f(x)\right)\left(\lim_{x \to a} g(x)\right)$
5. **Quotient** $\lim_{x \to a} \left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, provided $\lim_{x \to a} g(x) \neq 0$
6. **Power** $\lim_{x \to a} (f(x))^n = \left(\lim_{x \to a} f(x)\right)^n$
7. **Root** $\lim_{x \to a} (f(x))^{1/n} = \left(\lim_{x \to a} f(x)\right)^{1/n}$, provided $f(x) > 0$, for x near a, if n is even

A proof of Law 1 is given in Example 6 of Section 2.7; the proofs of Laws 2 and 3 are asked for in Exercises 43 and 44 of the same section. Laws 4 and 5 are proved in Appendix A. Law 6 is proved from Law 4 as follows.

For a positive integer n, if $\lim_{x \to a} f(x)$ exists, we have

$$\begin{aligned}\lim_{x\to a}(f(x))^n &= \lim_{x\to a}\underbrace{(f(x)\cdot f(x)\cdots f(x))}_{n\text{ factors of } f(x)}\\ &= \underbrace{\left(\lim_{x\to a} f(x)\right)\cdot\left(\lim_{x\to a} f(x)\right)\cdots\left(\lim_{x\to a} f(x)\right)}_{n\text{ factors of } \lim_{x\to a} f(x)} \quad \text{Repeated use of Law 4}\\ &= \left(\lim_{x\to a} f(x)\right)^n.\end{aligned}$$

Law 7 is a direct consequence of Theorem 2.11 (Section 2.6).

EXAMPLE 2 Evaluating limits

Suppose $\lim_{x\to 2} f(x) = 4$, $\lim_{x\to 2} g(x) = 5$, and $\lim_{x\to 2} h(x) = 8$. Use the limit laws in Theorem 2.3 to compute each limit.

a. $\lim_{x\to 2} \dfrac{f(x) - g(x)}{h(x)}$

b. $\lim_{x\to 2} (6\, f(x)\, g(x) + h(x))$

c. $\lim_{x\to 2} (g(x))^3$

SOLUTION »

a.

$$\begin{aligned}\lim_{x\to 2}\frac{f(x)-g(x)}{h(x)} &= \frac{\lim_{x\to 2}(f(x)-g(x))}{\lim_{x\to 2} h(x)} && \text{Law 5}\\ &= \frac{\lim_{x\to 2} f(x) - \lim_{x\to 2} g(x)}{\lim_{x\to 2} h(x)} && \text{Law 2}\\ &= \frac{4-5}{8} = -\frac{1}{8}\end{aligned}$$

b.

$$\begin{aligned}\lim_{x\to 2}(6\, f(x)\, g(x) + h(x)) &= \lim_{x\to 2}(6\, f(x)\, g(x)) + \lim_{x\to 2} h(x) && \text{Law 1}\\ &= 6\cdot\lim_{x\to 2}(f(x)\, g(x)) + \lim_{x\to 2} h(x) && \text{Law 3}\\ &= 6\cdot\left(\lim_{x\to 2} f(x)\right)\cdot\left(\lim_{x\to 2} g(x)\right) + \lim_{x\to 2} h(x) && \text{Law 4}\\ &= 6\cdot 4\cdot 5 + 8 = 128\end{aligned}$$

c.

$$\lim_{x\to 2}(g(x))^3 = \left(\lim_{x\to 2} g(x)\right)^3 = 5^3 = 125 \quad \text{Law 6}$$

Related Exercises 11–12 ◆

Limits of Polynomial and Rational Functions »

The limit laws are now used to find the limits of polynomial and rational functions. For example, to evaluate the limit of the polynomial $p(x) = 7x^3 + 3x^2 + 4x + 2$ at an arbitrary point a, we proceed as follows:

$$\begin{aligned}\lim_{x\to a} p(x) &= \lim_{x\to a}\left(7x^3+3x^2+4x+2\right)\\ &= \lim_{x\to a}\left(7x^3\right)+\lim_{x\to a}\left(3x^2\right)+\lim_{x\to a}(4x+2) && \text{Law 1}\\ &= 7\lim_{x\to a}\left(x^3\right)+3\lim_{x\to a}\left(x^2\right)+\lim_{x\to a}(4x+2) && \text{Law 3}\\ &= 7(\underbrace{\lim_{x\to a}x}_{a})^3+3(\underbrace{\lim_{x\to a}x}_{a})^2+\underbrace{\lim_{x\to a}(4x+2)}_{4a+2} && \text{Law 6}\\ &= 7a^3+3a^2+4a+2=p(a). && \text{Theorem 2.2}\end{aligned}$$

As in the case of linear functions, the limit of a polynomial is found by direct substitution; that is, $\lim_{x\to a} p(x)=p(a)$ (Exercise 105).

It is now a short step to evaluating limits of rational functions of the form $f(x)=\frac{p(x)}{q(x)}$, where p and q are polynomials. Applying Law 5, we have

$$\lim_{x\to a}\frac{p(x)}{q(x)}=\frac{\lim_{x\to a}p(x)}{\lim_{x\to a}q(x)}=\frac{p(a)}{q(a)},\quad \text{provided } q(a)\neq 0,$$

which shows that limits of rational functions are also evaluated by direct substitution.

Note »

The conditions under which direct substitution ($\lim_{x\to a} f(x)=f(a)$) can be used to evaluate a limit become clear in Section 2.6, when the important property of *continuity* is discussed.

THEOREM 2.4 Limits of Polynomial and Rational Functions

Assume p and q are polynomials and a is a constant.

a. Polynomial functions: $\lim_{x\to a} p(x)=p(a)$

b. Rational functions: $\lim_{x\to a}\frac{p(x)}{q(x)}=\frac{p(a)}{q(a)}$, provided $q(a)\neq 0$

Quick Check 1 Use Theorem 2.4 to evaluate $\lim_{x\to 2}\left(2x^4-8x-16\right)$ and $\lim_{x\to -1}\frac{x-1}{x}$. ◆

Answer »

0, 2

EXAMPLE 3 Limit of a rational function

Evaluate $\lim_{x\to 2}\frac{3x^2-4x}{5x^3-36}$.

SOLUTION »

Notice that the denominator of this function is nonzero at $x=2$. Using Theorem 2.4b, we find that

$$\lim_{x\to 2} \frac{3x^2 - 4x}{5x^3 - 36} = \frac{3\left(2^2\right) - 4(2)}{5\left(2^3\right) - 36} = 1.$$

Related Exercise 25 ◆

Quick Check 2 Use Theorem 2.4 to compute $\lim_{x\to 1} \frac{5x^4 - 3x^2 + 8x - 6}{x+1}$. ◆

Answer »

2

EXAMPLE 4 An algebraic function

Evaluate $\lim_{x\to 2} \frac{\sqrt{2x^3+9} + 3x - 1}{4x+1}$.

SOLUTION »

Using Theorems 2.3 and 2.4, we have

$$\begin{aligned}
\lim_{x\to 2} \frac{\sqrt{2x^3+9} + 3x - 1}{4x+1} &= \frac{\lim_{x\to 2}\left(\sqrt{2x^3+9} + 3x - 1\right)}{\lim_{x\to 2}(4x+1)} && \text{Law 5} \\
&= \frac{\sqrt{\lim_{x\to 2}\left(2x^3+9\right)} + \lim_{x\to 2}(3x-1)}{\lim_{x\to 2}(4x+1)} && \text{Laws 1 and 7} \\
&= \frac{\sqrt{\left(2(2)^3+9\right)} + (3(2)-1)}{(4(2)+1)} && \text{Theorem 2.4} \\
&= \frac{\sqrt{25} + 5}{9} = \frac{10}{9}.
\end{aligned}$$

Notice that the limit at $x = 2$ equals the value of the function at $x = 2$.

Related Exercises 26–27 ◆

One-Sided Limits »

Theorem 2.2, Limit Laws 1–6, and Theorem 2.4 also hold for left-sided and right-sided limits. In other words, these laws remain valid if we replace $\lim_{x\to a}$ with $\lim_{x\to a^+}$ or $\lim_{x\to a^-}$. Law 7 must be modified slightly for one-sided limits, as shown below.

THEOREM 2.3 (CONTINUED) **Limit Laws for One-Sided Limits**

Laws 1–6 hold with $\lim_{x\to a}$ replaced by $\lim_{x\to a^+}$ or $\lim_{x\to a^-}$. Law 7 is modified as follows. Assume $n>0$ is an integer.

7. Root

a. $\lim_{x\to a^+}(f(x))^{1/n}=\left(\lim_{x\to a^+}f(x)\right)^{1/n}$, provided $f(x)\ge 0$, for x near a with $x>a$, if n is even

b. $\lim_{x\to a^-}(f(x))^{1/n}=\left(\lim_{x\to a^-}f(x)\right)^{1/n}$, provided $f(x)\ge 0$, for x near a with $x<a$, if n is even

EXAMPLE 5 Calculating left- and right-sided limits

Let

$$f(x)=\begin{cases}-2x+4 & \text{if } x\le 1\\ \sqrt{x-1} & \text{if } x>1.\end{cases}$$

Find the values of $\lim_{x\to 1^-}f(x)$, $\lim_{x\to 1^+}f(x)$, and $\lim_{x\to 1}f(x)$, or state that they do not exist.

SOLUTION »

Notice that $f(x)=-2x+4$, for $x\le 1$. Therefore,

$$\lim_{x\to 1^-}f(x)=\lim_{x\to 1^-}(-2x+4)=2. \quad \text{Theorem 2.2}$$

For $x>1$, note that $x-1>0$; it follows that

$$\lim_{x\to 1^+}f(x)=\lim_{x\to 1^+}\sqrt{x-1}=\sqrt{\lim_{x\to 1^+}(x-1)}=0. \quad \text{Law 7 for one-sided limits}$$

Because $\lim_{x\to 1^-}f(x)\ne\lim_{x\to 1^+}f(x)$, $\lim_{x\to 1}f(x)$ does not exist by Theorem 2.1. The graph of f (**Figure 2.16**) is consistent with these findings.

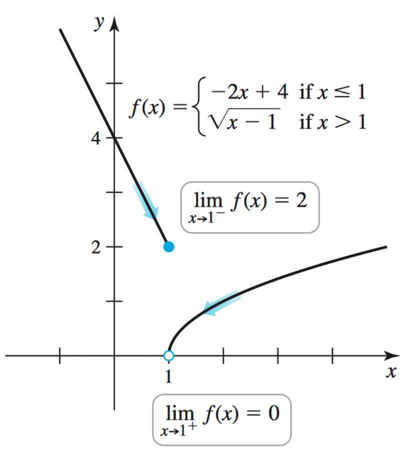


Figure 2.16

Related Exercises 72–73 ◆

Other Techniques »

So far, we have evaluated limits by direct substitution. A more challenging problem is finding $\lim_{x\to a} f(x)$ when the limit exists, but $\lim_{x\to a} f(x) \neq f(a)$. Two typical cases are shown in **Figure 2.17**. In the first case, $f(a)$ is defined, but it is not equal to $\lim_{x\to a} f(x)$; in the second case, $f(a)$ is not defined at all. In both cases, direct substitution does not work—we need a new strategy. One way to evaluate a challenging limit is to replace it with an equivalent limit that can be evaluated by direct substitution. Example 6 illustrates two common scenarios.

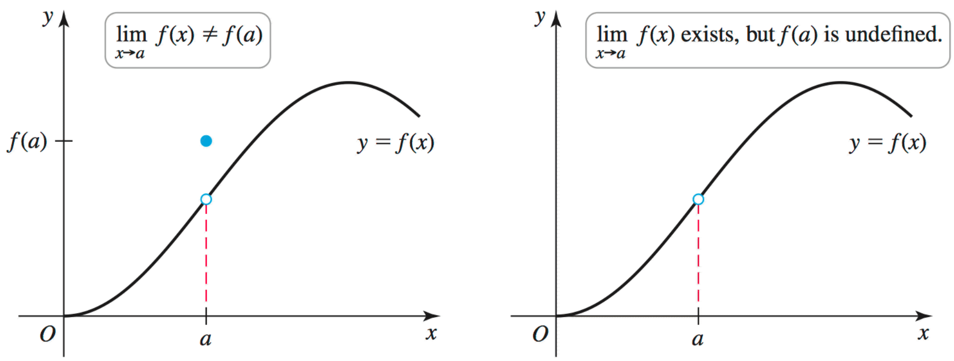


Figure 2.17

EXAMPLE 6 **Other techniques**

Evaluate the following limits.

a. $\lim_{x\to 2} \dfrac{x^2 - 6x + 8}{x^2 - 4}$

b. $\lim_{x\to 1} \dfrac{\sqrt{x} - 1}{x - 1}$

SOLUTION »

a. **Factor and cancel** This limit cannot be found by direct substitution because the denominator is zero when $x = 2$. Instead, the numerator and denominator are factored; then, assuming $x \neq 2$, we cancel like factors:

$$\frac{x^2 - 6x + 8}{x^2 - 4} = \frac{(x-2)(x-4)}{(x-2)(x+2)} = \frac{x-4}{x+2}.$$

Note »

Because $\dfrac{x^2 - 6x + 8}{x^2 - 4} = \dfrac{x-4}{x+2}$ whenever $x \neq 2$, the two functions have the same limit as x approaches 2 (**Figure 2.18**). Therefore,

$$\lim_{x\to 2} \frac{x^2 - 6x + 8}{x^2 - 4} = \lim_{x\to 2} \frac{x-4}{x+2} = \frac{2-4}{2+2} = -\frac{1}{2}.$$

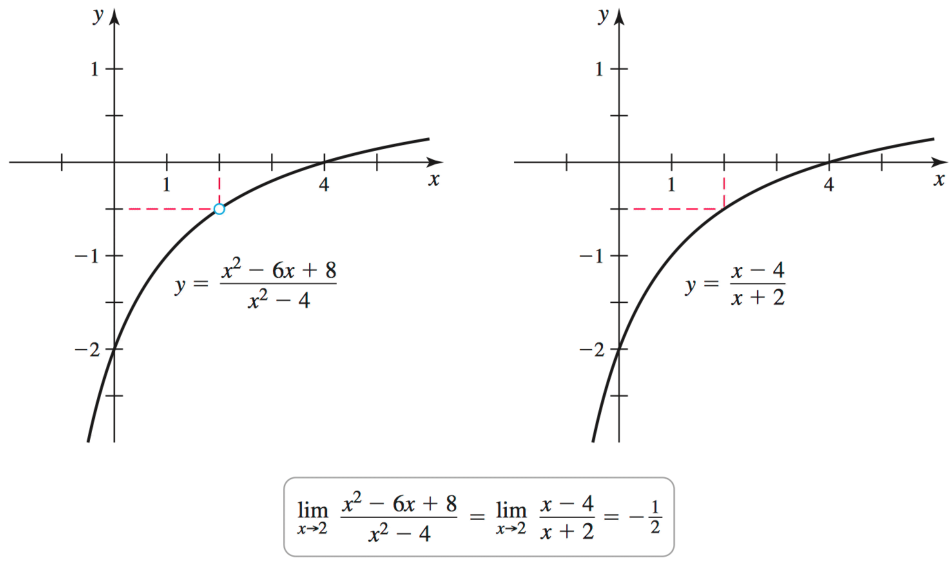


Figure 2.18

b. Use conjugates This limit was approximated numerically in Example 2 of Section 2.2; we made a conjecture that the value of the limit is $1/2$. Using direct substitution to verify this conjecture fails in this case, because the denominator is zero at $x = 1$. Instead, we first simplify the function by multiplying the numerator and denominator by the *algebraic conjugate* of the numerator. The conjugate of $\sqrt{x} - 1$ is $\sqrt{x} + 1$; therefore,

Note »

$$\frac{\sqrt{x} - 1}{x - 1} = \frac{(\sqrt{x} - 1)(\sqrt{x} + 1)}{(x - 1)(\sqrt{x} + 1)} \quad \text{Rationalize the numerator; multiply by 1.}$$

$$= \frac{x + \sqrt{x} - \sqrt{x} - 1}{(x - 1)(\sqrt{x} + 1)} \quad \text{Expand the numerator.}$$

$$= \frac{x - 1}{(x - 1)(\sqrt{x} + 1)} \quad \text{Simplify.}$$

$$= \frac{1}{\sqrt{x} + 1}. \quad \text{Cancel like factors when } x \neq 1.$$

The limit can now be evaluated:

$$\lim_{x \to 1} \frac{\sqrt{x} - 1}{x - 1} = \lim_{x \to 1} \frac{1}{\sqrt{x} + 1} = \frac{1}{1 + 1} = \frac{1}{2}.$$

Related Exercises 34, 41 ◆

EXAMPLE 7 Finding limits

Let $f(x) = \frac{x^3 - 6x^2 + 8x}{\sqrt{x - 2}}$. Find the values of $\lim_{x \to 2^-} f(x)$, $\lim_{x \to 2^+} f(x)$, and $\lim_{x \to 2} f(x)$, or state that they do not exist.

SOLUTION »

Because the denominator of f is $\sqrt{x-2}$, $f(x)$ is defined only when $x-2>0$. Therefore, the domain of f is $x>2$ and it follows that $\lim_{x\to 2^-} f(x)$ does not exist, which in turn implies that $\lim_{x\to 2} f(x)$ does not exist (Theorem 2.1). To evaluate $\lim_{x\to 2^+} f(x)$, factor the numerator of f and simplify:

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{x(x-2)(x-4)}{(x-2)^{1/2}} \qquad \text{Factor numerator; } \sqrt{x-2} = (x-2)^{1/2}.$$

$$= \lim_{x\to 2^+} x(x-4)(x-2)^{1/2} \qquad \text{Simplify; } \frac{(x-2)}{(x-2)^{1/2}} = (x-2)^{1/2}.$$

$$= \lim_{x\to 2^+} x \cdot \lim_{x\to 2^+} (x-4) \cdot \lim_{x\to 2^+} (x-2)^{1/2} \qquad \text{Theorem 2.3}$$

$$= 2(-2)(0) = 0. \qquad \text{Theorem 2.2, Law 7 for one-sided limits}$$

Related Exercises 69–70 ◆

Quick Check 3 Evaluate $\lim_{x\to 5} \dfrac{x^2-7x+10}{x-5}$. ◆

Answer »

3

The Squeeze Theorem »

The *Squeeze Theorem* provides another useful method for calculating limits. Suppose the functions f and h have the same limit L at a and assume the function g is trapped between f and h (**Figure 2.19**). The Squeeze Theorem says that g must also have the limit L at a. A proof of this theorem is assigned in Exercise 68 of Section 2.7.

Note »

The Squeeze Theorem is also called the Pinching Theorem or the Sandwich Theorem.

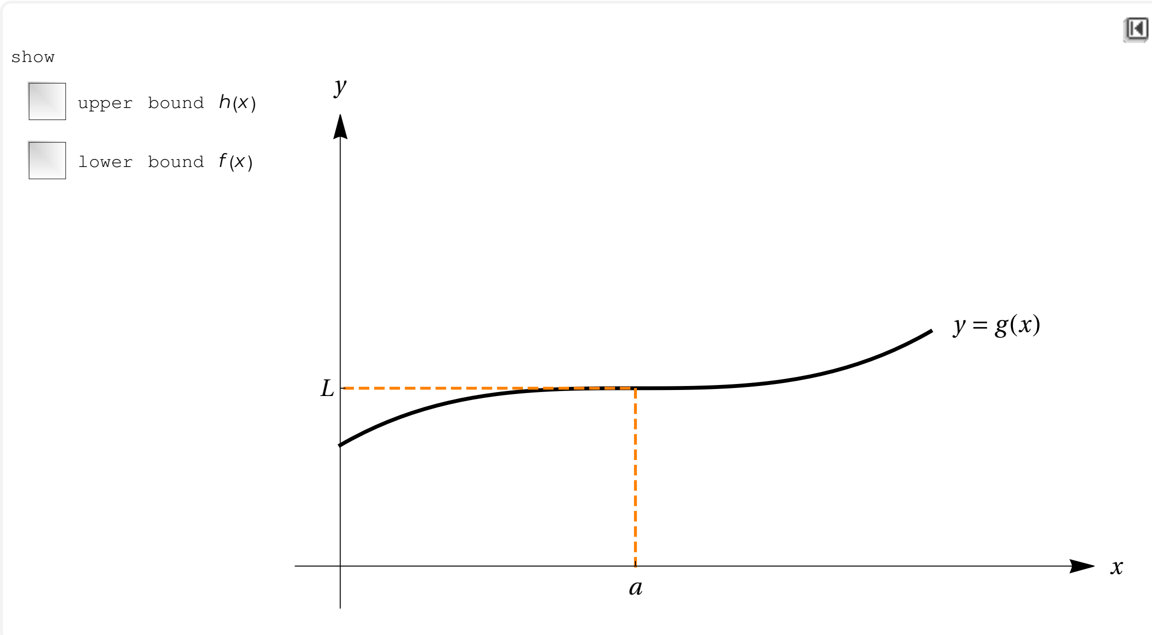


Figure 2.19

THEOREM 2.5 The Squeeze Theorem

Assume the functions f, g, and h satisfy $f(x) \le g(x) \le h(x)$ for values of x near a, except possibly at a. If $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$.

EXAMPLE 8 Applying the Squeeze Theorem

Use the Squeeze Theorem to verify that $\lim_{x\to 0} x^2 \sin(1/x) = 0$.

SOLUTION »

For any real number θ, $-1 \le \sin\theta \le 1$. Letting $\theta = 1/x$ for $x \ne 0$, it follows that

$$-1 \le \sin\frac{1}{x} \le 1.$$

Noting that $x^2 > 0$ for $x \ne 0$, each term in this inequality is multiplied by x^2 :

$$-x^2 \le x^2 \sin\frac{1}{x} \le x^2.$$

These inequalities are illustrated in **Figure 2.20** . Because $\lim_{x\to 0} x^2 = \lim_{x\to 0}\left(-x^2\right) = 0$, the Squeeze Theorem implies that $\lim_{x\to 0} x^2 \sin(1/x) = 0$.

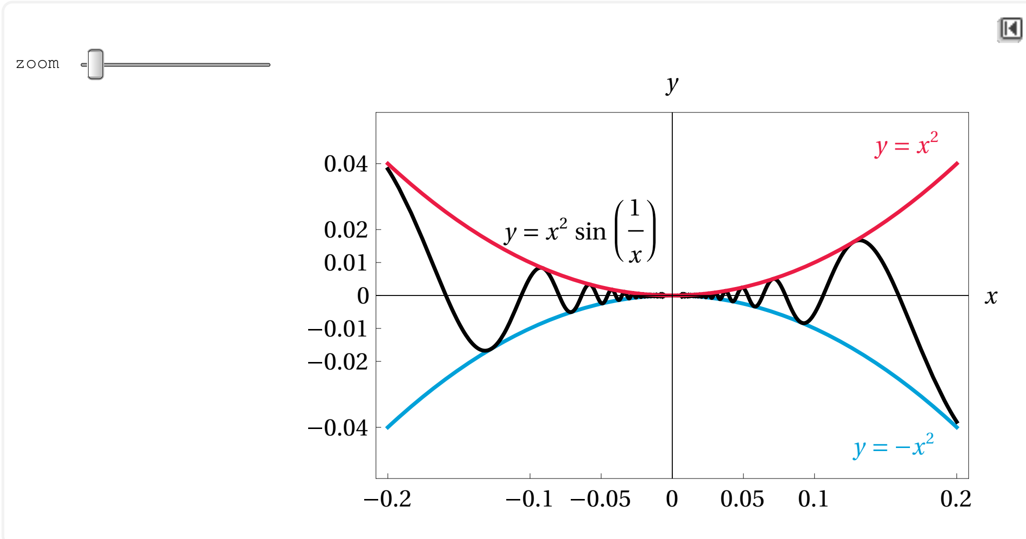


Figure 2.20

Related Exercises 81–82 ◆

Quick Check 4 Suppose f satisfies $1 \leq f(x) \leq 1 + \dfrac{x^2}{6}$ for all values of x near zero. Find $\lim_{x \to 0} f(x)$, if possible. ◆

Answer »

1

Trigonometric Limits »

The Squeeze Theorem is used to evaluate two important limits that play a crucial role in establishing fundamental properties of the trigonometric functions in Section 2.6. These limits are

$$\lim_{x \to 0} \sin x = 0 \quad \text{and} \quad \lim_{x \to 0} \cos x = 1.$$

To verify these limits, the following inequalities, both valid on $-\pi/2 < x < \pi/2$, are used:

$$-|x| \leq \sin x \leq |x| \text{ (**Figure 2.21 a**)} \quad \text{and} \quad 1 - |x| \leq \cos x \leq 1 \text{ (**Figure 2.21 b**)}.$$

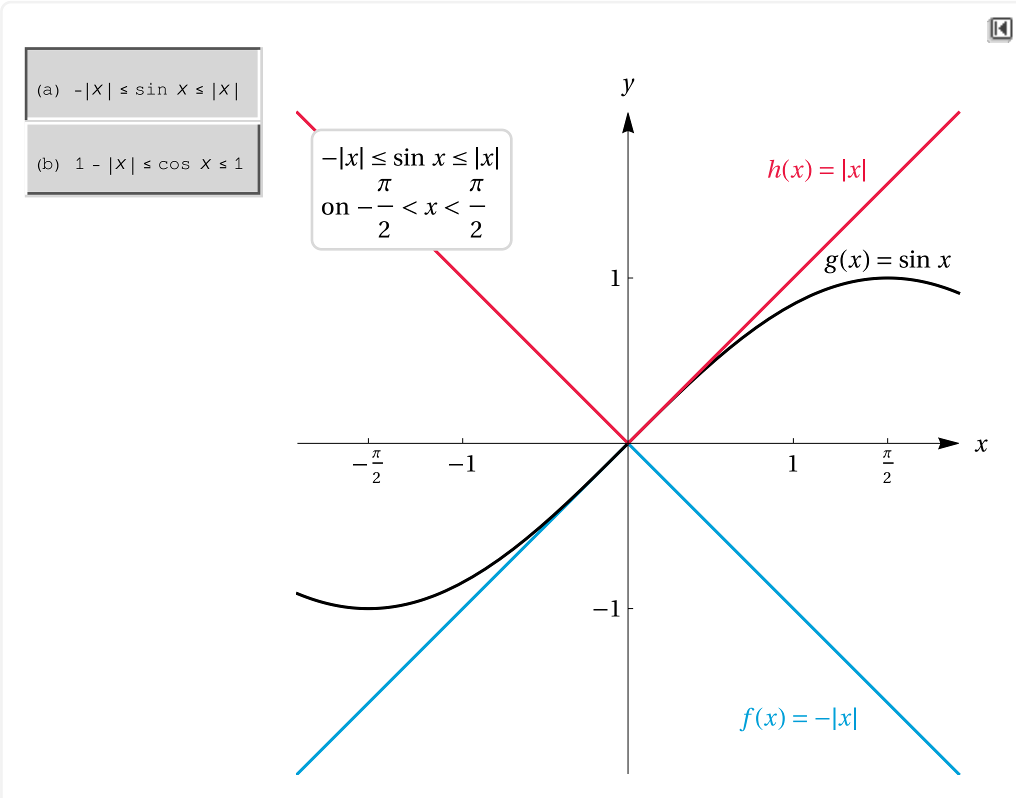


Figure 2.21

Note »

See Exercise 104 for a geometric derivation of the inequalities illustrated in Figure 2.21.

Let's begin with the inequality illustrated in Figure 2.21a. Letting $f(x) = -|x|$, $g(x) = \sin x$, and $h(x) = |x|$, we see that g is trapped between f and h on $-\pi/2 < x < \pi/2$. Because $\lim_{x \to 0} f(x) = \lim_{x \to 0} h(x) = 0$ (Exercise 85), the Squeeze Theorem implies that $\lim_{x \to 0} g(x) = \lim_{x \to 0} \sin x = 0$.

To evaluate $\lim_{x \to 0} \cos x$, let $f(x) = 1 - |x|$, $g(x) = \cos x$, and $h(x) = 1$, and notice that g is again trapped between f and h on $-\pi/2 < x < \pi/2$ (Figure 2.21b). Because $\lim_{x \to 0} f(x) = \lim_{x \to 0} h(x) = 1$, the Squeeze Theorem implies that $\lim_{x \to 0} g(x) = \lim_{x \to 0} \cos x = 1$.

Having established that $\lim_{x \to 0} \sin x = 0$ and $\lim_{x \to 0} \cos x = 1$, we can evaluate more complicated limits involving trigonometric functions.

Note »

> Notice that direct substitution can be used to evaluate the important trigonometric limits just derived. That is, $\lim_{x \to 0} \sin x = \sin 0 = 0$ and $\lim_{x \to 0} \cos x = \cos 0 = 1$. In Section 2.6, these limits are used to show that
>
> $$\lim_{x \to a} \sin x = \sin a \quad \text{and}$$
> $$\lim_{x \to a} \cos x = \cos a.$$
>
> In other words, direct substitution may be used to evaluate limits of the sine and cosine functions for any value of a.

EXAMPLE 9 Trigonometric limits

Evaluate the following trigonometric limits.

a. $\lim_{x \to 0} \dfrac{\sin^2 x}{1 - \cos x}$

b. $\lim_{x \to 0} \dfrac{1 - \cos 2x}{\sin x}$

SOLUTION »

Direct substitution does not work for either limit because the denominator is zero at x = 0 in both parts (a) and (b). Instead, trigonometric identities and limit laws are used to simplify the trigonometric function.

a. The Pythagorean identity $\sin^2 x + \cos^2 x = 1$ is used to simplify the function.

$$\begin{aligned}
\lim_{x \to 0} \frac{\sin^2 x}{1 - \cos x} &= \lim_{x \to 0} \frac{1 - \cos^2 x}{1 - \cos x} && \sin^2 x = 1 - \cos^2 x \\
&= \lim_{x \to 0} \frac{(1 - \cos x)(1 + \cos x)}{1 - \cos x} && \text{Factor the numerator.} \\
&= \lim_{x \to 0} (1 + \cos x) && \text{Simplify.} \\
&= \underbrace{\lim_{x \to 0} 1}_{1} + \underbrace{\lim_{x \to 0} \cos x}_{1} = 2 && \text{Theorem 2 (Law 1)}
\end{aligned}$$

b. We use the identity $\cos 2x = \cos^2 x - \sin^2 x$ to simplify the function.

$$\begin{aligned}
\lim_{x \to 0} \frac{1 - \cos 2x}{\sin x} &= \lim_{x \to 0} \frac{1 - \left(\cos^2 x - \sin^2 x\right)}{\sin x} && \cos 2x = \cos^2 x - \sin^2 x \\
&= \lim_{x \to 0} \frac{1 - \cos^2 x + \sin^2 x}{\sin x} && \text{Distribute.} \\
&= \lim_{x \to 0} \frac{2 \sin^2 x}{\sin x} && 1 - \cos^2 x = \sin^2 x \\
&= 2 \lim_{x \to 0} \sin x = 0 && \text{Simplify; Theorem 2.3 (Law 3).}
\end{aligned}$$

Related Exercises 60–61 ◆

Exercises »

Getting Started »

Practice Exercises »

19–70. Evaluating limits *Find the following limits or state that they do not exist. Assume that a, b, c, and k are fixed real numbers.*

19. $\lim_{x \to 4} (3x - 7)$

20. $\lim_{x \to 1} (-2x + 5)$

21. $\lim_{x \to -9} 5x$

22. $\lim_{x \to 6} 4$

23. $\lim_{x \to 1} \left(2x^3 - 3x^2 + 4x + 5\right)$

24. $\lim_{t \to -2} \left(t^2 + 5t + 7\right)$

25. $\lim_{x \to 1} \frac{5x^2 + 6x + 1}{8x - 4}$

26. $\lim_{t \to 3} \sqrt[3]{t^2 - 10}$

27. $\lim_{p \to 2} \frac{3p}{\sqrt{4p + 1} - 1}$

28. $\lim_{x \to 2} \left(x^2 - x\right)^5$

29. $\lim_{x \to 3} \frac{-5x}{\sqrt{4x - 3}}$

30. $\lim_{h \to 0} \frac{3}{\sqrt{16 + 3h} + 4}$

31. $\lim_{x \to 2} (5x - 6)^{3/2}$

32. $\lim_{h \to 0} \frac{100}{(10h - 1)^{11} + 2}$

33. $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$

34. $\lim_{x \to 3} \frac{x^2 - 2x - 3}{x - 3}$

35. $\lim_{x \to 4} \frac{x^2 - 16}{4 - x}$

36. $\lim_{t \to 2} \frac{3\,t^2 - 7\,t + 2}{2 - t}$

37. $\lim_{x \to b} \frac{(x - b)^{50} - x + b}{x - b}$

38. $\lim_{x \to -b} \frac{(x + b)^7 + (x + b)^{10}}{4\,(x + b)}$

39. $\lim_{x \to -1} \frac{(2\,x - 1)^2 - 9}{x + 1}$

40. $\lim_{h \to 0} \frac{\frac{1}{5+h} - \frac{1}{5}}{h}$

41. $\lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9}$

42. $\lim_{w \to 1} \left(\frac{1}{w^2 - w} - \frac{1}{w - 1}\right)$

43. $\lim_{t \to 5} \left(\frac{1}{t^2 - 4\,t - 5} - \frac{1}{6\,(t - 5)}\right)$

44. $\lim_{t \to 3} \left(\left(4\,t - \frac{2}{t - 3}\right)\left(6 + t - t^2\right)\right)$

45. $\lim_{x \to a} \frac{x - a}{\sqrt{x} - \sqrt{a}}$, $a > 0$

46. $\lim_{x \to a} \frac{x^2 - a^2}{\sqrt{x} - \sqrt{a}}$, $a > 0$

47. $\lim_{h \to 0} \frac{\sqrt{16 + h} - 4}{h}$

48. $\lim_{x \to c} \frac{x^2 - 2\,c\,x + c^2}{x - c}$

49. $\lim_{x \to 4} \frac{\frac{1}{x} - \frac{1}{4}}{x - 4}$

50. $\lim_{x \to 3} \frac{\frac{1}{x^2+2\,x} - \frac{1}{15}}{x - 3}$

51. $\lim_{x\to 1} \frac{\sqrt{10\,x-9}-1}{x-1}$

52. $\lim_{x\to 2} \left(\frac{1}{x-2} - \frac{2}{x^2-2\,x}\right)$

53. $\lim_{h\to 0} \frac{(5+h)^2-25}{h}$

54. $\lim_{w\to -k} \frac{w^2+5\,k\,w+4\,k^2}{w^2+k\,w}$, $k \neq 0$

55. $\lim_{x\to 1} \frac{x-1}{\sqrt{x}-1}$

56. $\lim_{x\to 1} \frac{x-1}{\sqrt{4\,x+5}-3}$

57. $\lim_{x\to 4} \frac{3\,(x-4)\,\sqrt{x+5}}{3-\sqrt{x+5}}$

58. $\lim_{x\to 0} \frac{x}{\sqrt{c\,x+1}-1}$, $c \neq 0$

59. $\lim_{x\to 0} x\cos x$

60. $\lim_{x\to 0} \frac{\sin 2\,x}{\sin x}$

61. $\lim_{x\to 0} \frac{1-\cos x}{\cos^2 x-3\cos x+2}$

62. $\lim_{x\to 0} \frac{\cos x-1}{\cos^2 x-1}$

63. $\lim_{x\to 0^-} \frac{x^2-x}{|x|}$

64. $\lim_{w\to 3^-} \frac{|w-3|}{w^2-7\,w+12}$

65. $\lim_{t\to 2^+} \frac{|2\,t-4|}{t^2-4}$

66. $\lim_{x\to -1} g(x)$, where $g(x) = \begin{cases} \frac{x^2-1}{x+1} & \text{if } x < -1 \\ -2 & \text{if } x \geq -1 \end{cases}$

67. $\lim_{x\to 3} \frac{x-3}{|x-3|}$

68. $\lim_{x\to 5} \frac{|x-5|}{x^2-25}$

69. $\lim_{x\to 1^-} \frac{x^3+1}{\sqrt{x-1}}$

70. $\lim_{x\to 1^+} \frac{x-1}{\sqrt{x^2-1}}$

71. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample. Assume a and L are finite numbers.

a. If $\lim_{x\to a} f(x) = L$, then $f(a) = L$.

b. If $\lim_{x\to a^-} f(x) = L$, then $\lim_{x\to a^+} f(x) = L$.

c. If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = L$, then $f(a) = g(a)$.

d. The limit $\lim_{x\to a} \frac{f(x)}{g(x)}$ does not exist if $g(a) = 0$.

e. If $\lim_{x\to 1^+} \sqrt{f(x)} = \sqrt{\lim_{x\to 1^+} f(x)}$, it follows that $\lim_{x\to 1} \sqrt{f(x)} = \sqrt{\lim_{x\to 1} f(x)}$.

72. **One-sided limits** Let

$$g(x) = \begin{cases} 5x-15 & \text{if } x < 4 \\ \sqrt{6x+1} & \text{if } x \geq 4. \end{cases}$$

Compute the following limits or state that they do not exist.

a. $\lim_{x\to 4^-} g(x)$

b. $\lim_{x\to 4^+} g(x)$

c. $\lim_{x\to 4} g(x)$

73. **One-sided limits** Let

$$f(x) = \begin{cases} x^2+1 & \text{if } x < -1 \\ \sqrt{x+1} & \text{if } x \geq -1. \end{cases}$$

Compute the following limits or state that they do not exist.

a. $\lim_{x\to -1^-} f(x)$

b. $\lim_{x\to -1^+} f(x)$

c. $\lim_{x\to -1} f(x)$

74. One-sided limits Let

$$f(x)=\begin{cases} 0 & \text{if } x \le -5 \\ \sqrt{25-x^2} & \text{if } -5 < x < 5 \\ 3x & \text{if } x \ge 5. \end{cases}$$

Compute the following limits or state that they do not exist.

a. $\lim\limits_{x \to -5^-} f(x)$

b. $\lim\limits_{x \to -5^+} f(x)$

c. $\lim\limits_{x \to -5} f(x)$

d. $\lim\limits_{x \to 5^-} f(x)$

e. $\lim\limits_{x \to 5^+} f(x)$

f. $\lim\limits_{x \to 5} f(x)$

75. One-sided limits

a. Evaluate $\lim\limits_{x \to 2^+} \sqrt{x-2}$.

b. Explain why $\lim\limits_{x \to 2^-} \sqrt{x-2}$ does not exist.

76. One-sided limits

a. Evaluate $\lim\limits_{x \to 3^-} \sqrt{\dfrac{x-3}{2-x}}$.

b. Explain why $\lim\limits_{x \to 3^+} \sqrt{\dfrac{x-3}{2-x}}$ does not exist.

T **77. Electric field** The magnitude of the electric field at a point x meters from the midpoint of a 0.1-m line of charge is given by $E(x)=\dfrac{4.35}{x\sqrt{x^2+0.01}}$ (in units of newtons per coulomb, N/C). Evaluate $\lim\limits_{x \to 10} E(x)$.

78. Torricelli's law A cylindrical tank is filled with water to a depth of 9 meters. At $t=0$, a drain in the bottom of the tank is opened and water flows out of the tank. The depth of water in the tank (measured from the bottom of the tank) t seconds after the drain is opened is approximated by $d(t)=(3-0.015\,t)^2$, for $0 \le t \le 200$. Evaluate and interpret $\lim\limits_{t \to 200^-} d(t)$.

79. Limit of the radius of a cylinder A right circular cylinder with a height of 10 cm and a surface area of S cm^2 has a radius given by

$$r(S)=\frac{1}{2}\left(\sqrt{100+\frac{2S}{\pi}}-10\right).$$

Find $\lim\limits_{S \to 0^+} r(S)$ and interpret your result.

80. A problem from relativity theory Suppose a spaceship of length L_0 travels at a high speed v relative to an observer. To the observer, the ship appears to have a smaller length given by the *Lorentz contraction formula*

$$L = L_0 \sqrt{1 - \frac{v^2}{c^2}},$$

where c is the speed of light.

a. What is the observed length L of the ship if it is traveling at 50% of the speed of light?

b. What is the observed length L of the ship if it is traveling at 75% of the speed of light?

c. In parts (a) and (b), what happens to L as the speed of the ship increases?

d. Find $\lim_{v \to c^-} L_0 \sqrt{1 - \frac{v^2}{c^2}}$ and explain the significance of this limit.

T **81.** **a.** Show that $-|x| \leq x \sin \frac{1}{x} \leq |x|$, for $x \neq 0$.

b. Illustrate the inequalities in part (a) with a graph.

c. Show that $\lim_{x \to 0} x \sin \frac{1}{x} = 0$.

T **82. A cosine limit** It can be shown that $1 - \frac{x^2}{2} \leq \cos x \leq 1$, for x near 0.

a. Illustrate these inequalities with a graph.

b. Use these inequalities to evaluate $\lim_{x \to 0} \cos x$.

T **83. A sine limit** It can be shown that $1 - \frac{x^2}{6} \leq \frac{\sin x}{x} \leq 1$, for x near 0.

a. Illustrate these inequalities with a graph.

b. Use these inequalities to evaluate $\lim_{x \to 0} \frac{\sin x}{x}$.

T **84. A secant limit**

a. Draw a graph to verify that $0 \leq x^2 \sec x^2 \leq x^4 + x^2$ for x near 0.

b. Use the Squeeze Theorem to determine $\lim_{x \to 0} x^2 \sec x^2$.

85. Absolute value Show that $\lim_{x \to 0} |x| = 0$ by first evaluating $\lim_{x \to 0^-} |x|$ and $\lim_{x \to 0^+} |x|$. Recall that

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

86. Absolute value limit Show that $\lim_{x \to a} |x| = |a|$, for any real number a. (*Hint:* Consider the cases $a < 0$, $a = 0$, and $a > 0$.)

87. Finding a constant Suppose

$$f(x)=\begin{cases}\dfrac{x^2-5x+6}{x-3} & \text{if } x\neq 3\\ a & \text{if } x=3.\end{cases}$$

Determine the value of the constant a for which $\lim_{x\to 3} f(x)=f(3)$.

88. Finding a constant Suppose

$$f(x)=\begin{cases}3x+b & \text{if } x\leq 2\\ x-2 & \text{if } x>2.\end{cases}$$

Determine the value of the constant b for which $\lim_{x\to 2} f(x)$ exists and state the value of the limit, if possible.

89. Finding a constant Suppose

$$g(x)=\begin{cases}x^2-5x & \text{if } x\leq -1\\ ax^3-7 & \text{if } x>-1.\end{cases}$$

Determine the value of the constant a for which $\lim_{x\to -1} g(x)$ exists and state the value of the limit, if possible.

90–94. Useful factorization formula *Calculate the following limits using the factorization formula*

$$x^n-a^n=(x-a)\left(x^{n-1}+x^{n-2}a+x^{n-3}a^2+\cdots+xa^{n-2}+a^{n-1}\right),$$

where n is a positive integer and a is a real number.

90. $\lim_{x\to 2}\dfrac{x^5-32}{x-2}$

91. $\lim_{x\to 1}\dfrac{x^6-1}{x-1}$

92. $\lim_{x\to -1}\dfrac{x^7+1}{x+1}$ (*Hint:* Use the formula for x^7-a^7 with $a=-1$.)

93. $\lim_{x\to a}\dfrac{x^5-a^5}{x-a}$

94. $\lim_{x\to a}\dfrac{x^n-a^n}{x-a}$, for any positive integer n

Explorations and Challenges »

95. Even function limits Suppose f is an even function where $\lim_{x\to 1^-} f(x)=5$ and $\lim_{x\to 1^+} f(x)=6$. Find $\lim_{x\to -1^-} f(x)$ and $\lim_{x\to -1^+} f(x)$.

96. Odd function limits Suppose g is an odd function where $\lim_{x\to 1^-} g(x)=5$ and $\lim_{x\to 1^+} g(x)=6$. Find $\lim_{x\to -1^-} g(x)$ and $\lim_{x\to -1^+} g(x)$.

97. Evaluate $\lim_{x \to 1} \frac{\sqrt[3]{x} - 1}{x - 1}$. (*Hint:* $x - 1 = \left(\sqrt[3]{x}\right)^3 - (1)^3$.)

98. Evaluate $\lim_{x \to 16} \frac{\sqrt[4]{x} - 2}{x - 16}$.

99. Creating functions satisfying given limit conditions Find functions f and g such that $\lim_{x \to 1} f(x) = 0$ and $\lim_{x \to 1} (f(x)\, g(x)) = 5$.

100. Creating functions satisfying given limit conditions Find a function f satisfying $\lim_{x \to 1} \frac{f(x)}{x - 1} = 2$.

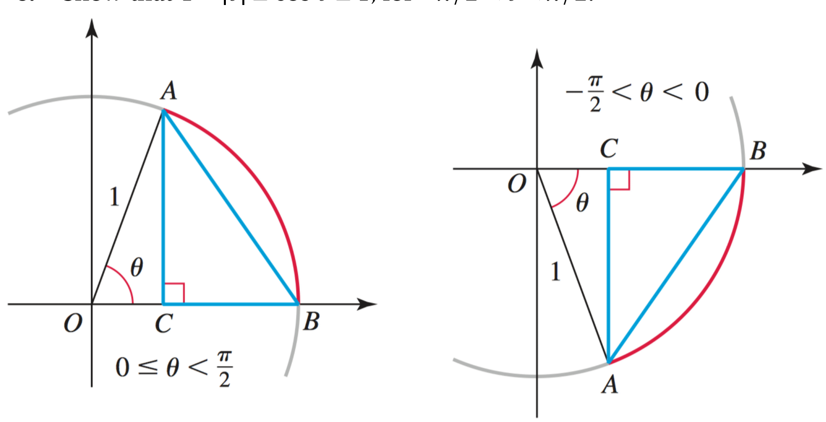
101. Finding constants Find constants b and c in the polynomial $p(x) = x^2 + bx + c$ such that $\lim_{x \to 2} \frac{p(x)}{x - 2} = 6$. Are the constants unique?

102. If $\lim_{x \to 1} f(x) = 4$, find $\lim_{x \to -1} f\left(x^2\right)$.

103. Suppose $g(x) = f(1 - x)$ for all x, $\lim_{x \to 1^+} f(x) = 4$, and $\lim_{x \to 1^-} f(x) = 6$. Find $\lim_{x \to 0^+} g(x)$ and $\lim_{x \to 0^-} g(x)$.

104. Two trigonometric inequalities Consider the angle θ in standard position in a unit circle where $0 \le \theta < \pi/2$ or $-\pi/2 < \theta < 0$ (use both figures).

- **a.** Show that $|AC| = |\sin \theta|$, for $-\pi/2 < \theta < \pi/2$. (*Hint:* Consider the cases $0 < \theta < \pi/2$ and $-\pi/2 < \theta < 0$ separately.)
- **b.** Show that $|\sin \theta| < |\theta|$, for $-\pi/2 < \theta < \pi/2$. (*Hint:* The length of arc AB is θ if $0 \le \theta < \pi/2$, and is $-\theta$ if $-\pi/2 < \theta < 0$.)
- **c.** Conclude that $-|\theta| \le \sin \theta \le |\theta|$, for $-\pi/2 < \theta < \pi/2$.
- **d.** Show that $0 \le 1 - \cos \theta \le |\theta|$, for $-\pi/2 < \theta < \pi/2$.
- **e.** Show that $1 - |\theta| \le \cos \theta \le 1$, for $-\pi/2 < \theta < \pi/2$.



105. Theorem 2.4a Given the polynomial

$$p(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0,$$

prove that $\lim_{x \to a} p(x) = p(a)$ for any value of a.