# 2.2 Definitions of Limits

Computing tangent lines and instantaneous velocities (Section 2.1) are just two of many important calculus problems that rely on limits. We now put these two problems aside until Chapter 3 and begin with a preliminary definition of the limit of a function.

## Limit of a Function »

**DEFINITION** **Limit of a Function (Preliminary)**

Suppose the function $f$ is defined for all $x$ near $a$ except possibly at $a$. If $f(x)$ is arbitrarily close to $L$ (as close to $L$ as we like) for all $x$ sufficiently close (but not equal) to $a$, we write

$$\lim_{x\to a} f(x) = L$$

and say the limit of $f(x)$ as $x$ approaches $a$ equals $L$.

**Note »**

The terms *arbitrarily close* and *sufficiently close* will be made precise when rigorous definitions of limits are given in Section 2.7.

Informally, we say that $\lim_{x\to a} f(x) = L$ if $f(x)$ gets closer and closer to $L$ as $x$ gets closer and closer to $a$ from both sides of $a$. The value of $\lim_{x\to a} f(x)$ (if it exists) depends upon the values of $f$ near $a$, but it does not depend on the value of $f(a)$. In some cases, the limit $\lim_{x\to a} f(x)$ equals $f(a)$. In other instances, $\lim_{x\to a} f(x)$ and $f(a)$ differ, or $f(a)$ may not even be defined.

**EXAMPLE 1** **Finding limits from a graph**

Use the graph of $f$ (**Figure 2.7**) to determine the following values, if possible.

**a.** $f(1)$ and $\lim_{x\to 1} f(x)$

**b.** $f(2)$ and $\lim_{x\to 2} f(x)$

**c.** $f(3)$ and $\lim_{x\to 3} f(x)$

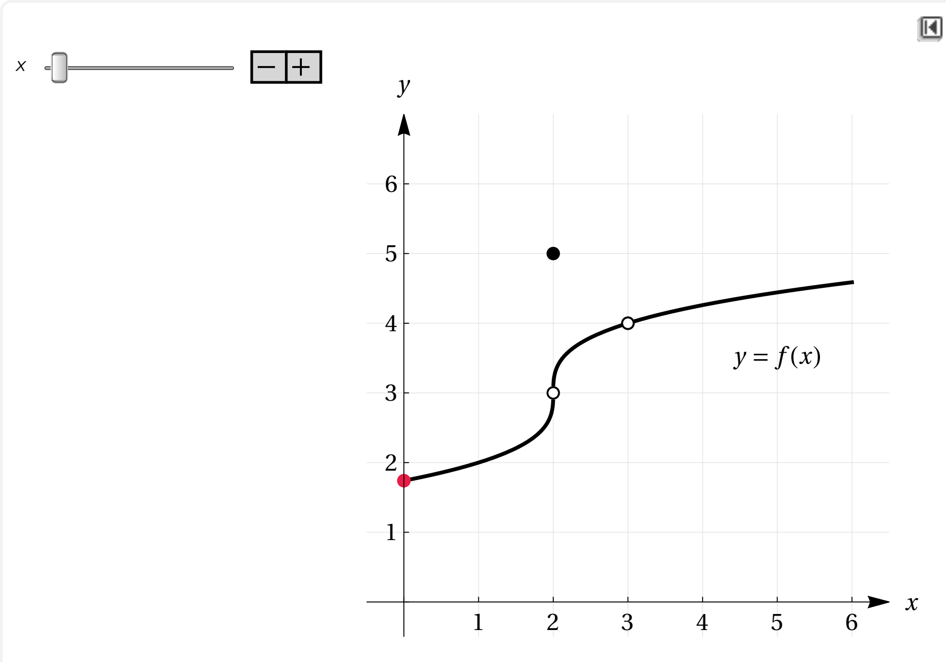



**Figure 2.7**

**SOLUTION »**

**a.** We see that $f(1) = 2$. As $x$ approaches 1 from either side, the values of $f(x)$ approach 2 (**Figure 2.8**). Therefore, $\lim_{x \to 1} f(x) = 2$.

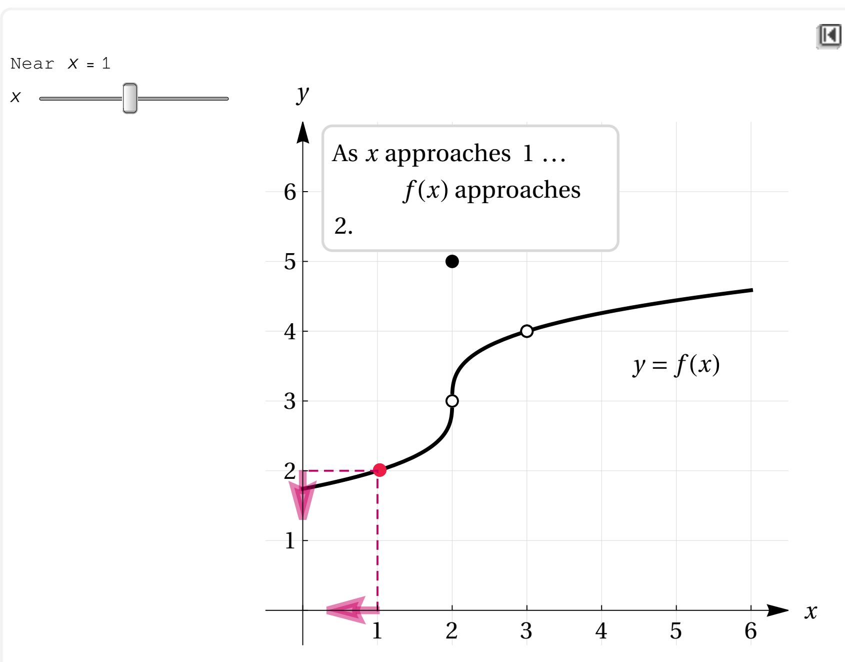



**Figure 2.8**

**b.** We see that $f(2) = 5$. However, as $x$ approaches 2 from either side, $f(x)$ approaches 3 because the points on the graph of $f$ approach the open circle at (2, 3) (**Figure 2.9**). Therefore, $\lim_{x\to 2} f(x) = 3$ even though $f(2) = 5$.

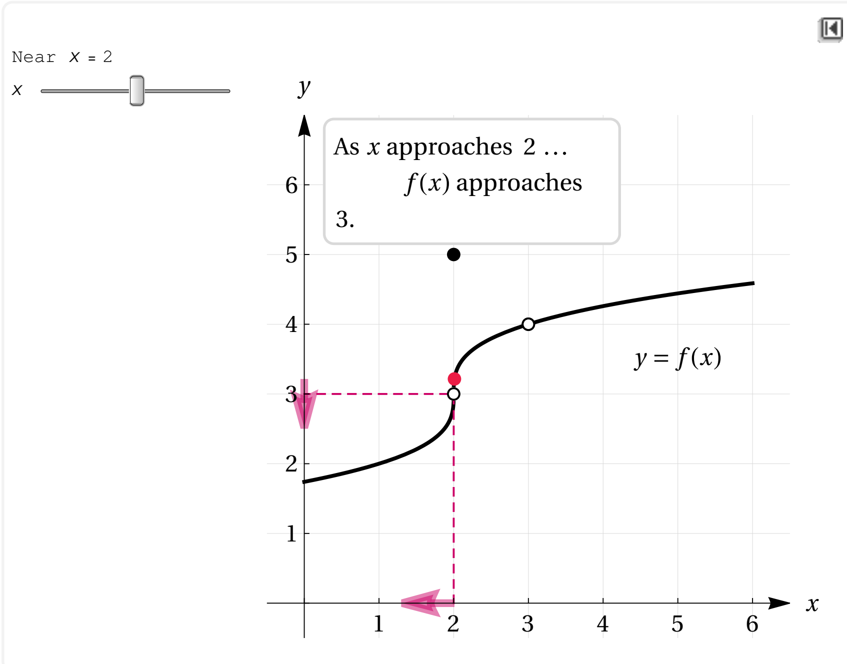



**Figure 2.9**

**c.** In this case, $f(3)$ is undefined. We see that $f(x)$ approaches 4 as $x$ approaches 3 from either side (**Figure 2.10**). Therefore, $\lim_{x\to 3} f(x) = 4$ even though $f(3)$ does not exist.

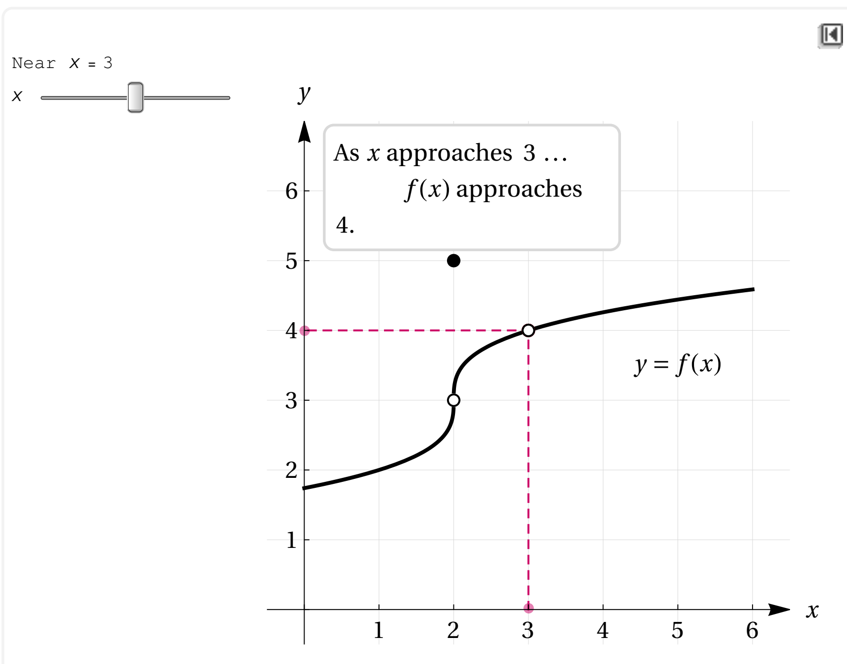



**Figure 2.10**

*Related Exercises 3–4* ◆

**Quick Check 1** In Example 1, suppose we redefine the function at one point so that $f(1)=1$. Does this change the value of $\lim_{x\to 1} f(x)$? ◆

**Answer »**

> The value of $\lim_{x\to 1} f(x)$ depends only on the values of $f$ *near* 1, not at 1. Therefore, changing the value of $f(1)$ will not change the value of $\lim_{x\to 1} f(x)$.

In Example 1, we worked with the graph of a function to estimate limits. Let's now estimate limits using tabulated values of a function.

### EXAMPLE 2 Finding limits from a table

Create a table of values of $f(x)=\dfrac{\sqrt{x}-1}{x-1}$ corresponding to values of $x$ near 1. Then make a conjecture about the value of $\lim_{x\to 1} f(x)$.

**SOLUTION »**

Table 2.2 lists values of $f$ corresponding to values of $x$ approaching 1 from both sides. The numerical evidence suggests that $f(x)$ approaches 0.5 as $x$ approaches 1. Therefore, we make the conjecture that $\lim_{x\to 1} f(x)=0.5$.

**Note »**

**Table 2.2**

| $x$ | $f(x)=\dfrac{\sqrt{x}-1}{x-1}$ |
|---|---|
| 0.9 | 0.5131670 |
| 0.99 | 0.5012563 |
| 0.999 | 0.5001251 |
| 0.9999 | 0.5000125 |
| 1 | |
| 1.0001 | 0.4999875 |
| 1.001 | 0.4998751 |
| 1.01 | 0.4987562 |
| 1.1 | 0.4880885 |

*Related Exercises 7–8* ◆

## One-Sided Limits »

The limit $\lim_{x\to a} f(x)=L$ is referred to as a *two-sided* limit because $f(x)$ approaches $L$ as $x$ approaches $a$ for values of $x$ less than $a$ *and* for values of $x$ greater than $a$. For some functions, it makes sense to examine *one-sided* limits called *right-sided* and *left-sided* limits.

**Note »**

**DEFINITION** **One-Sided Limits**

**1.** **Right-sided limit** Suppose $f$ is defined for all $x$ near $a$ with $x > a$. If $f(x)$ is arbitrarily close to $L$ for all $x$ sufficiently close to $a$ with $x > a$, we write

$$\lim_{x\to a^+} f(x) = L$$

and say the limit of $f(x)$ as $x$ approaches $a$ from the right equals $L$.

**2.** **Left-sided limit** Suppose $f$ is defined for all $x$ near $a$ with $x < a$. If $f(x)$ is arbitrarily close to $L$ for all $x$ sufficiently close to $a$ with $x < a$, we write

$$\lim_{x\to a^-} f(x) = L$$

and say the limit of $f(x)$ as $x$ approaches $a$ from the left equals $L$.

## EXAMPLE 3 Examining limits graphically and numerically

Let $f(x) = \dfrac{x^3 - 8}{4(x-2)}$. Use tables and graphs to make a conjecture about the values of $\lim_{x\to 2^+} f(x)$, $\lim_{x\to 2^-} f(x)$, and $\lim_{x\to 2} f(x)$, if they exist.

**Note »**

## SOLUTION »

**Figure 2.11** shows the graph of $f$ obtained with a graphing utility. The graph is misleading because $f(2)$ is undefined, which means there should be a hole in the graph at (2, 3).

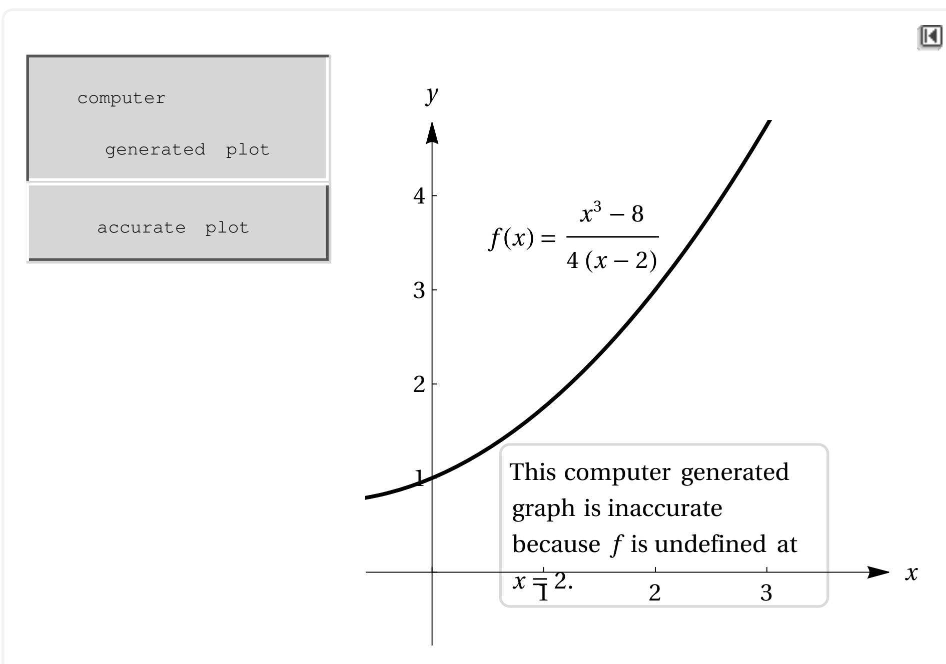



**Figure 2.11**

The graph in **Figure 2.12** and the function values in Table 2.3 suggest that $f(x)$ approaches 3 as $x$ approaches 2 from the right. Therefore, we write

$$\lim_{x \to 2^+} f(x) = 3,$$

which says the limit of $f(x)$ as $x$ approaches 2 from the right equals 3.

**Table 2.3**

| $x$ | $f(x) = \dfrac{x^3 - 8}{4(x-2)}$ |
|---|---|
| 1.9 | 2.8525 |
| 1.99 | 2.985025 |
| 1.999 | 2.99850025 |
| 1.9999 | 2.99985000 |
| 2 | |
| 2.0001 | 3.00015000 |
| 2.001 | 3.00150025 |
| 2.01 | 3.015025 |
| 2.1 | 3.1525 |

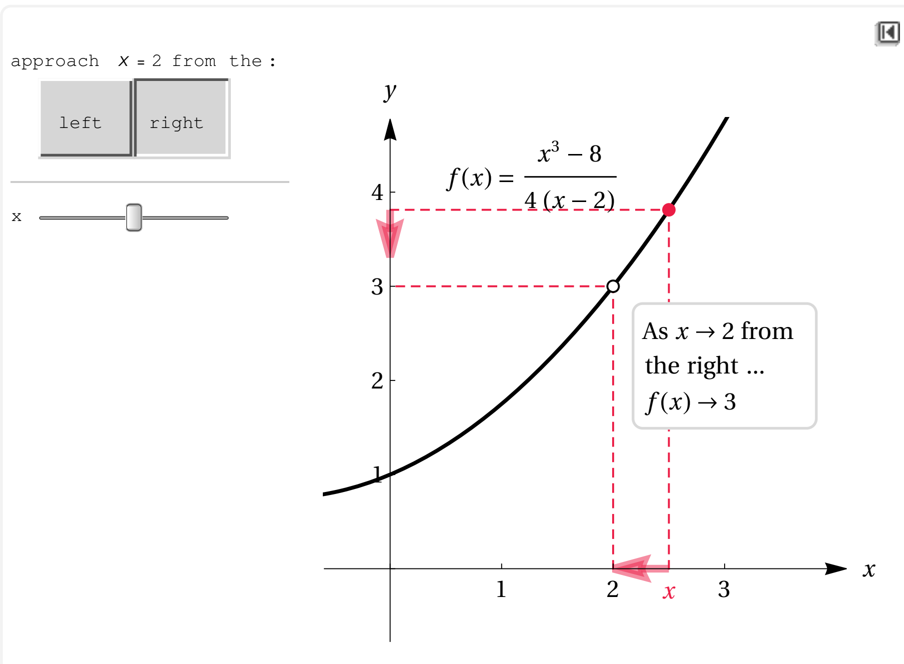



**Figure 2.12**

Similarly, Figure 2.12 and Table 2.3 suggest that as $x$ approaches 2 from the left, $f(x)$ approaches 3. So, we write

$$\lim_{x \to 2^-} f(x) = 3,$$

which says the limit of $f(x)$ as $x$ approaches 2 from the left equals 3. Because $f(x)$ approaches 3 as $x$ approaches 2 from either side, we write the two-sided limit $\lim_{x \to 2} f(x) = 3$.

**Note »**

> Remember that the value of the limit does not depend upon the value of $f(2)$. In this case, $\lim_{x\to 2} f(x) = 3$ despite the fact that $f(2)$ is undefined.

*Related Exercises 27–28* ◆

Based on the previous example, you might wonder whether the limits $\lim_{x\to a^-} f(x)$, $\lim_{x\to a^+} f(x)$, and $\lim_{x\to a} f(x)$ always exist and are equal. The remaining examples demonstrate that these limits sometimes have different values and in other cases, some or all of these limits do not exist. The following result is useful when comparing one-sided and two-sided limits.

**THEOREM 2.1 Relationship Between One-Sided and Two-Sided Limits**

Assume $f$ is defined for all $x$ near $a$ except possibly at $a$. Then $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^+} f(x) = L$ and $\lim_{x\to a^-} f(x) = L$.

**Note »**

> Suppose P and Q are statements. We write $P$ *if and only if* $Q$ when $P$ implies $Q$ *and* $Q$ implies $P$.

A proof of Theorem 2.1 is outlined in Exercise 56 of Section 2.7. Using this theorem, it follows that $\lim_{x\to a} f(x) \neq L$ if either $\lim_{x\to a^+} f(x) \neq L$ or $\lim_{x\to a^-} f(x) \neq L$ (or both). Furthermore, if either $\lim_{x\to a^+} f(x)$ or $\lim_{x\to a^-} f(x)$ does not exist, then $\lim_{x\to a} f(x)$ does not exist. We put these ideas to work in the next two examples.

## EXAMPLE 4 A function with a jump

Sketch the graph of $g(x) = \dfrac{2x^2 - 6x + 4}{|x-1|}$ and use the graph to find the values of $\lim_{x\to 1^-} g(x)$, $\lim_{x\to 1^+} g(x)$, and $\lim_{x\to 1} g(x)$, if they exist.

### SOLUTION »

Sketching the graph of $g$ is straightforward if we first rewrite $g$ as a piecewise function. Factoring the numerator of $g$, we obtain

$$g(x) = \frac{2x^2 - 6x + 4}{|x-1|} = \frac{2(x-1)(x-2)}{|x-1|}.$$

Observe that $g$ is undefined at $x = 1$. For $x > 1$, $|x-1| = x-1$ and

$$g(x) = \frac{2(x-1)(x-2)}{(x-1)} = 2x - 4.$$

Note that $x - 1 < 0$ when $x < 1$, which implies that $|x-1| = -(x-1)$ and

$$g(x) = \frac{2(x-1)(x-2)}{-(x-1)} = -2x + 4.$$

Therefore, $g(x) = \begin{cases} 2x - 4 & \text{if } x > 1 \\ -2x + 4 & \text{if } x < 1 \end{cases}$ and the graph of $g$ consists of two linear pieces with a jump at $x = 1$ (**Figure 2.13**).

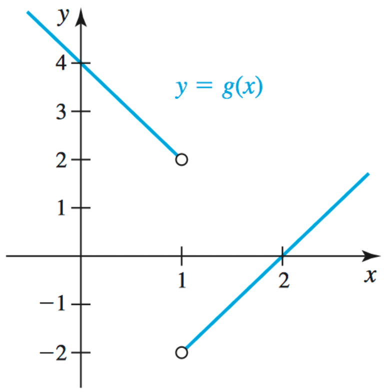



**Figure 2.13**

Examining the graph of $g$, we see that as $x$ approaches 1 from the left, $g(x)$ approaches 2. Therefore, $\lim_{x \to 1^-} g(x) = 2$. As $x$ approaches 1 from the right, $g(x)$ approaches −2 and $\lim_{x \to 1^+} g(x) = -2$. By Theorem 2.1, $\lim_{x \to 1} g(x)$ does not exist because $\lim_{x \to 1^-} g(x) \neq \lim_{x \to 1^+} g(x)$.

*Related Exercises 19–20* ◆

## EXAMPLE 5 Some strange behavior

Examine $\lim_{x \to 0} \cos \dfrac{1}{x}$.

**SOLUTION** »

**Quick Check 2** Why is the graph of $y = \cos \dfrac{1}{x}$ difficult to plot near $x = 0$, as suggested by Figure 2.14? ◆

**Answer** »

> A graphing device has difficulty plotting $y = \cos \dfrac{1}{x}$ near 0 because values of the function vary between −1 and 1 over shorter and shorter intervals as $x$ approaches 0.

Using tables and graphs to make conjectures for the values of limits worked well until Example 5. The limitation of technology in this example is not an isolated incident. For this reason, analytical techniques (paper-and-pencil methods) for finding limits are developed in the next section.

## Exercises »

### Getting Started »

### Practice Exercises »

**19–26. Evaluating limits graphically** *Sketch a graph of $f$ and use it to make a conjecture about the values of $f(a)$, $\lim_{x \to a^-} f(x)$, $\lim_{x \to a^+} f(x)$, and $\lim_{x \to a} f(x)$ or state that they do not exist.*

**19.** $f(x) = \begin{cases} x^2 + 1 & \text{if } x \le -1 \\ 3 & \text{if } x > -1 \end{cases}$; $a = -1$

**20.** $f(x) = \begin{cases} 3 - x & \text{if } x < 2 \\ x - 1 & \text{if } x > 2 \end{cases}$; $a = 2$

**21.** $f(x) = \begin{cases} \sqrt{x} & \text{if } x < 4 \\ 3 & \text{if } x = 4 \\ x + 1 & \text{if } x > 4 \end{cases}$; $a = 4$

**22.** $f(x) = |x + 2| + 2$; $a = -2$

**23.** $f(x) = \dfrac{x^2 - 25}{x - 5}$; $a = 5$

T **24.** $f(x) = \dfrac{x - 100}{\sqrt{x} - 10}$; $a = 100$

**25.** $f(x) = \dfrac{x^2 + x - 2}{x - 1}$; $a = 1$

**26.** $f(x) = \dfrac{1 - x^4}{x^2 - 1}$; $a = 1$

T **27–32. Estimating limits graphically and numerically** *Use a graph of f to estimate* $\lim_{x \to a} f(x)$ *or to show that the limit does not exist. Evaluate* $f(x)$ *near* $x = a$ *to support your conjecture.*

**27.** $f(x) = \dfrac{x - 2}{\sin(x - 2)}$; $a = 2$

**28.** $f(x) = \dfrac{\tan^2(\sin x)}{1 - \cos x}$; $a = 0$

**29.** $f(x) = \dfrac{1 - \cos(2x - 2)}{(x - 1)^2}$; $a = 1$

**30.** $f(x) = \dfrac{3 \sin x - 2 \cos x + 2}{x}$; $a = 0$

**31.** $f(x) = \dfrac{\sin(x + 1)}{|x + 1|}$; $a = -1$

**32.** $f(x) = \dfrac{x^3 - 4x^2 + 3x}{|x - 3|}$; $a = 3$

**33. Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.

**a.** The value of $\lim_{x\to 3} \frac{x^2-9}{x-3}$ does not exist.

**b.** The value of $\lim_{x\to a} f(x)$ is always found by computing $f(a)$.

**c.** The value of $\lim_{x\to a} f(x)$ does not exist if $f(a)$ is undefined.

**d.** $\lim_{x\to 0} \sqrt{x} = 0$. (*Hint:* Graph $y = \sqrt{x}$.)

**e.** $\lim_{x\to \pi/2} \cot x = 0$. (*Hint:* Graph $y = \cot x$.)

**34. The Heaviside function** The Heaviside function is used in engineering applications to model flipping a switch. It is defined as

$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0. \end{cases}$$

**a.** Sketch a graph of $H$ on the interval $[-1, 1]$.

**b.** Does $\lim_{x\to 0} H(x)$ exist?

**35. Postage rates** Assume that postage for sending a first-class letter in the United States is \$0.47 for the first ounce (up to and including 1 oz) plus \$0.21 for each additional ounce (up to and including each additional ounce).

**a.** Graph the function $p = f(w)$ that gives the postage $p$ for sending a letter that weighs $w$ ounces, for $0 < w \leq 3.5$.

**b.** Evaluate $\lim_{w\to 2.3} f(w)$.

**c.** Does $\lim_{w\to 3} f(w)$ exist? Explain.

T **36–42. Calculator limits** *Estimate the following limits using graphs or tables.*

**36.** $\lim_{h\to 0} \frac{\sin h}{h}$

**37.** $\lim_{x\to \pi/2} \frac{\cot 3x}{\cos x}$

**38.** $\lim_{x\to 1} \frac{18\left(\sqrt[3]{x}-1\right)}{x^3-1}$

**39.** $\lim_{x\to 1} \frac{9\left(\sqrt{2x-x^4}-\sqrt[3]{x}\right)}{1-x^{3/4}}$

**40.** $\lim_{x\to 0} \frac{1-\cos^5 2x}{1-\cos x}$

**41.** $\lim_{x\to 8} \frac{\left(x^{4/3}-1\right)\left(x^{2/3}-4\right)}{x-8}$

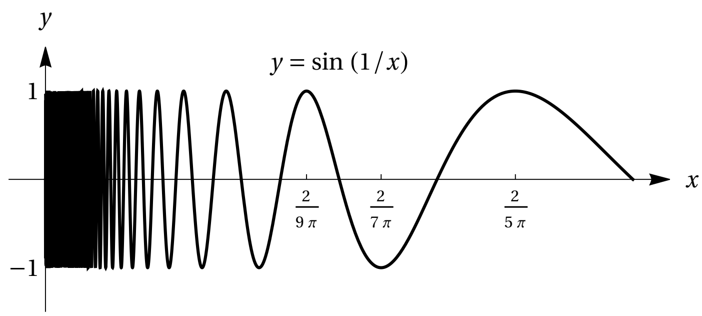
**42.** $\lim_{x\to 3} \frac{x^4-7x^3+15x^2-9x}{x-3}$

T **43. Strange behavior near $x = 0$**

**a.** Create a table of values of $\sin(1/x)$, for $x = 2/\pi,\ 2/(3\pi),\ 2/(5\pi),\ 2/(7\pi),\ 2/(9\pi)$, and $2/(11\pi)$. Describe the pattern of values you observe.

**b.** Why does a graphing utility have difficulty plotting the graph of $y = \sin(1/x)$ near $x = 0$ (see figure)?

**c.** What do you conclude about $\lim_{x\to 0} \sin(1/x)$?




T **44. Strange behavior near $x = 0$**

**a.** Create a table of values of $\tan(3/x)$, for $x = 12/\pi,\ 12/(3\pi),\ 12/(5\pi), \ldots, 12/(11\pi)$. Describe the general pattern in the values you observe.

**b.** Use a graphing utility to graph $y = \tan(3/x)$. Why do graphing utilities have difficulty plotting the graph near $x = 0$?

**c.** What do you conclude about $\lim_{x\to 0} \tan(3/x)$?

**45–49. Sketching graphs of functions** *Sketch the graph of a function with the given properties. You do not need to find a formula for the function.*

**45.** $f(2) = 1,\ \lim_{x\to 2} f(x) = 3$

**46.** $f(1) = 0,\ f(2) = 4,\ f(3) = 6,\ \lim_{x\to 2^-} f(x) = -3,\ \lim_{x\to 2^+} f(x) = 5$

**47.** $g(1) = 0,\ g(2) = 1,\ g(3) = -2,\ \lim_{x\to 2} g(x) = 0,\ \lim_{x\to 3^-} g(x) = -1,\ \lim_{x\to 3^+} g(x) = -2$

**48.** $h(-1) = 2,\ \lim_{x\to -1^-} h(x) = 0,\ \lim_{x\to -1^+} h(x) = 3,\ h(1) = \lim_{x\to 1^-} h(x) = 1,\ \lim_{x\to 1^+} h(x) = 4$

**49.** $p(0) = 2,\ \lim_{x\to 0} p(x) = 0,\ \lim_{x\to 2} p(x)$ does not exist, $p(2) = \lim_{x\to 2^+} p(x) = 1$

**50. A step function** Let $f(x) = \dfrac{|x|}{x}$, for $x \neq 0$.

**a.** Sketch a graph of $f$ on the interval $[-2, 2]$.

**b.** Does $\lim_{x\to 0} f(x)$ exist? Explain your reasoning after first examining $\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0^+} f(x)$.

**51. The floor function** For any real number $x$, the *floor function* (or *greatest integer function*) $\lfloor x \rfloor$ is the greatest integer less than or equal to $x$ (see figure).

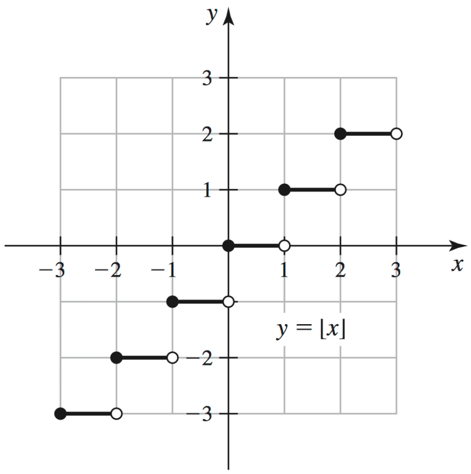
**a.** Compute $\lim_{x\to -1^-} \lfloor x \rfloor,\ \lim_{x\to -1^+} \lfloor x \rfloor,\ \lim_{x\to 2^-} \lfloor x \rfloor$, and $\lim_{x\to 2^+} \lfloor x \rfloor$.

**b.** Compute $\lim_{x\to 2.3^-} \lfloor x \rfloor$, $\lim_{x\to 2.3^+} \lfloor x \rfloor$, and $\lim_{x\to 2.3} \lfloor x \rfloor$.

**c.** For a given integer $a$, state the values of $\lim_{x\to a^-} \lfloor x \rfloor$ and $\lim_{x\to a^+} \lfloor x \rfloor$.

**d.** In general, if $a$ is not an integer, state the values of $\lim_{x\to a^-} \lfloor x \rfloor$ and $\lim_{x\to a^+} \lfloor x \rfloor$.

**e.** For what values of $a$ does $\lim_{x\to a} \lfloor x \rfloor$ exist? Explain?




**52. The ceiling function** For any real number $x$, the *ceiling function* $\lceil x \rceil$ is the smallest integer greater than or equal to $x$.

**a.** Graph the ceiling function $y = \lceil x \rceil$, for $-2 \le x \le 3$.

**b.** Evaluate $\lim_{x\to 2^-} \lceil x \rceil$, $\lim_{x\to 1^+} \lceil x \rceil$, and $\lim_{x\to 1.5} \lceil x \rceil$.

**c.** For what values of $a$ does $\lim_{x\to a} \lceil x \rceil$ exist? Explain.

## Explorations and Challenges »

**53. Limits of even functions** A function $f$ is even if $f(-x) = f(x)$, for all $x$ in the domain of $f$. Suppose $f$ is even, with $\lim_{x\to 2^+} f(x) = 5$ and $\lim_{x\to 2^-} f(x) = 8$. Evaluate the following limits.

**a.** $\lim_{x\to -2^+} f(x)$

**b.** $\lim_{x\to -2^-} f(x)$

**54. Limits of odd functions** A function $g$ is odd if $g(-x) = -g(x)$, for all $x$ in the domain of $g$. Suppose $g$ is odd, with $\lim_{x\to 2^+} g(x) = 5$ and $\lim_{x\to 2^-} g(x) = 8$. Evaluate the following limits.

**a.** $\lim_{x\to -2^+} g(x)$

**b.** $\lim_{x\to -2^-} g(x)$

**T 55. Limits by graphs**

**a.** Use a graphing utility to estimate $\lim_{x\to 0} \frac{\tan 2x}{\sin x}$, $\lim_{x\to 0} \frac{\tan 3x}{\sin x}$, and $\lim_{x\to 0} \frac{\tan 4x}{\sin x}$.

**b.** Make a conjecture about the value of $\lim_{x\to 0} \frac{\tan px}{\sin x}$, for any real constant $p$.

**T 56. Limits by graphs** Graph $f(x) = \dfrac{\sin nx}{x}$, for $n = 1, 2, 3$, and 4 (four graphs). Use the window $[-1, 1] \times [0, 5]$.

**a.** Estimate $\lim\limits_{x \to 0} \dfrac{\sin x}{x}$, $\lim\limits_{x \to 0} \dfrac{\sin 2x}{x}$, $\lim\limits_{x \to 0} \dfrac{\sin 3x}{x}$, and $\lim\limits_{x \to 0} \dfrac{\sin 4x}{x}$.

**b.** Make a conjecture about the value of $\lim\limits_{x \to 0} \dfrac{\sin px}{x}$, for any real constant $p$.

**T 57. Limits by graphs** Use a graphing utility to plot $y = \dfrac{\sin px}{\sin qx}$ for at least three different pairs of nonzero constants $p$ and $q$ of your choice. Estimate $\lim\limits_{x \to 0} \dfrac{\sin px}{\sin qx}$ in each case. Then use your work to make a conjecture about the value of $\lim\limits_{x \to 0} \dfrac{\sin px}{\sin qx}$ for any nonzero values of $p$ and $q$.