2 Limits

Chapter Preview All of calculus is based on the idea of a *limit*. Not only are limits important in their own right, but they underlie the two fundamental operations of calculus: differentiation (calculating derivatives) and integration (evaluating integrals). Derivatives enable us to talk about the instantaneous rate of change of a function, which, in turn, leads to concepts such as velocity and acceleration, population growth rates, marginal cost, and flow rates. Integrals enable us to compute areas under curves, surface areas, and volumes. Because of the incredible reach of this single idea, it is essential to develop a solid understanding of limits. We first present limits intuitively by showing how they arise in computing instantaneous velocities and finding slopes of tangent lines. As the chapter progresses, we build more rigor into the definition of the limit, and we examine different ways in which limits arise. The chapter concludes by introducing the important property called *continuity* and by giving the formal definition of a limit.

2.1 The Idea of Limits

This brief opening section illustrates how limits arise in two seemingly unrelated problems: finding the instantaneous velocity of a moving object and finding the slope of a line tangent to a curve. These two problems provide important insights into limits on an intuitive level. In the remainder of the chapter, we develop limits carefully and fill in the mathematical details.

Average Velocity »

Suppose you want to calculate your average velocity as you travel along a straight highway. If you pass milepost 100 at noon and milepost 130 at 12:30 P.M., you travel 30 mi in a half-hour, so your **average velocity** over this time interval is (30 mi)/(0.5 hr) = 60 mi/hr. By contrast, even though your average velocity may be 60 mi/hr, it's almost certain that your **instantaneous velocity**, the speed indicated by the speedometer, varies from one moment to the next.

EXAMPLE 1 Average velocity

A rock is launched vertically upward from the ground with a speed of 96 ft/s. Neglecting air resistance, a well-known formula from physics states that the position of the rock after t seconds is given by the function

$$s(t) = -16\,t^2 + 96\,t.$$

The position s is measured in feet with $s = 0$ corresponding to the ground. Find the average velocity of the rock between each pair of times.

a. $t = 1$ s and $t = 3$ s

b. $t = 1$ s and $t = 2$ s

SOLUTION »

Figure 2.1 shows the position of the rock on the time interval $0 \le t \le 3$. The graph is *not* the path of the rock. The rock travels up and down on a vertical line.

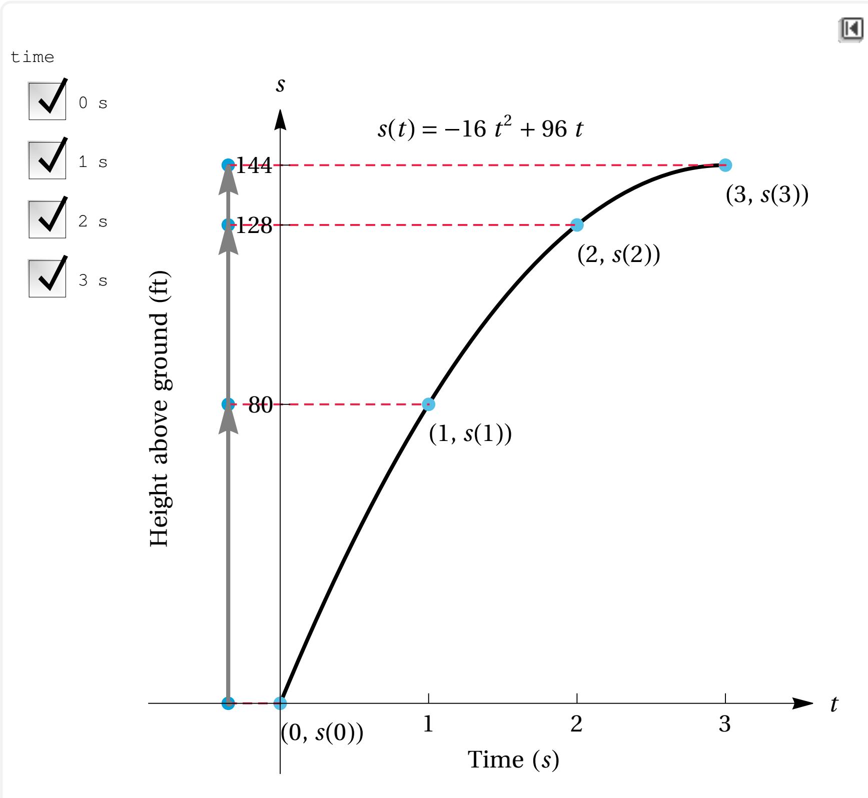


Figure 2.1

a. The average velocity of the rock over any time interval $[t_0, t_1]$ is the change in position divided by the elapsed time:

$$v_{av} = \frac{s(t_1) - s(t_0)}{t_1 - t_0}.$$

Therefore, the average velocity over the interval $[1, 3]$ is

$$v_{av} = \frac{s(3) - s(1)}{3 - 1} = \frac{144\text{ ft} - 80\text{ ft}}{3\text{ s} - 1\text{ s}} = \frac{64\text{ ft}}{2\text{ s}} = 32\text{ ft/s}.$$

Here is an important observation: As shown in **Figure 2.2**, the average velocity is the slope of the line joining the points $(1, s(1))$ and $(3, s(3))$ on the graph of the position function.

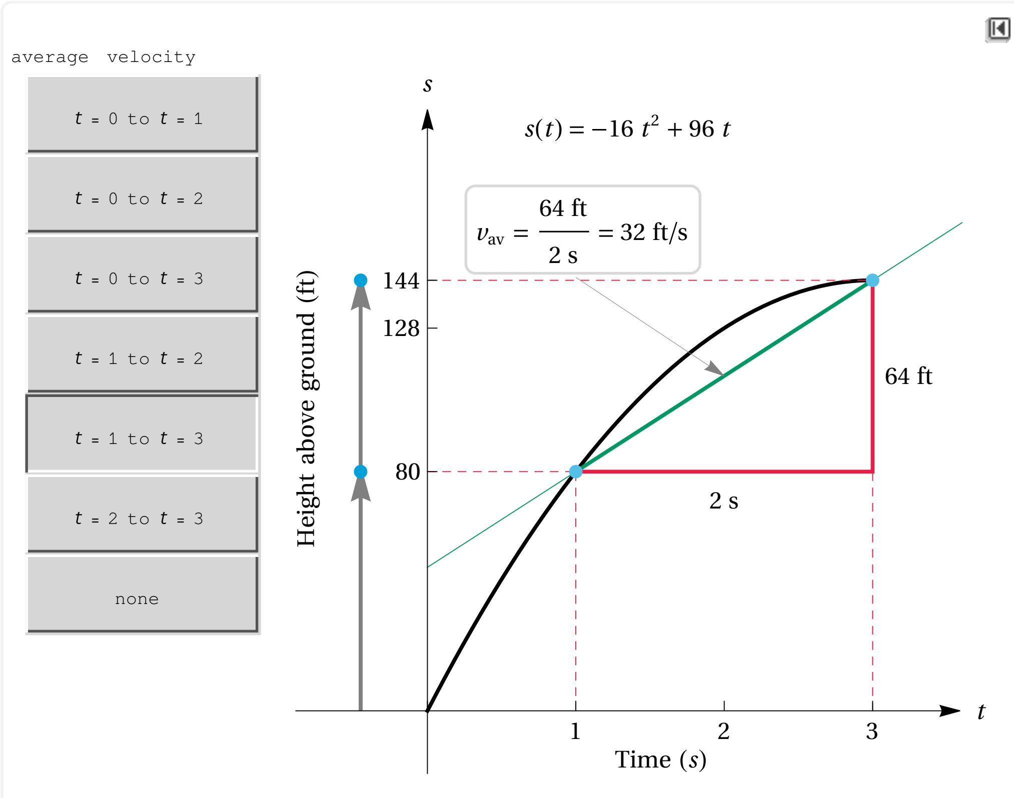


Figure 2.2

b. The average velocity of the rock over the interval [1, 2] is

$$v_{av} = \frac{s(2) - s(1)}{2 - 1} = \frac{128\text{ ft} - 80\text{ ft}}{2\text{ s} - 1\text{ s}} = \frac{48\text{ ft}}{1\text{ s}} = 48\text{ ft/s}.$$

Again, the average velocity is the slope of the line joining the points $(1, s(1))$ and $(2, s(2))$ on the graph of the position function (Figure 2.2).

Related Exercises 13, 15 ◆

Quick Check 1 In Example 1, what is the average velocity between $t = 2$ and $t = 3$? ◆

Answer »

16 ft/s.

In Example 1, we computed slopes of lines passing through two points on a curve. Any such line joining two points on a curve is called a **secant line**. The slope of the secant line, denoted m_{sec}, for the position function in Example 1 on the interval $[t_0, t_1]$ is

$$m_{sec} = \frac{s(t_1) - s(t_0)}{t_1 - t_0}.$$

Example 1 demonstrates that the average velocity is the slope of a secant line on the graph of the position function; that is, $v_{av} = m_{sec}$ (**Figure 2.3**).

Note »

See Section 1.1 for a discussion of secant lines.

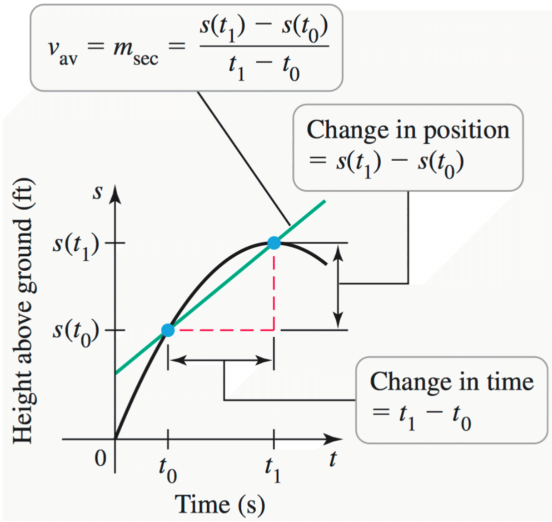


Figure 2.3

Instantaneous Velocity »

To compute the average velocity, we use the position of the object at *two* distinct points in time. How do we compute the instantaneous velocity at a *single* point in time? As illustrated in Example 2, the instantaneous velocity at a point $t = t_0$ is determined by computing average velocities over intervals $[t_0, t_1]$ that decrease in length. As t_1 approaches t_0, the average velocities typically approach a unique number, which is the instantaneous velocity. This single number is called a **limit**.

Quick Check 2 Explain the difference between average velocity and instantaneous velocity. ◆

Answer »

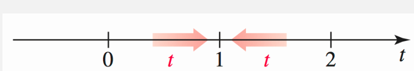
Average velocity is the velocity over an interval of time. Instantaneous velocity is the velocity at one point of time.

EXAMPLE 2 **Instantaneous velocity**

Estimate the *instantaneous velocity* of the rock in Example 1 at the *single* point $t = 1$.

Note »

The same instantaneous velocity is obtained as t approaches 1 from the left (with $t < 1$) and as t approaches 1 from the right (with $t > 1$).



SOLUTION »

We are interested in the instantaneous velocity at $t = 1$, so we compute the average velocity over smaller and smaller time intervals $[1, t]$ using the formula

$$v_{av} = \frac{s(t) - s(1)}{t - 1}.$$

Notice that these average velocities are also slopes of secant lines, several of which are shown in Table 2.1. For example, the average velocity on the interval $[1, 1.0001]$ is 63.9984 ft/s. Because this time interval is so short, the average velocity gives a good approximation to the instantaneous velocity at $t = 1$. We see that as t approaches 1, the average velocities appear to approach 64 ft/s. In fact, we could make the average velocity as close to 64 ft/s as we like by taking t sufficiently close to 1. Therefore, 64 ft/s is a reasonable estimate of the instantaneous velocity at $t = 1$.

Table 2.1

Time interval	Average velocity
[1, 2]	48 ft/s
[1, 1.5]	56 ft/s
[1, 1.1]	62.4 ft/s
[1, 1.01]	63.84 ft/s
[1, 1.001]	63.984 ft/s
[1, 1.0001]	63.9984 ft/s

Related Exercises 17, 19 ◆

In language to be introduced in Section 2.2, we say that the limit of v_{av} as t approaches 1 equals the instantaneous velocity v_{inst}, which is 64 ft/s. This statement is illustrated in **Figure 2.4** and written compactly as

$$v_{inst} = \lim_{t \to 1} v_{av} = \lim_{t \to 1} \frac{s(t) - s(1)}{t - 1} = 64 \text{ ft/s}.$$

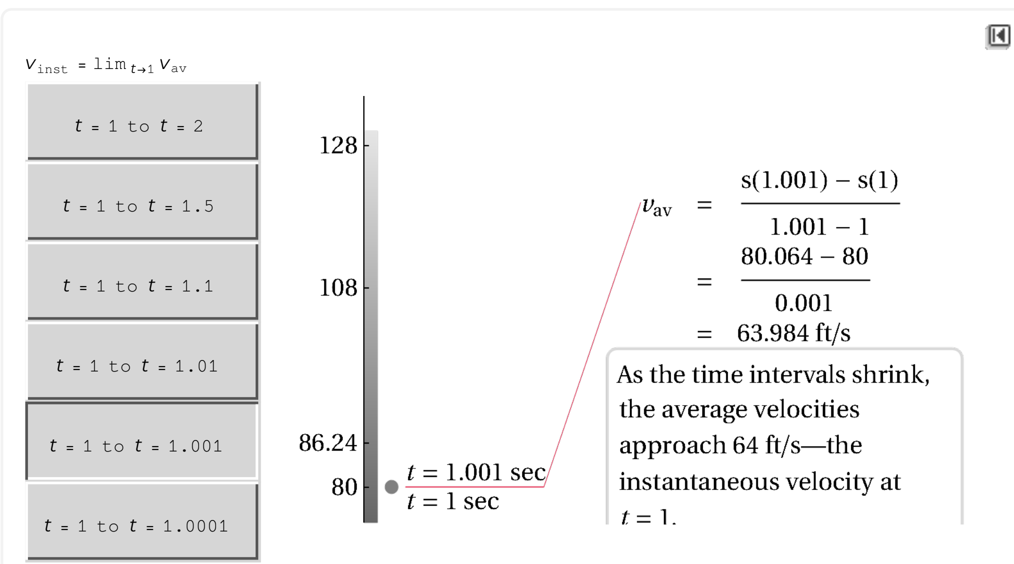


Figure 2.4

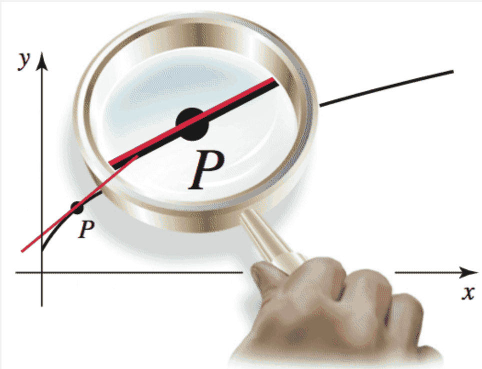
Slope of the Tangent Line »

Several important conclusions follow from Examples 1 and 2. Each average velocity in Table 2.1 corresponds to the slope of a secant line on the graph of the position function (**Figure 2.5**). Just as the average velocities approach a limit as t approaches 1, the slopes of the secant lines approach the same limit as t approaches 1. Specifically, as t approaches 1, two things happen:

1. The secant lines approach a unique line called the **tangent line**.

Note »

We define tangent lines carefully in Section 3.1. For the moment, imagine zooming in on a point P on a smooth curve. As you zoom in, the curve appears more and more like a line passing through P. This line is the *tangent line* at P. Because a smooth curve approaches a line as we zoom in on a point, a smooth curve is said to be *locally linear* at any given point.



2. The slopes of the secant lines m_{sec} approach the slope of the tangent line m_{tan} at the point $(1, s(1))$. Therefore, the slope of the tangent line is also expressed as a limit:

$$m_{tan} = \lim_{t \to 1} m_{sec} = \lim_{t \to 1} \frac{s(t) - s(1)}{t - 1} = 64.$$

This limit is the same limit that defines the instantaneous velocity. Therefore, the instantaneous velocity at $t = 1$ is the slope of the line tangent to the position curve at $t = 1$.

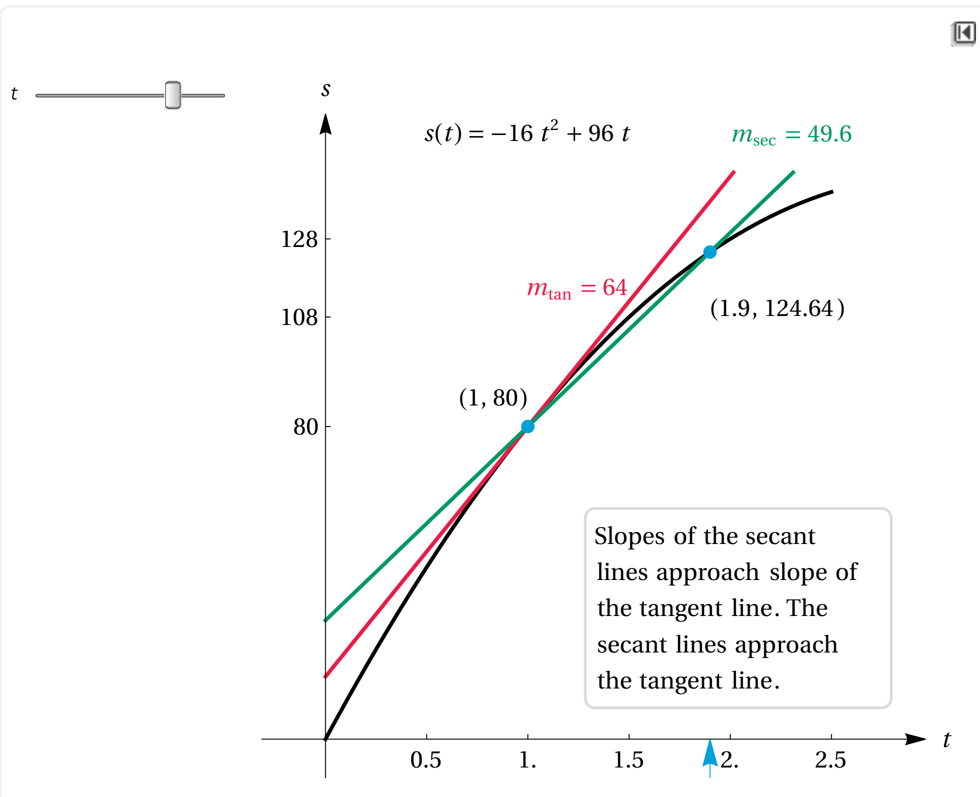


Figure 2.5

Quick Check 3 In Figure 2.5, is $m_{\tan}$ at $t = 2$ greater than or less than $m_{\tan}$ at $t = 1$? ◆

Answer »

> Less than.

The parallels between average and instantaneous velocities, on one hand, and between slopes of secant lines and tangent lines, on the other, illuminate the power behind the idea of a limit. As $t \to 1$, slopes of secant lines approach the slope of a tangent line. And as $t \to 1$, average velocities approach an instantaneous velocity. **Figure 2.6** summarizes these two parallel limit processes. These ideas lie at the foundation of what follows in the coming chapters.

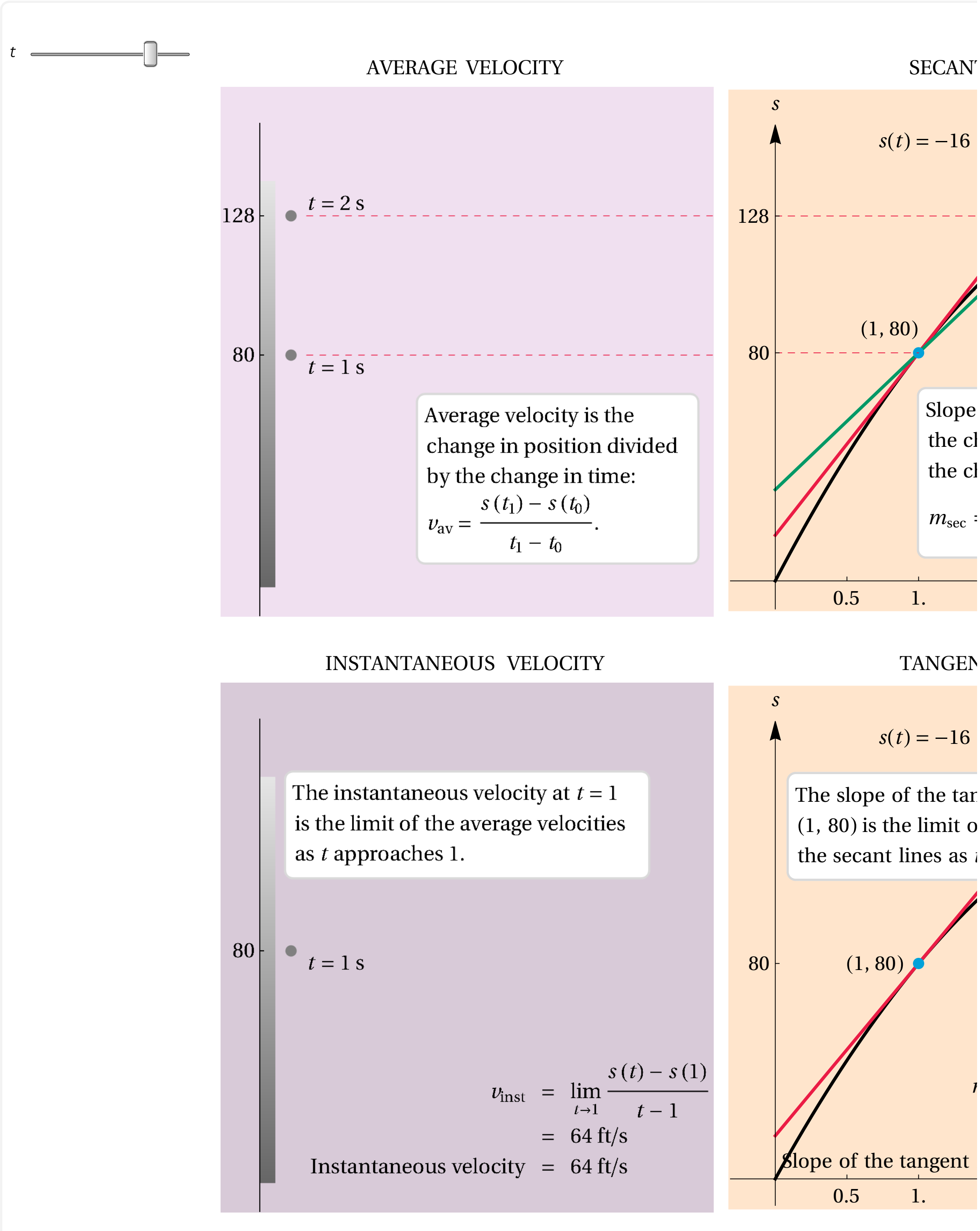


Figure 2.6

Exercises »

Getting Started »

Practice Exercises »

T **13. Average velocity** The position of an object moving vertically along a line is given by the function $s(t) = -16t^2 + 128t$. Find the average velocity of the object over the following intervals.

a. [1, 4]
b. [1, 3]
c. [1, 2]
d. $[1, 1 + h]$, where $h > 0$ is a real number

T **14. Average velocity** The position of an object moving vertically along a line is given by the function $s(t) = -4.9t^2 + 30t + 20$. Find the average velocity of the object over the following intervals.

a. [0, 3]
b. [0, 2]
c. [0, 1]
d. $[0, h]$, where $h > 0$ is a real number

T **15. Average velocity** Consider the position function $s(t) = -16t^2 + 100t$ representing the position of an object moving vertically along a line. Sketch a graph of s with the secant line passing through $(0.5, s(0.5))$ and $(2, s(2))$. Determine the slope of the secant line and explain its relationship to the moving object.

16. Average velocity Consider the position function $s(t) = \sin \pi t$ representing the position of an object moving along a line on the end of a spring. Sketch a graph of s together with the secant line passing through $(0, s(0))$ and $(0.5, s(0.5))$. Determine the slope of the secant line and explain its relationship to the moving object.

T **17. Instantaneous velocity** Consider the position function $s(t) = -16t^2 + 128t$ (Exercise 13). Complete the following table with the appropriate average velocities. Then make a conjecture about the value of the instantaneous velocity at $t = 1$.

Time interval	[1, 2]	[1, 1.5]	[1, 1.1]	[1, 1.01]	[1, 1.001]
Average velocity					

T **18. Instantaneous velocity** Consider the position function $s(t) = -4.9t^2 + 30t + 20$ (Exercise 14). Complete the following table with the appropriate average velocities. Then make a conjecture about the value of the instantaneous velocity at $t = 2$.

Time interval	[2, 3]	[2, 2.5]	[2, 2.1]	[2, 2.01]	[2, 2.001]
Average velocity					

T **19. Instantaneous velocity** Consider the position function $s(t) = -16t^2 + 100t$. Complete the following table with the appropriate average velocities. Then make a conjecture about the value of the instantaneous velocity at $t = 3$.

Time interval	Average velocity
$[2, 3]$	
$[2.9, 3]$	
$[2.99, 3]$	
$[2.999, 3]$	
$[2.9999, 3]$	

T **20. Instantaneous velocity** Consider the position function $s(t) = 3 \sin t$ that describes a block bouncing vertically on a spring. Complete the following table with the appropriate average velocities. Then make a conjecture about the value of the instantaneous velocity at $t = \pi/2$.

Time interval	Average velocity
$[\pi/2, \pi]$	
$[\pi/2, \pi/2 + 0.1]$	
$[\pi/2, \pi/2 + 0.01]$	
$[\pi/2, \pi/2 + 0.001]$	
$[\pi/2, \pi/2 + 0.0001]$	

T **21–24. Instantaneous velocity** *For the following position functions, make a table of average velocities similar to those in Exercises 19–20 and make a conjecture about the instantaneous velocity at the indicated time.*

21. $s(t) = -16t^2 + 80t + 60$ at $t = 3$

22. $s(t) = 20 \cos t$ at $t = \pi/2$

23. $s(t) = 40 \sin 2t$ at $t = 0$

24. $s(t) = 20/(t + 1)$ at $t = 0$

T **25–28. Slopes of tangent lines** *For the following functions, make a table of slopes of secant lines and make a conjecture about the slope of the tangent line at the indicated point.*

25. $f(x) = 2x^2$ at $x = 2$

26. $f(x) = 3 \cos x$ at $x = \pi/2$

27. $f(x) = 1/\left(1 + x^2\right)$ at $x = -1$

28. $f(x) = x^3 - x$ at $x = 1$

Explorations and Challenges »

T **29. Tangent lines with zero slope**

a. Graph the function $f(x) = x^2 - 4x + 3$.

b. Identify the point $(a, f(a))$ at which the function has a tangent line with zero slope.

c. Confirm your answer to part (b) by making a table of slopes of secant lines to approximate the slope of the tangent line at this point.

T **30. Tangent lines with zero slope**

a. Graph the function $f(x) = 4 - x^2$.

b. Identify the point $(a, f(a))$ at which the function has a tangent line with zero slope.

c. Consider the point $(a, f(a))$ found in part (b). Is it true that the secant line between $(a - h, f(a - h))$ and $(a + h, f(a + h))$ has slope zero for any value of $h \neq 0$?

T **31.** **Zero velocity** A projectile is fired vertically upward and has a position given by $s(t) = -16t^2 + 128t + 192$, for $0 \le t \le 9$.

a. Graph the position function, for $0 \le t \le 9$.

b. From the graph of the position function, identify the time at which the projectile has an instantaneous velocity of zero; call this time $t = a$.

c. Confirm your answer to part (b) by making a table of average velocities to approximate the instantaneous velocity at $t = a$.

d. For what values of t on the interval $[0, 9]$ is the instantaneous velocity positive (the projectile moves upward)?

e. For what values of t on the interval $[0, 9]$ is the instantaneous velocity negative (the projectile moves downward)?

T **32.** **Impact speed** A rock is dropped off the edge of a cliff, and its distance s (in feet) from the top of the cliff after t seconds is $s(t) = 16t^2$. Assume the distance from the top of the cliff to the ground is 96 ft.

a. When will the rock strike the ground?

b. Make a table of average velocities and approximate the velocity at which the rock strikes the ground.

T **33.** **Slope of tangent line** Given the function $f(x) = 1 - \cos x$ and the points $A(\pi/2, f(\pi/2))$, $B(\pi/2 + 0.05, f(\pi/2 + 0.05))$, $C(\pi/2 + 0.5, f(\pi/2 + 0.5))$, and $D(\pi, f(\pi))$ (see figure), find the slopes of the secant lines through A and D, A and C, and A and B. Use your calculations to make a conjecture about the slope of the line tangent to the graph of f at $x = \pi/2$.