Math. 150a, Spring 2024

Optimization problems

Example 1

Every morning Tom leaves his house, gets water from the river, and takes it to the farm (see Figure 1 to the right). What is the shortest possible path that Tom has to walk?

Solution

With the notation use in Figure 1, the total distance is $T(x)$ is

$$T(x) = d(x) + D(x) = \sqrt{x^2 + 10^2} + \sqrt{(30 - x)^2 + 20^2},$$

where d denotes the distance from the house to the river and D denotes the distance from the river to the farm. Note that the domain of $T(x)$ is the interval $[0, 30]$ (*Do you know why?*) The derivative of $T(x)$ is

$$T'(x) = \frac{x}{\sqrt{x^2 + 10^2}} - \frac{30 - x}{\sqrt{(30 - x)^2 + 20^2}}$$

There are **no** critical numbers corresponding to the points where $T'(x)$ does not exist. The only critical numbers of $T(x)$ are points where $T'(x) = 0$. Here is the process of solving for those x where $T'(x) = 0$:

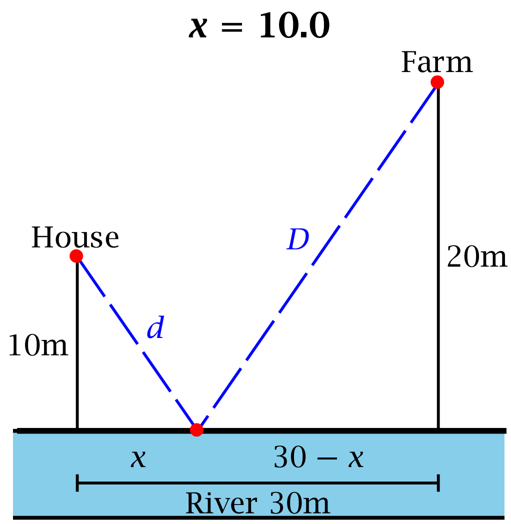

FIGURE 1

$$\begin{aligned}
x\sqrt{(30 - x)^2 + 20^2} - (30 - x)\sqrt{x^2 + 10^2} &= 0 \\
(30 - x)\sqrt{x^2 + 10^2} &= x\sqrt{(30 - x)^2 + 20^2} \\
(30 - x)^2\left(\sqrt{x^2 + 10^2}\right)^2 &= x^2\left(\sqrt{(30 - x)^2 + 20^2}\right)^2 \\
(30 - x)^2)(x^2 + 10^2) &= x^2\left((30 - x)^2 + 20^2\right) \\
(x^2 - 60x + 900)(x^2 + 100) &= x(x^2 - 60x + 900 + 400)^2 \\
x^4 - 60x^3 + 1000x^2 - 6000x + 90000 &= x^4 - 60x^3 + 1300x^2 \\
-300x^2 - 6000x + 90000 &= 0 \\
-300(x + 30)(x - 10) &= 0
\end{aligned}$$

Since $x = -30$ is negative, it does not belong to the domain of $T(x)$, thus the only solution is $x = 10$. Now,

$$T(0) = 10 + 10\sqrt{13} \approx 46.0555, \quad T(10) = 30\sqrt{2} \approx 42.4264, \quad T(30) = 10\sqrt{10} + 20 \approx 51.6228.$$

and the smallest of these three values is $T(10) = 30\sqrt{2}$. Thus, the shortest possible path for Tom is to walk to the point along the river that is 10m downstream from the house, get water there, and then continue along the straight line to the farm.

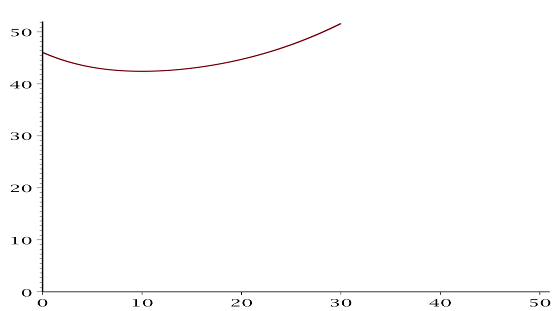

FIGURE 2. The graph of $T(x) = \sqrt{x^2 + 10^2} + \sqrt{(30 - x)^2 + 20^2}$ for $x \in [0, 30]$.

Example 2

Find the dimensions of a rectangle with area $1000\,\mathrm{m}^2$ whose perimeter is as small as possible.

Solution

If the rectangle has dimensions x and y (see Figure 3 to the right), then its area is $1000 = A = xy$ and its perimeter $P = 2x + 2y$. Eliminating y from $1000 = xy$, we obtain $y = 1000/x$ and thus the perimeter has the form

$$P(x) = 2x + \frac{2000}{x}.$$

The domain of $P(x)$ is $(0, \infty)$. We want to minimize $P(x)$ on $(0, \infty)$. Since $P'(x) = 2 - \dfrac{2000}{x^2} = (x^2 - 1000)\dfrac{2}{x^2}$, the only critical number in the domain of $P(x)$ is $x = \sqrt{1000}$. Furthermore, $P''(x) = \dfrac{4000}{x^3} > 0$ for $x \in (0, \infty)$ and by the Second Derivative Test, $x = \sqrt{1000}$ is a local minimum. However, since $P(x) \to \infty$ as $x \to 0^+$ and $P(x) \to \infty$ as $x \to \infty$, so there must be a minimum value of $P(x)$, which must occur at the critical number $x = \sqrt{1000}$. The dimensions of the rectangle with minimal perimeter are $x = y = \sqrt{1000} = 10\sqrt{10}$m, i,e., the rectangle is a square. The resulting perimeter is $P(\sqrt{1000}) = 4\sqrt{1000}$.

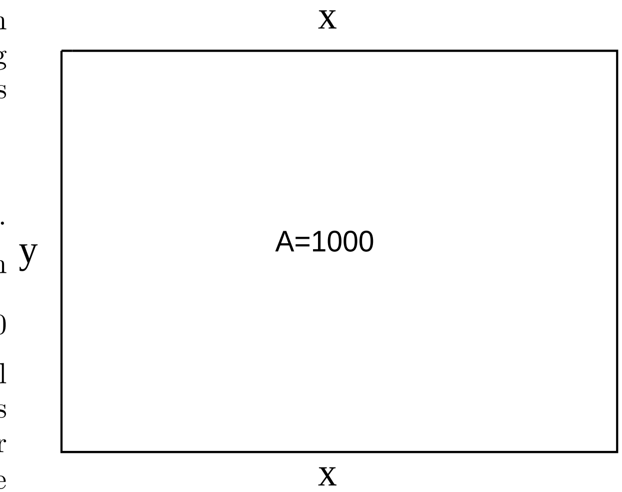

FIGURE 3

Here is a variant of the *First Derivative Test* that can be used to justify the absolute minimum or absolute maximum for functions defined **not necessarily** on closed intervals.

First Derivative Test for Absolute Extreme Values *Suppose that c is a critical number of a continuous function f defined on an interval.*

(a) *If $f'(x) > 0$ for **all** $x < c$ and $f'(x) < 0$ for **all** $x > c$, then $f(x)$ is the absolute maximum value of f.*

(b) *If $f'(x) < 0$ for **all** $x < c$ and $f'(x) > 0$ for **all** $x > c$, then $f(x)$ is the absolute minimum value of f.*

And here is an alternative argument that can be used in Problem 2 to show that the critical number $x = \sqrt{1000}$ must give rise to an *absolute* minimum for $P(x)$. We observe that $P'(x) = (x^2 - 1000)\dfrac{2}{x^2} < 0$ for $0 < x < \sqrt{1000}$ and $P'(x) = (x^2 - 1000)\dfrac{2}{x^2} > 0$ for $x > \sqrt{1000}$. (*Do you know why?*) Thus, using (b) part of the **First Derivative Test for Absolute Extreme Values**, we obtain again that the absolute minimum for $P(x)$ must occur at $x = \sqrt{1000}$.

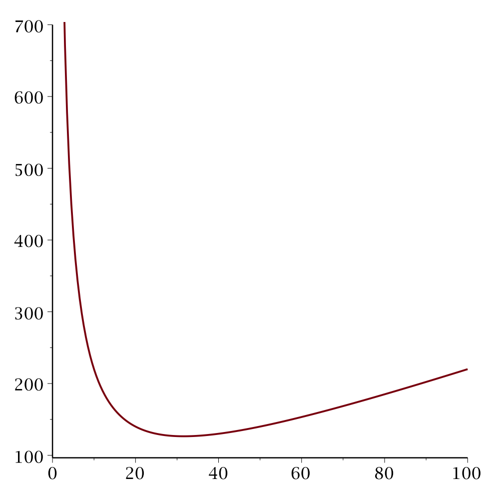

FIGURE 4. The graph of $P(x) = 2x + \dfrac{2000}{x}$ for $x \in [0, 100]$.

Example 3

A rectangular storage container with an open top is to have a volume of $10\,\mathrm{m}^3$. The length of its base is twice the width Material for the base costs \$10 per square meter. Material for the sides costs \$6 per square meter. Find the cost of the materials for the cheapest such container.

Solution

The volume is $V = lwh \quad \Longrightarrow \quad 10 = (2w)(w)h = 2w^2h$, so $h = \dfrac{5}{w^2}$.

The cost is

$$C = 10(2w^2) + 6[2(2wh) + 2(hw)] = 20w^2 + 36wh,$$

and eliminating $h = 5/w^2$, we obtain

$$C(w) = 20w^2 + 36w\frac{5}{w^2} = 20w^2 + \frac{180}{w}.$$

The domain of $C(w)$ is $(0, \infty)$.

$C'(w) = 40w - \dfrac{180}{w^2} = \left(w^3 - \dfrac{9}{2}\right)\dfrac{40}{w^2} \quad \Longrightarrow \quad w = \sqrt[3]{\dfrac{9}{2}}$ is the only critical number. Since $C'(w) < 0$ for $0 < w < \sqrt[3]{\dfrac{9}{2}}$ and $C'(w) > 0$ for $w > \sqrt[3]{\dfrac{9}{2}}$.

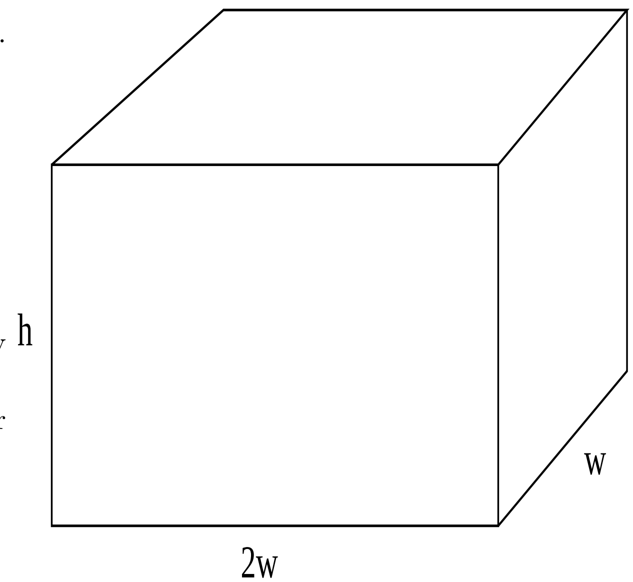

FIGURE 5

The **First Derivative Test for Absolute Extreme Values** yields that there is an absolute minimum for $C(w)$ when $w = \sqrt[3]{\dfrac{9}{2}}$ and

$$C\left(\sqrt[3]{\frac{9}{2}}\right) = 20\left(\sqrt[3]{\frac{9}{2}}\right)^2 + \frac{180}{\sqrt[3]{9/2}} \approx \$163.54.$$

Example 4

(a) Show that of all rectangles with a given area, the one with smallest perimeter is a square.

(b) Show that of all rectangles with a given perimeter, the one with the greatest area is a square.

Solution

(a) If the rectangle has sides x and y and the area A, so $A = xy$ or $y = A/x$. The problem is to minimize the perimeter $P = 2x + 2y = 2x + 2A/x$. The domain of $P(x)$ is $(0, \infty)$. Now $P'(x) = 2 - 2A/x^2 = 2(x^2 - A)/x^2$. The only critical number is $x = \sqrt{A}$. Since $P'(x) < 0$ for $0 < x < \sqrt{A}$ and $P'(x) > 0$ for $x > \sqrt{A}$. Therefore by the **First Derivative Test for Absolute Extreme Values**, $P(x)$ has an absolute minimum at $x = \sqrt{A}$. The sides of the rectangle are $\sqrt{A}$ and $A/\sqrt{A} = \sqrt{A}$, so the rectangle is a square.

(b) Let p be the perimeter and x and y the lengths of the sides, so $p = 2x + 2y \quad \Longrightarrow \quad 2y = p - 2x \quad \Longrightarrow \quad y = \dfrac{1}{2}p - x$. The area is $A(x) = x\left(\dfrac{1}{2}p - x\right) = \dfrac{1}{2}px - x^2$. The domain of $A(x)$ is $(0, p/2)$ (*Do you know why?*). Now, $A'(x) = \dfrac{1}{2}p - 2x = 0$ when $x = \dfrac{p}{4}$. Since $A'(x) > 0$ for $0 < x < p/4$ and $A'(x) < 0$ for $p/4 < x < p/2$, the **First Derivative Test for Absolute Extreme Values** implies that $A(x)$ has an absolute maximum at $x = \dfrac{p}{4}$. The sides of the rectangle are $\dfrac{p}{4}$ and $\dfrac{p}{2} - \dfrac{p}{4} = \dfrac{p}{4}$, so the rectangle is a square.

Example 5

Find the area of the largest rectangle that can be inscribed in a right traingle with legs of lengths 3 cm and 4 cm if two sides of rectangle lie along the legs.

Solution

The rectangle has area xy. (See Figure 6 to the right.) By similar triangles

$$\frac{3-y}{x}=\frac{3}{4} \implies -4y=12-3x \text{ or } y=-\frac{3}{4}x+3.$$

So the area is

$$A(x)=x(-\frac{3}{4}x+3)=-\frac{3}{4}x^2+3x, \quad \text{with } x\in[0,4].$$

Th critical number of $A(x)$ is a solution to $A'(x)=-\frac{3}{2}x+3=0 \implies x=2$ and $y=\frac{3}{2}$. Since $A(0)=A(4)=0$, the maximum area is $A(2)=3\text{cm}^2$.

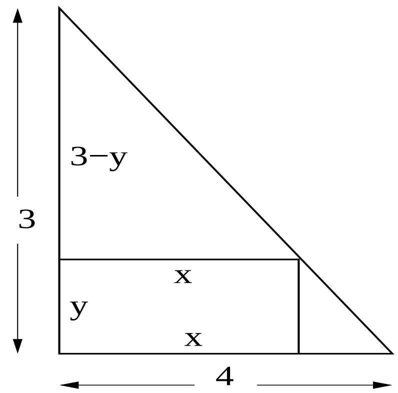

Figure 6

Example 6

A right circular cylinder is inscribed in a sphere of radius r. Find the largest possible volume of such cylinder.

Solution

The cylinder has volume $\pi y^2(2x)$. (See Figure 7 to the right.) Also $x^2+y^2=r^2 \implies y^2=r^2-x^2$, so

$V(x)=\pi(r^2-x^2)(2x)=2\pi(r^2x-x^3)$, where $0\le x\le r$. (*Do you know why?*)

$$V'(x)=2\pi(r^2-3x^2)=0 \implies x=\frac{r}{\sqrt{3}}.$$

Now $V(0)=V(r)=0$, so there is a maximum when $x=r/\sqrt{3}$ and

$$V(r/\sqrt{3})=\pi(r^2-r^2/\sqrt{3})(2r/\sqrt{3})=4\pi r^3/(3\sqrt{3}).$$

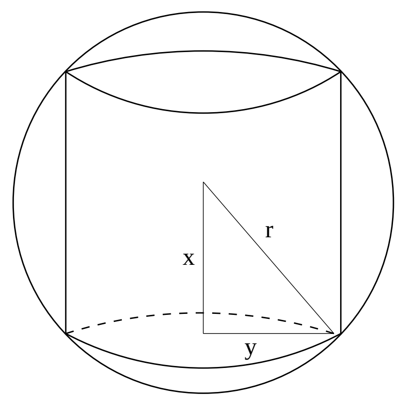

Figure 7

Example 7

A piece of wire 10m long is cut into two pieces. One piece is bent into a square and the other into a circle. How should the wire be cut so that the total area enclosed is (a) a maximum? (b) A minimum?

Solution

With the radius r of a circle (see Figure 8 to the right) equal to $\frac{10-x}{2\pi}$, the total area is

$$A(x)=\left(\frac{x}{4}\right)^2+\pi\left(\frac{10-x}{2\pi}\right)^2=\frac{x^2}{16}+\frac{(10-x)^2}{4\pi}, \quad 0\le x\le 10.$$

$$A'(x)=\frac{x}{8}-\frac{10-x}{2\pi}=\left(\frac{1}{2\pi}+\frac{1}{8}\right)x-\frac{5}{\pi}=0 \implies x=\frac{40}{4+\pi}.$$

$A(0)=25/\pi\approx 7.96$, $A(10)=6.25$ and $A(40/(4+\pi))\approx 3.5$, so the maximum occurs when $x=0$ (no cut) and the minimum occurs when $x=40/(4+\pi)\,\text{m}$.

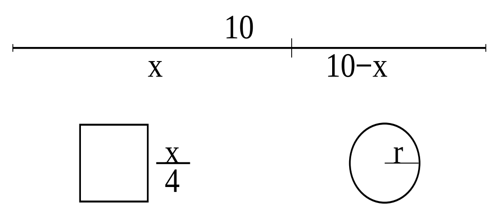

Figure 8

Example 8

A fence 8 ft tall runs parallel to a tall building at a distance of 4 ft from the building. What is the length of the shortest ladder that will reach from the ground over the fence to the wall of the building?

Solution

With the notation as in Figure 9,

$$L = 8\csc\theta + 4\sec\theta, \quad \text{with } 0 < \theta < \pi/2. \quad (\textit{Do you know why?})$$

$$\frac{dL}{d\theta} = -8\csc\cot\theta + 4\sec\theta\tan\theta = 0 \iff -8\frac{\cos\theta}{\sin^2\theta} + 4\frac{\sin\theta}{\cos^2\theta} = 0$$

$$\iff \tan^3\theta = 2 \iff \tan\theta = \sqrt[3]{2} \iff \theta = \tan^{-1}(\sqrt[3]{2}).$$

Next, $dL/d\theta < 0$ when $0 < \theta < \tan^{-1}(\sqrt[3]{2})$ and $dL/d\theta > 0$ when $\tan^{-1}(\sqrt[3]{2}) < \theta < \pi.2$, so L has an absolute minimum when $\theta == \tan^{-1}(\sqrt[3]{2})$ and the shortest ladder has length

$$L = 8\frac{\sqrt{1+2^{2/3}}}{2^{1/3}} + 4\sqrt{1+2^{2/3}} \approx 16.65\,\text{ft}.$$

Note that when $\tan\theta = \sqrt[3]{2}$ *then* $sin\theta = \dfrac{2^{1/3}}{\sqrt{1+2^{2/3}}}$ *and* $\cos\theta = \dfrac{1}{\sqrt{1+2^{2/3}}}$.

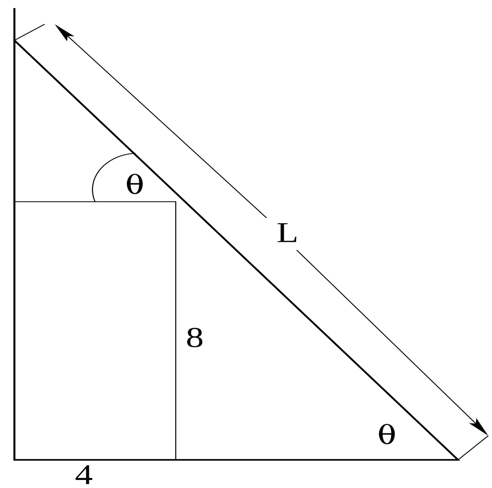

FIGURE 9

Second, more complicated method

With the notation as in Figure 10 to the right, we must minimize

$$L = \sqrt{x^2 + (4+y)^2},$$

where $\dfrac{x}{4+y} = \dfrac{8}{y}$, It is simpler to find an absolute minimum of L^2 (*Do you know why minimizing* L^2 *instead of* L *is allowed here?*) Eliminating x, we obtain

$$L^2 = 64\left(\frac{4+y}{y}\right)^2 + (4+y)^2, \quad \text{with } y > 0.$$

$$(L^2)'(y) = \frac{2(y^4 + 4y^3 - 256y - 1024)}{y^3} = 2\frac{(4+y)(y^3 - 256)}{y^3} = 0.$$

The only critical number is $y = (256)^{1/3} = 42^{2/3}$. Since $L^2 \to \infty$ as $y \to \infty$ and $L^2 \to \infty$ as $y \to 0^+$ and $(L^2)''(y) = \dfrac{2(y^4 + 512y + 3072)}{y^4} > 0$ for all $y > 0$, L^2 has a minimum at $y = (256)^{1/3} = 42^{2/3}$. The minimum is

$$L((256)^{1/3}) = 4\sqrt{62^{1/3} + 32^{2/3} + 5} = \underbrace{8\frac{\sqrt{1+2^{2/3}}}{2^{1/3}} + 4\sqrt{1+2^{2/3}}}_{\text{the same answer as in the first method. Can you check it?}}$$

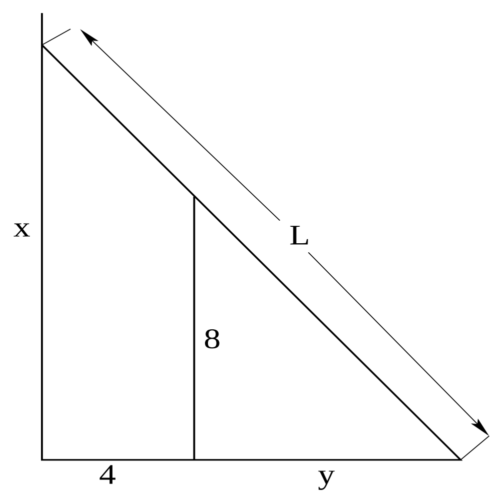

FIGURE 10

Example 9

For a fish swimming at a speed v relative to water, the energy expenditure per unit time is proportional to v^3. It is believed that migrating fish try to minimize the total energy required to swim a fixed distance. If the fish swimming against a current u ($u < v$), then the time required to swim a distance L is $\dfrac{L}{v-u}$ and the total energy required to swim the distance is given by

$$E(v) = \alpha v^3 \cdot \frac{L}{v-u}, \quad \text{where } \alpha > 0 \text{ is a proportionality constant.}$$

(a) Determine the value that minimize E.

(b) Sketch the graph of E.

Solution

(a)

$$E(v) = \frac{\alpha L v^3}{v-u}, \quad \text{with } v > u$$

implies that

$$E'(v) = \alpha L \frac{(v-u)3v^2 - v^3}{(v-u)^2} = \alpha L \frac{v^2(2v-3u)}{(v-u)^2} = 0$$

when $2v = 3u \implies v = \frac{3}{2}u$. Since $E'(v) < 0$ for $u < v < \frac{3}{2}u$ and $E'(v) > 0$ for $v > \frac{3}{2}u$. Therefore, by the **First Derivative Test for Absolute Extreme Values**, $E(v)$ has an absolute minimum at $v = \frac{3}{2}u$.

Note: *This result has been verified experimentally; migrating fish swim against a current at a speed 50% greater than the current speed.*

(b)

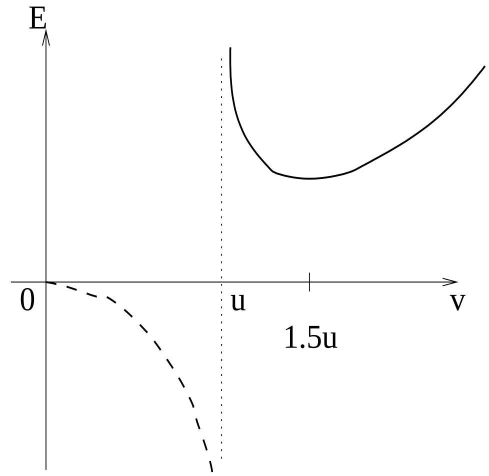

FIGURE 11

Example 10

A boat leaves a dock at 2:00 pm and travels due south at a speed of 20 km/h. Another boat has been heading due east at 15 km/h and reaches the same dock at 3:00 pm/ At what time were the two boats closest together?

Solution

Let t be the time, in hours, after 2:00 pm. The position of the boat heading south at time t is $(0, -20t)$. The position of the boat heading east at time t is $(-15 + 15t, 0)$. (*Do you know why?*) Distance $D(t)$ between the boats is

$$D(t) = \sqrt{20^2t^2 + 15^2(t-1)^2} \implies D^2(t) = 20^2t^2 + 15^2(t-1)^2.$$

Note that it is simpler to minimize $f(t) = D^2(t) = 20^2t^2 + 15^2(t-1)^2$.

$$f'(t) = 800t + 450(t-1) = 1250t - 450 = 0 \quad \text{when } t = \frac{450}{1250} = 0.36\,\text{h}.$$

Since $f''(t) > 0$, $t = 0.36$ gives a minimum. Now,

$$0.36\,\text{h} \times \frac{60\,\text{min}}{\text{h}} = 21\,\text{min}\,36\,\text{s}$$

so the boats are closest together at 2:21:36 pm.

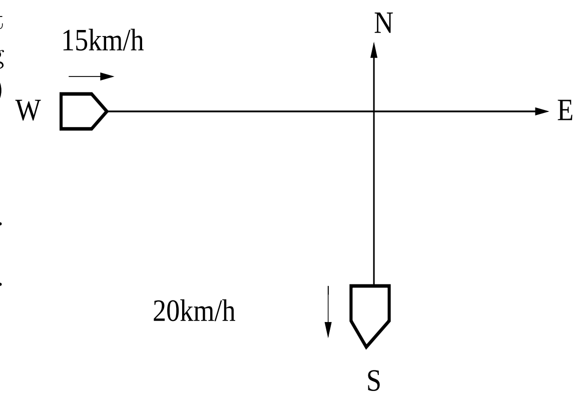

FIGURE 12