Math. 150A, Spring 2024
A selection of solved problems for Chapter 6

Problem 1.

Sketch the region bounded by the graphs of the given equations and calculate the area of the region: $x = y^2 - 3y,\ x - y + 3 = 0$.

$$\int_1^3 [(y-3) - (y^2 - 3y)]\,dy = \int_1^3 [4y - 3 - y^2]\,dy = \left(2y^2 - 3y - \frac{1}{3}y^3\right)\Big|_1^3 = (18 - 9 - 9) - (2 - 3 - \frac{1}{3}) = \frac{4}{3}.$$

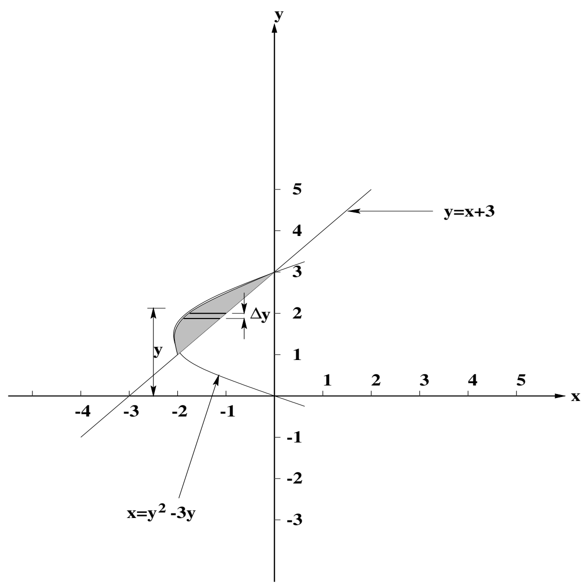


Figure 1. The region between the graphs of $x = y^2 - 3y$ and $y = x + 3$ together with a typical rectangle.

Problem 2

Find the volume of the solid generated when the region to the right is revolved about the specified axis.

(a) About x-axis.

$$V = \pi \int_0^2 [(4 - x^2)^2 - 0^2]\,dx$$
$$= \pi \int_0^2 [16 - 8x^2 + x^4]\,dx$$
$$= \pi \left(16x - \frac{8}{3}x^3 + \frac{1}{5}x^5\right)\Big|_0^2 = \frac{256\pi}{15} \approx 53.6165.$$

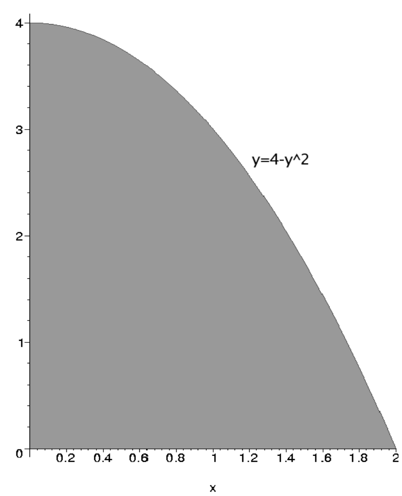


Figure 2. The region between $x = 0$, $y = 0$, and the graph of $y = 4 - x^2$ that is being revolved about the specified axis.

(b) About y-axis.

$$V = \pi \int_0^4 [(\sqrt{4-y})^2 - 0^2]\,dy = \pi \int_0^4 (4 - y)\,dy = \pi \left(4y - \frac{1}{2}y^2\right)\Big|_0^4 = 8\pi \approx 25.1327.$$

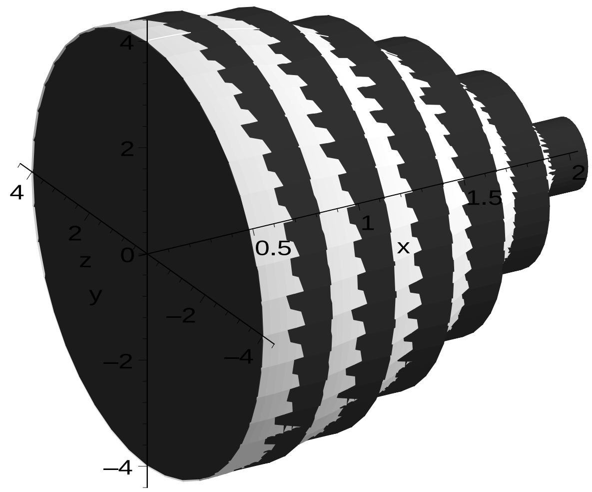


Figure 3. The volume together with approximating disks for the region revolved about the x-axis.

Problem 3

(a) Find the volume of the solid generated by revolving about the x-axis the region bounded by the line $y = 3x$ and parabola $y = 4 - x^2$.

$$V = \pi \int_0^1 [(4-x^2)^2 - (3x)^2]\,dx = \pi \int_0^1 [(16 - 17x^2 + x^4]\,dx = \pi \left(16x - \frac{17}{3}x^3 + \frac{1}{5}x^5\right)\Big|_0^1 = \frac{158\pi}{15} \approx 33.0914.$$

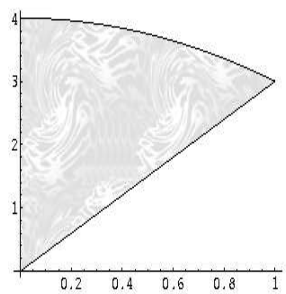


Figure 4. The region bounded by the line $y = 3x$ and parabola $y = 4 - x^2$.

Figure 5. Two different views on the solid of revolution.

(b) Find the volume of the solid generated by revolving about $y = 7$ the region bounded by the line $y = 3x$ and parabola $y = 4 - x^2$.

$$V = \pi \int_0^1 \left|\{(7-3x)^2 - [7-(4-x^2)]^2\}\right| dx = \pi \int_0^1 |x^4 - 3x^2 + 42x - 40|\, dx = \pi \int_0^1 (-x^4 + 3x^2 - 42x + 40)\, dx = \frac{99\pi}{5}.$$

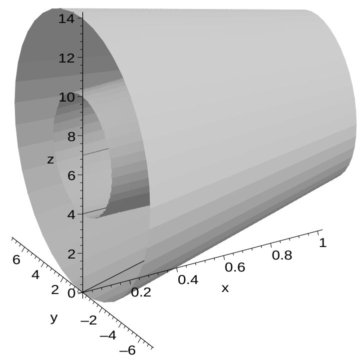


Figure 6. A view of the solid of revolution.

Problem 4

The region R is shown in Figure 7 below. Set up an integral for the volume of the solid of revolution when R is revolved about each line. Use the indicated method.
(a) The y-axis (washers); (b) The x-axis (shells); (c) The line $y = 3$ (shells).

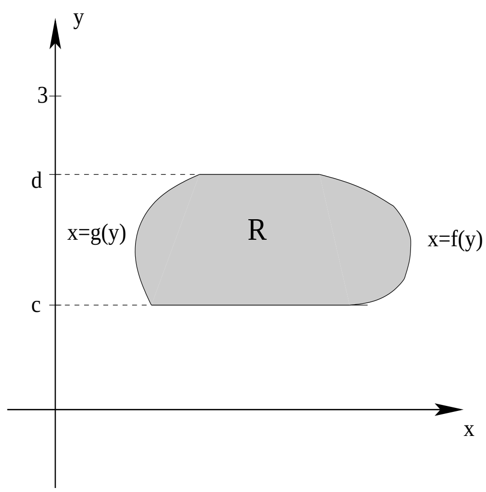


Figure 7. The region R.

(a) $V = \pi \displaystyle\int_c^d [f^2(y) - g^2(y)]\, dy$; (b) $V = 2\pi \displaystyle\int_c^d y[f(y) - g(y)]\, dy$; (c) $V = 2\pi \displaystyle\int_c^d (3-y)[f(y) - g(y)]\, dy$.

Problem 5

Sketch the region R bounded by $y = x^3 + 1$ and $y = 0$ and between $x = 0$ and $x = 2$. Find
(a) Area of R.
(b) Volume of the solid obtained when R is revolved about the y-axis.
(c) Volume of the solid obtained when R is revolved about $y = -1$.
(d) Volume of the solid obtained when R is revolved about $x = 4$.

(a) $A(R) = \int_0^2 (x^3+1)\,dx = 6.$

(b) $V = 2\pi \int_0^2 x(x^3+1)\,dx = \dfrac{84\pi}{5} \approx 52.7788.$

(c) $V = \pi \int_0^2 \left[(x^3+2)^2 - (0-1)^2\right]\,dx = \dfrac{282\pi}{7} \approx 126.5613.$

(d) $V = 2\pi \int_0^2 (4-x)(x^3+1)\,dx = \dfrac{156\pi}{5} \approx 98.0177.$

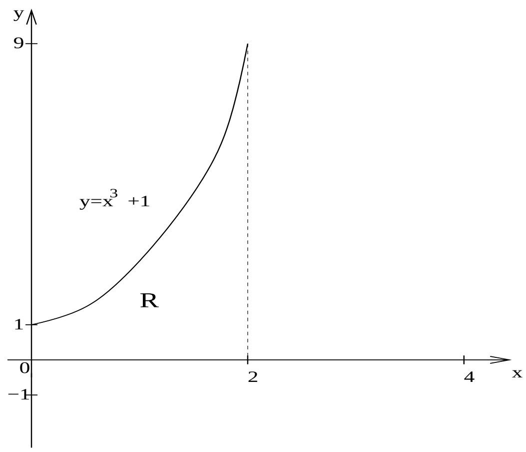


Figure 8. The region R from part (a)

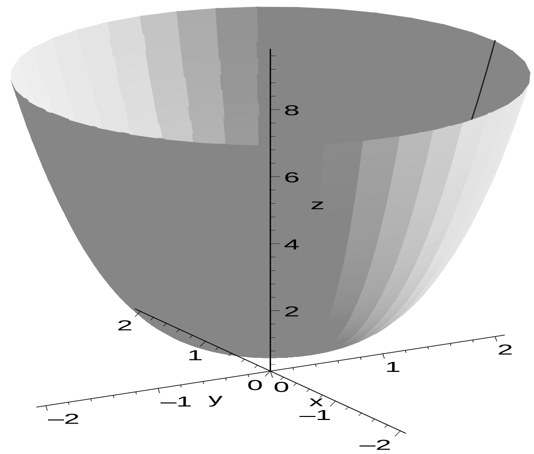


Figure 9. The solid of revolution from part (b).

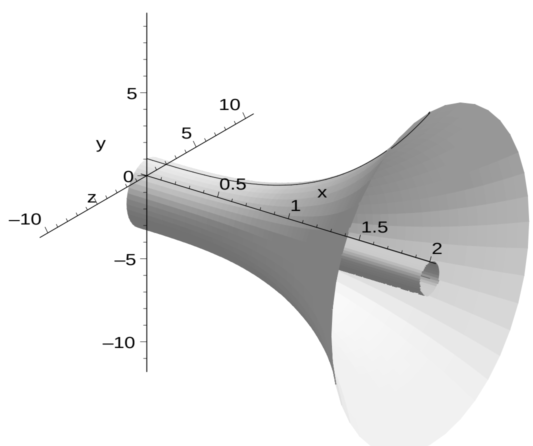


Figure 10. The solid of revolution from part (c).

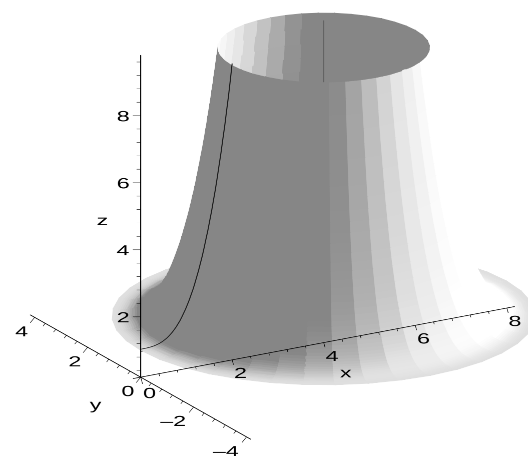


Figure 11. The solid of revolution from part (d).

Problem 6

A round hole of radius a is drilled through the center of a solid sphere of radius b (assume that $b > a$). Find the volume of the solid that remains.

Let R be the region bounded by $y = \sqrt{b^2 - x^2}$, $y = -\sqrt{b^2 - x^2}$, and $x = a$. When R is revolved about the y-axis, it produces the desired solid.

$$V = 2\pi \int_a^b x\left(\sqrt{b^2-x^2} + \sqrt{b^2-x^2}\right)\,dx = 4\pi \int_a^b x\sqrt{b^2-x^2}\,dx = 4\pi \left(-\frac{1}{3}(b^2-x^2)^{3/2}\right)\Big|_a^b = \frac{4\pi}{3}(b^2-a^2)^{3/2}.$$

Note: *The value of V becomes the volume of a solid sphere with radius b when $a = 0$.*

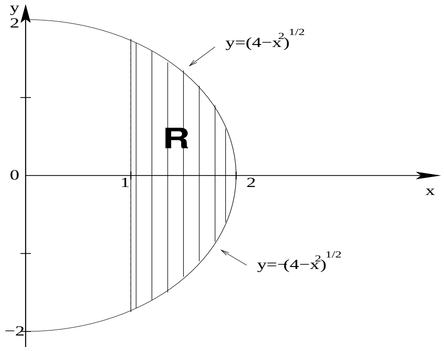


Figure 12. The region R with $b = 2$ and $a = 1$.

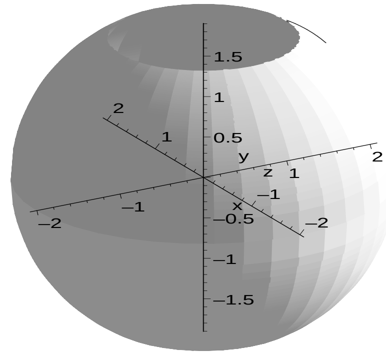


Figure 13. A solid sphere with radius $b = 2$ with a round hole of radius $a = 1$ drilled trough the center of it.

Problem 7

The region R is bounded by $y = x^2$, $y = 0$, and between $x = 0$ and $x = 1$.
(a) Use the *method of disks* to compute the volume of the solid of revolution when R is revolved about the x-axis.
(b) Use the *method of shells* to compute the volume of the solid of revolution when R is revolved about the x-axis.

$$\text{(a)}\quad V = \pi \int_0^1 (x^2)^2\,dx = \pi \left[\frac{1}{5}x^5\right]\Bigg|_0^1 = \frac{\pi}{5} \approx 0.6283.$$

$$\text{(b)}\quad V = \pi - 2\pi \int_0^1 y\sqrt{y}\,dy = \pi - 2\pi \left[\frac{2}{5}y^{5/2}\right]\Bigg|_0^1 = \frac{\pi}{5} \approx 0.6283.$$

Note: *π is volume of the cylinder of height 1 and radius $r = 1$. Part (b) can be also setup as* $2\pi \int_0^1 y(1 - \sqrt{y})\,dy$.

Problem 8

For a certain type of nonlinear spring, the force required to keep the spring stretched a distance s is given by the formula $F = ks^{4/3}$. If the force required to keep it stretched 8 inches is 2 pounds, how much work is done in stretching this spring 27 inches.

We have $F(8) = 2,\ k \cdot 16 = 2$ and $k = 1/8$. Thus,

$$W = \int_0^{27} \frac{1}{8}s^{4/3}\,ds = \frac{1}{8}\left(\frac{3}{7}s^{7/3}\right)\Bigg|_0^{27} = \frac{6561}{56} \approx 117.1607$$

Problem 9

A tank with the vertical cross section shown in Figure 14 is 10 feet long and is full of water. The water is to be pumped to a height 5 feet above the top of the tank. Find the work done in emptying the tank.

A slab of thickness Δy at height y has width $2\sqrt{3^2 - (3-y)^2} = 2\sqrt{6y - y^2}$ (*Do you know why?*) and length 10. The slab will be lifted a distance $5 + 3 - y = 8 - y$.

$$\Delta W \approx g \cdot \delta \cdot 10 \cdot 2\sqrt{6y - y^2} \cdot \Delta y \cdot (8 - y) = 20 \cdot g\delta(8 - y)\sqrt{6y - y^2}\Delta y,$$

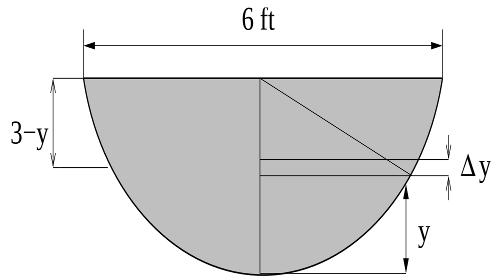


Figure 14. A vertical cross section of the tank (a semi-circle of radius 3)

where $\delta = 62.4$ pounds per cubic foot is the (weight) density of water and $g = 32$ ft/sec^2 is the acceleration due to gravity. The work done in emptying the tank is

$$W = 20 \cdot g \cdot \delta \int_0^3 (8-y)\sqrt{6y-y^2}\,dy = 20 \cdot g \cdot \delta \int_0^3 (3-y)\sqrt{6y-y^2}\,dy + 100 \cdot g \cdot \delta \int_0^3 \sqrt{6y-y^2}\,dy$$

$$= 20 \cdot g \cdot \delta \left(\frac{1}{3}(6y-y^2)^{3/2}\right)\Bigg|_0^3 + 100 \cdot g \cdot \delta \int_0^3 \sqrt{6y-y^2}\,dy$$

Notice that $\int_0^3 \sqrt{6y-y^2}\,dy$ is the area of a quarter of a circle with radius 3. Thus,

$$W = 20 \cdot g \cdot \delta(9) + 100 \cdot g \cdot \delta \left(\frac{9\pi}{4}\right) = \cdot g \cdot 62.4(180 + 225\pi) \approx 1,770,880\,\text{ft-lb}.$$

Problem 10

One cubic foot of air under a pressure of 80 pounds per square inch expands adiabatically (i.e., no heat is transferred to or from air in this process) to 4 cubic feet according to the law: $pv^{1/4} = c$. Find the work done by this gas.

We have, $80\,\dfrac{\text{lb}}{\text{in}^2} = 11,520\,\dfrac{\text{lb}}{\text{ft}^2}$. Thus, $c = 11,520(1)^{1/4} = 11,520$ and $\Delta W \approx p(v)\Delta v = 11,520\,v^{-1/4}\Delta v$.

$$W = \int_1^4 11,520\,v^{-1/4}\,dv = \left(-28,800\,v^{-0.4}\right)\Big|_1^4 = -28,800\left(4^{-0.4} - 1^{-0.4}\right) \approx 12,259\,\text{ft-lb}.$$

Problem 11

A 10 pound monkey hangs at the end of a 20-foot chain that weighs 1/2 pounds per foot. How much work does it do in climbing the chain to the top? Assume that the end of the chain is attached to the monkey.

The total work is equal to the work W_1 to lift the monkey plus the work W_2 to lift the chain.

Now, $W_1 = 10 \cdot 20 = 200$ ft-lb. Let $y = 20$ represent the top. As the monkey climbs the chain, the piece of chain at height y $(0 \le y \le 10)$ will be lifted $20 - 2y$ ft. (*Do you know why?*).

$$\Delta W_2 \approx \frac{1}{2}(20 - 2y)\Delta y = (10 - y)\Delta y.$$

$$W_2 = \int_0^{10} (10 - y)\,dy = \left(10y - \frac{1}{2}y^2\right)\Bigg|_0^{10} = 100 - 50 = 50\,\text{ft-lb}.$$

The total work

$$W = W_1 + W_2 = 250\,\text{ft-lb}.$$

Problem 12

Sketch the region bounded by the graphs of the given equations and calculate the area of the region.

(a) $x = y^2, \quad x - y - 2 = 0;$

$$\int_{-1}^{2} [(y+2) - y^2]\, dy = \left(\frac{1}{2}y^2 + 2y - \frac{1}{3}y^3\right)\Bigg|_{-1}^{2} = \left(2 + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right) = \frac{9}{2} = 4.5.$$

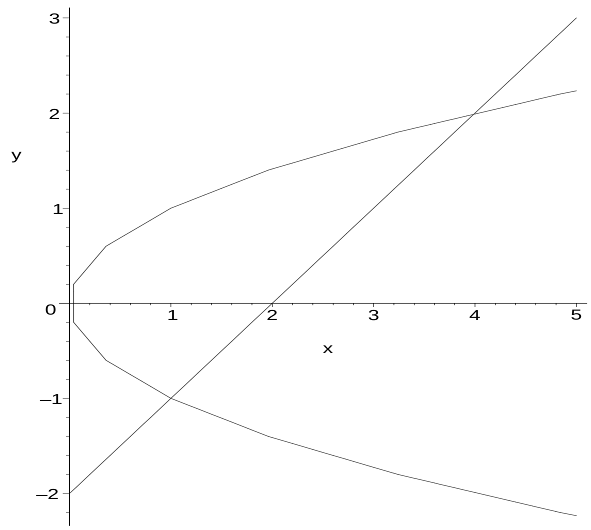


Figure 15. The region between the graphs of $x = y^2$ and $y = x - 2$.

(b) $y = x^2 - 4x + 3 = (x-1)(x-3), \quad x - y - 1 = 0.$

$$\text{(b)} \quad \int_{1}^{4} [(x-1) - (x^2 - 4x + 3)]\, dx = \int_1^4 (5x - x^2 - 4)\, dx = \left(\frac{5}{2}x^2 - \frac{1}{3}x^3 - 4x\right)\Bigg|_1^4$$

$$= \left(40 - \frac{64}{3} - 16\right) - \left(\frac{5}{2} - \frac{1}{3} - 4\right) = \frac{9}{2} = 4.5.$$

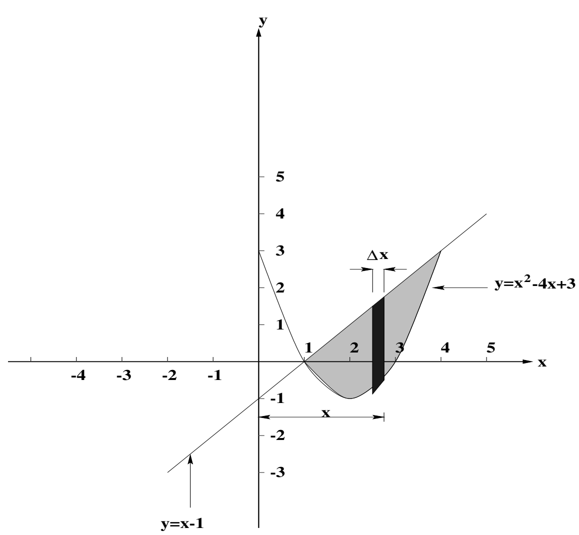
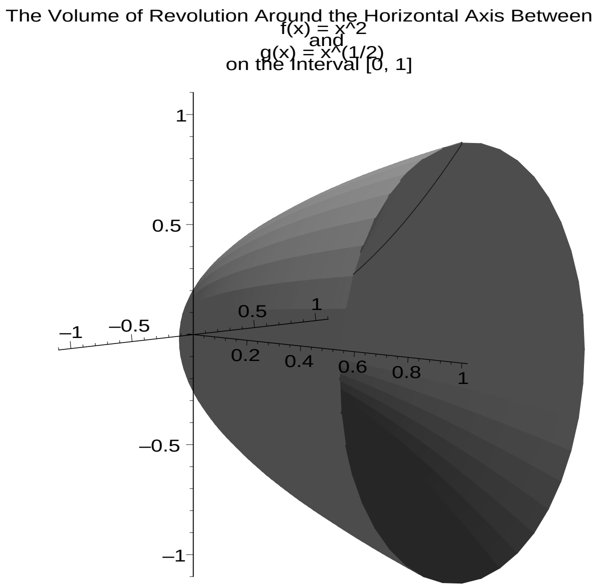


Figure 16. The region between the graphs of $y = x^2 - 4x + 3$ and $y = x - 1$ together with a typical rectangle.

Problem 13.

Find the volume of the solid generated by revolving about the x-axis the region bounded by two parabolas: $y = \sqrt{x}$ and $y = x^2$.

$$\pi \int_0^1 \left[(\sqrt{x})^2 - (x^2)^2\right] dx = \pi \int_0^1 (x - x^4)\, dx = \pi \left(\frac{1}{2}x^2 - \frac{1}{5}x^3\right)\Big|_0^1 = \pi\left(\frac{1}{2} - \frac{1}{5}\right) = \frac{3\pi}{10}.$$



Problem 14.

(a) Sketch the region R bounded by $y = x^2 + 1$ and $y = 0$ and between $x = 1$ and $x = 2$.

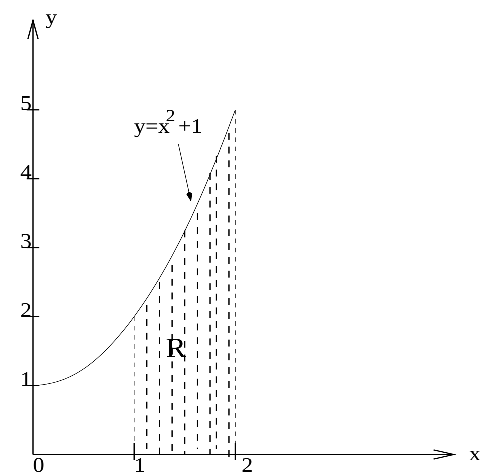


Figure 17. The region R.

Set up **(but do not evaluate!)** integrals for each of the following:

(b) Area of R.

$A = \int_1^2 (x^2 + 1)\, dx.$

(c) Volume of the solid obtained when R is revolved about the y-axis.

$V = 2\pi \int_1^2 x(x^2 + 1)\, dx.$

(d) Volume of the solid obtained when R is revolved about $y = -2$.

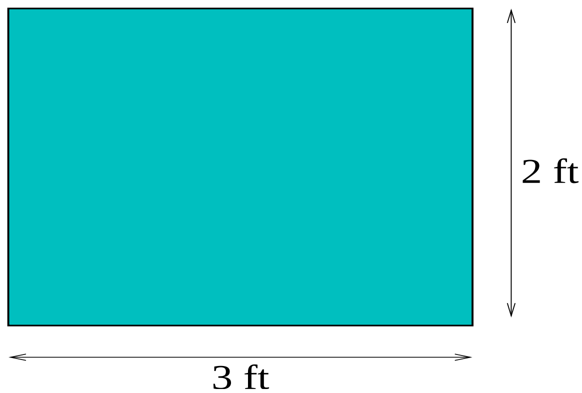
$V = \pi \int\limits_1^2 |(x^2+3)^2 - (0-2)^2| \; dx = \pi \int\limits_1^2 \left[(x^2+3)^2 - 4\right] \; dx$

(e) Volume of the solid obtained when R is revolved about $x = 6$.

$V = 2\pi \int\limits_1^2 (6-x)(x^2+1)\, dx.$

Problem 15.

A tank half-full of water is 20 feet long with its vertical cross section shown below. The water is to be pumped to a height of 3 feet above the tank. Find the work done in emptying the tank. The density of the water is $\delta = 62.4$ pounds per cubic foot.



A slab of thickness Δy at height y has width 3 ft and length 20 ft The slab will be lifted a distance $5 - y$. An approximate work done in lifting this slab of water is

$$\Delta W \approx g \cdot \delta \cdot 20 \cdot 3 \cdot (5-y)\Delta y,$$

where $\delta = 62.4$ pounds per cubic foot is the (weight) density of water and $g = 32$ ft/sec^2 is the acceleration due to gravity. The work done in emptying the tank is

$$W = 60 \cdot g \cdot \delta \int_0^1 (5-y)\, dy = 60 \cdot g \cdot \delta \left(5y - \frac{1}{2}y^2\right)\Bigg|_0^1 = 270 \cdot g \cdot \delta = 539,136 \text{ lb}\cdot\text{ft}.$$