Math. 150a, Fall Spring 2024

Solutions to Various Practice Problems II

Problem 1.
Consider $f(x) = x^{1/4}$, $x = 16$, and $\Delta x = -1$.

Using $f(x + \Delta x) \approx f(x) + f'(x)\Delta x$, we have $(15)^{1/4} \approx 2 - \frac{1}{4}\frac{1}{16^{3/4}} = 2 - \frac{1}{32} = \frac{63}{32} = 1.96875$.

Problem 2.
For $f(x) = (1 + 2x) - n$, $f'(x) = -2n(1 + 2x)^{-n-1}$.
Thus, with $a = 0$, $L(x) = f(a) + f'(a)(x - a) = 1 + (-2n)(x - 0) = 1 - 2nx$.

Problem 3.
$f(x) = \frac{1}{1+x}$, $x = 4$, $100\frac{\Delta x}{x} = 2\%$, $f'(x) = \frac{-1}{(1+x)^2}$.
Thus, $\Delta f \approx f'(x)\Delta x = \left(\frac{-1}{25}\right) \cdot (0.02) \cdot (4) = -0.0032$ and $100\frac{\Delta f}{f} = \pm 100\frac{0.0032}{\frac{1}{1+4}} = \pm 1.6\%$.

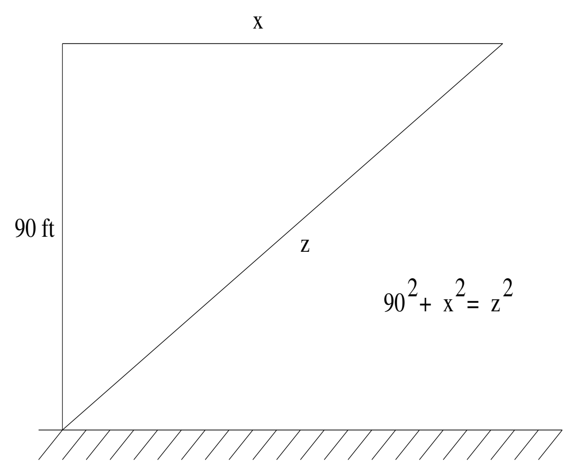
Problem 4.

$90^2 + x^2 = z^2; \quad \frac{dx}{dt} = 5$

$2x\frac{dx}{dt} = 2z\frac{dz}{dt}$

When $z = 150$, $x = 120$, so

$\frac{dz}{dt} = \frac{x}{z} \cdot \frac{dx}{dt} = \frac{120}{150} \cdot 5 = 4\,\text{ft/s}$



Problem 5.

We want to find $\frac{dy}{dt}$ when $y = 12$. For any t we have the relation $x^2(t) + y^2(t) = L^2 = 20^2 = 400$. Differentiating with respect to t the last identity, we obtain $2x(t)\frac{dx}{dt} + 2y(t)\frac{dy}{dt} = 0$, for all t. Hence, $\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$. Furthermore, when $y = 12\,\text{ft}$ $x = \sqrt{20^2 - 12^2} = \sqrt{256} = 16\,\text{ft}$, and since $\frac{dx}{dt} = 2\,\text{ft/sec}$, we obtain $\frac{dy}{dt} = -\frac{16}{12}(2) = -\frac{8}{3}$ ft/sec.
(*The rate of change is negative since the ladder slides down and the orientation of the y-axis is positive upwards.*)

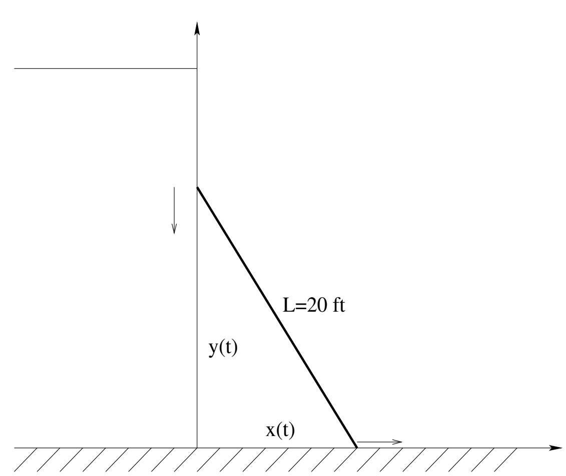


FIGURE 1. A ladder is sliding down the building.

Problem 6.
We use the following linear approximation formula

$$f(x) \approx f(a) + f'(a)(x-a)$$

in order to approximate $\sqrt[4]{81.6}$. Take $x = 81.6,\ a = 81$, and $f(x) = \sqrt[4]{x}$.

Since $f'(x) = \dfrac{1}{4\sqrt[4]{x^3}}$, and thus $f'(81) = \dfrac{1}{108}$, we have

$$\sqrt[4]{81.6} \approx 3 + \frac{0.6}{108} = 3 + \frac{1}{180} = \frac{541}{180}.$$

Problem 7.
The length of a circle with radius r is given by $L = 2\pi r$. Thus the change, ΔL, in the length of the circle can approximated by $2\pi\Delta r$ (since $L'(r) = 2\pi$). Since $\Delta r = 2$ feet, we have $\Delta L \approx 2\pi \cdot 2 = 4\pi$ ft. (≈ 12.56637 ft)

Problem 8.
For $f(x) = (1-x)^{-n}$, (n positive integer), $a = 0$; $f'(x) = (-n)(1-x)^{-n-1}(-1)$ and $L(x) = f(0) + f'(0)(x-0) = 1 + nx$.
In other words, for x close to zero, $(1-x)^{-n} \approx 1 + nx$.

Problem 9.
For $v(R) = cR^2$, $v'(R) = 2cR$. We have $100\frac{\Delta R}{R} = 5\%$; thus

$$100\frac{\Delta v}{v} \approx \pm 100\frac{2cR\Delta R}{cR^2} = \pm 2\left(100\frac{\Delta R}{R}\right) = \pm 10\%.$$

Problem 10.
For $f(R) = a\dfrac{R}{k+R}$, $f'(R) = a\dfrac{k+R-R}{(k+R)^2} = a\dfrac{k}{(k+R)^2}$. We have

$$\left(100\frac{\Delta f}{f}\right) \approx \pm 100\frac{f'(R)\Delta R}{f(R)} = \pm 100\frac{a\frac{k}{(k+R)^2}}{a\frac{R}{k+R}}\Delta R = \pm 100\frac{k}{k+R}\cdot\frac{\Delta R}{R} = \pm\frac{k}{k+R}\left(100\frac{\Delta R}{R}\right).$$

Problem 11.

$$f'(x) = \frac{(1+x^2) - x(2x)}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2} = \frac{(1-x)(1+x)}{(1+x^2)^2}$$

Since the denominator is always positive, $f'(x)$ has the same sign as $(1+x)(1-x)$. Solving the inequalities $(1+x)(1-x) < 0$ and $(1+x)(1-x) > 0$, we obtain that $f'(x) > 0$ on the interval $(-1, 1)$ and $f'(x) < 0$ on the set $(-\infty, -1) \cup (1, \infty)$. We conclude from Theorem 1 on page 269 that $f(x)$ is increasing on the interval $(-1, 1)$ and it is decreasing on the set $(-\infty, -1) \cup (1, \infty)$.

Problem 12.

(a): $f'(x) = 4x^3 - 4x = 4x(x+1)(x-1)$; $f''(x) = 12x^2 - 4 = 4(3x^2 - 1)$.
Critical points are $-1, 0$, and 1.
$f''(-1) = 8 > 0$ so -1 -1 gives a local minimum value of $f(-1) = 2$.
$f''(0) = -4 < 0$ so 0 gives a local maximum value of $f(0) = 3$.
$f''(1) = 8 > 0$ so 1 gives a local minimum value of $f(1) = 2$.
Points of inflections are $-\dfrac{1}{\sqrt{3}}$ and $\dfrac{1}{\sqrt{3}}$.

(b): $g'(x) = 4x^3 + 6x^2 = 2x^2(2x+3)$; $g''(x) = 12x^2 + 12x = 12x(x+1)$.
Critical points are 0 and $-\dfrac{3}{2}$.
$g'(x) < 0$ on $\left(-\infty, -\dfrac{3}{2}\right)$ and $g'(x) > 0$ on $\left(-\dfrac{3}{2}, 0\right) \cup (0, \infty)$. Using the First Derivative Test, $-\dfrac{3}{2}$ gives a local minimum of $g\left(-\dfrac{3}{2}\right) = -\dfrac{27}{16}$; 0 does **NOT** give a local extremum.
Points of inflections are -1 and 0.

(c): $h'(x) = 2 + \frac{2}{3}x^{-1/3} = \frac{2}{3}\frac{3x^{1/3}+1}{x^{1/3}}$; $h''(x) = -\frac{2}{9}x^{-4/3}$.

Critical points are $-\frac{1}{27}$ and 0.

$h''\left(-\frac{1}{27}\right) = -16 < 0$ so $-\frac{1}{27}$ gives a local maximum value of $h\left(-\frac{1}{27}\right) = \frac{1}{27}$.

$h'(x) < 0$ on $\left(-\frac{1}{27}, 0\right)$ and $h'(x) > 0$ on $\left(-\infty, -\frac{1}{27}\right) \cup (0, \infty)$. Therefore, using the First Derivative Test, 0 gives a local minimum value of $h(0) = 0$.

NOTE: *Since $h''(0)$ is not defined, one cannot use the Second Derivative Test to check $x = 0$ for a local extremum.*

There are **NO** points of inflections.

(d): $F'(x) = \frac{x(x^2+2)}{(x^2+1)^{3/2}}$; $F''(x) = \frac{2-x^2}{(x^2+1)^{5/2}}$; Critical point is 0.

$F''(0) = 2 > 0$ so 0 gives a local minimum value of $F(0) = 0$.

Points of inflections are $-\sqrt{2}$ and $\sqrt{2}$.

Problem 13.

(a) Critical points: $x = 0$ and $x = 1$.

(b) f is increasing on $(1, \infty)$ and f is decreasing on $(-\infty, 1)$.

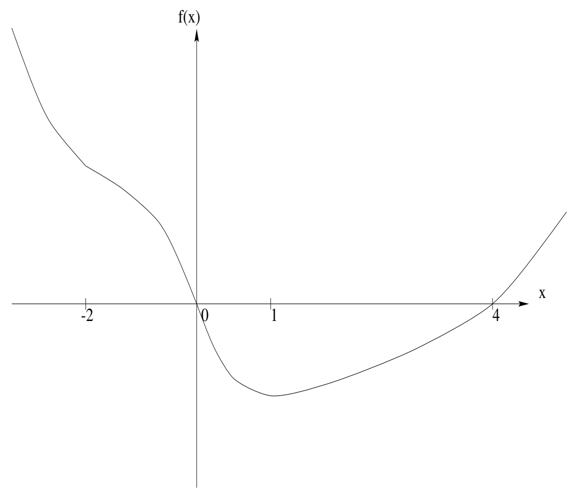
(c) Local minimum at $x = 1$.

Note: There is **no** local extremum at $x = 0$ since f' does not change sign at $x = 0$.

(d) f is concave up on $(-\infty, -2) \cup (0, \infty)$ and f is concave down on $(-2, 0)$.

(e) Inflection points: $x = -2$ and $x = 0$.

(f) For the graph see the figure to the right.



Problem 14.

The domain of the function is $D_f = (-\infty) \cup (-2, 2) \cup (2, \infty)$.

(a) $f'(x) = \frac{10x}{(x^2-4)^2}$ and$x = 0$ is the only critical point.

NOTE: $x = \pm 2$ *are not critical points since they do not belong to the domain* D_f.

(b) f' changes sign at $x = 0$ from $-$ to $+$ and therefore it is a relative minimum.

(c) $f''(x) = -10\frac{3x^2+4}{(x^2-4)^3}$ and therefore there are no points of inflection. Furthermore, $f'' > 0$ on $(-2, 2)$ (i.e., f is concave upward on $(-2, 2)$ and $f'' < 0$ on $(-\infty, -2) \cup (2, \infty)$ (i.e., f is concave downward on $(-\infty, -2) \cup (2, \infty)$).

(d) $\lim_{x\to\pm\infty} \frac{x^2-9}{x^2-4} = 1$ and therefore $y = 1$ is a horizontal asymptote. Since the numerator $(x^2 - 9)$ is not zero at $x = \pm 2$, we see that $x = -2$ and $x = 2$ are vertical asymptotes.

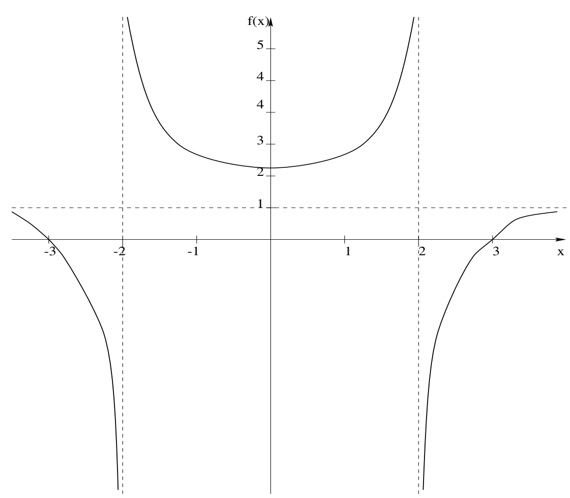
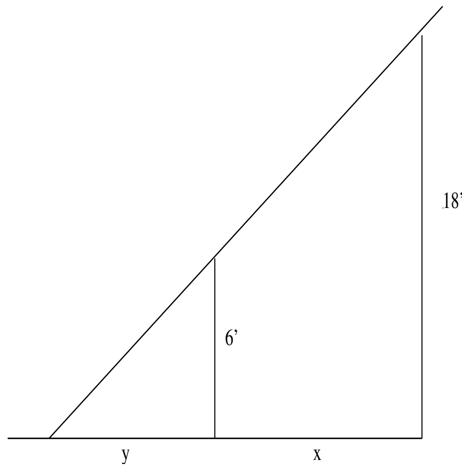


FIGURE 2. (e) The graph of f.

Problem 15.
Refer to the diagram at the right.

$$\begin{aligned}\frac{y}{6} &= \frac{y+x}{18} \qquad \textbf{(WHY?)}\\ 18y &= 6(y+x)\\ 3y &= y+x\\ 2y &= x\\ y &= \frac{1}{2}x.\end{aligned}$$



The distance of tip of man's shadow to the street light is $D = y + x = \frac{3}{2}x$. Differentiating implicitly, we have

$$\frac{dD}{dt} = \frac{3}{2}\frac{dx}{dt}$$

and when $\frac{dx}{dt} = 6$

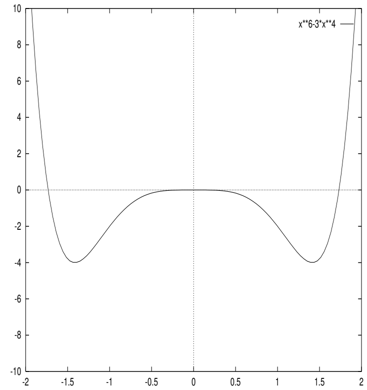
$$\frac{dD}{dt} = \frac{3}{2} \cdot 6 = 9, \quad \text{or} \quad 9 \text{ ft/sec.}$$

Problem 16.
(a) $f'(x) = 6x^3(x^2-2) = 6x^3(x-\sqrt{2})(x+\sqrt{2})$.
f is decreasing on $(-\infty, \sqrt{2}) \cup (0, \sqrt{2})$. f is increasing on $(-\sqrt{2}, 0) \cup (\sqrt{2}, \infty)$.
$f''(x) = 6x^2(5x^2-6) = 30x^2\left(x^2 - \frac{6}{5}\right) = 30x^2\left(x - \sqrt{\frac{6}{5}}\right)\left(x - \sqrt{\frac{6}{5}}\right)$.
f is concave down on $\left(-\sqrt{\frac{6}{5}}, \sqrt{\frac{6}{5}}\right)$. f is concave up on $\left(-\infty, -\sqrt{\frac{6}{5}}\right) \cup \left(\sqrt{\frac{6}{5}}, \infty\right)$.

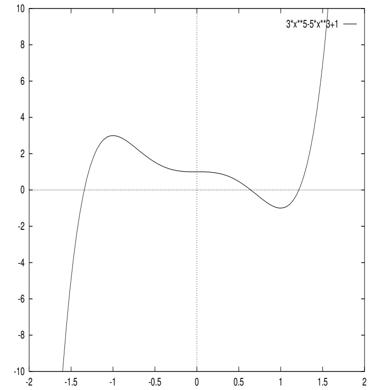
(b) $g'(x) = 15x^2(x+1)(x-1)$.
g is decreasing on $(-1, 1)$. g is increasing on $(-\infty, -1) \cup (1, \infty)$.
$g''(x) = 30x(2x^2-1) = 60x\left(x^2 - \frac{1}{2}\right) = 60x\left(x + \sqrt{\frac{1}{2}}\right)\left(x - \sqrt{\frac{1}{2}}\right)$
g is concave up on $\left(-\sqrt{\frac{1}{2}}, 0\right) \cup \left(\sqrt{\frac{1}{2}}, \infty\right)$. g is concave down on $\left(-\infty, -\sqrt{\frac{1}{2}}\right) \cup \left(0, \sqrt{\frac{1}{2}}\right)$

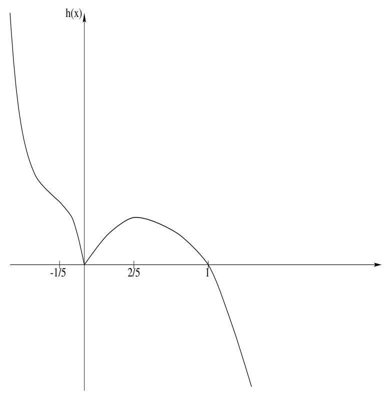
(c) $h(x) = x^{2/3} - x^{5/3}$.
$h'(x) = \frac{2}{3}x^{-1/3} - \frac{5}{3}x^{2/3} = \frac{2-5x}{3x^{1/3}}$.
h is decreasing on $(-\infty, 0) \cup \left(\frac{2}{5}, \infty\right)$. h is increasing on $\left(0, \frac{2}{5}\right)$
$h''(x) = -\frac{2}{9}x^{-4/3} - \frac{10}{9}x^{-1/3} = -\frac{2+10x}{9x^{4/3}}$.
h is concave up on $\left(-\infty, -\frac{1}{5}\right)$. h is concave down on $\left(-\frac{1}{5}, 0\right) \cup (0, \infty)$.

(a)



(b)





(c)

Problem 17.

For f with $f'(x) = 2(x+2)(x+1)^2(x-2)^4(x-3)^3$ we have
$f'(x) < 0 \quad \text{on} \quad (-2,-1) \cup (-1,2) \cup (2,3)$
$f'(x) > 0 \quad \text{on} \quad (-\infty,-2) \cup (3,\infty)$.
There is a relative maximum at $x = -2$ and there is a relative minimum at $x = 3$.

NOTE: *There are* ***NO*** *relative extrema at $x = -1$ and $x = 2$ since f' does* ***NOT*** *change sign at these points.*

Problem 18.

(1) $f(0) = 3$; $\quad f(3) = 0$; $\quad f(6) = 4$;
(2) $f'(x) < 0$ on $(0,3)$; $\quad f'(x) > 0$ on $(3,6)$;
(3) $f''(x) > 0$ on $(0,5)$; $\quad f''(x) < 0$ on $(5,6)$.

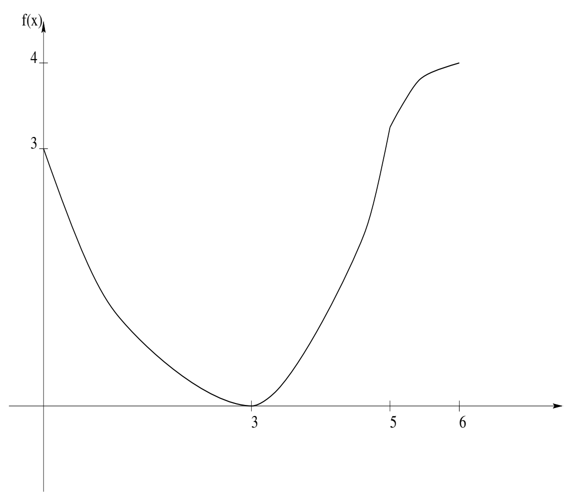


FIGURE 3. A possible graph of the function defined on the interval $[0,6]$ and satisfying the conditions (1)-(3).

Problem 19.
Let x denote 1/2 of the lower base of the rectangular, y the size of the vertical side. The area $A = 2xy = 2x(8-x^2)$ and $A'(x) = 16 - 6x^2$. Furthermore, $A''(x) = -12x$, and $x = \frac{2}{3}\sqrt{6}$ is a local maximum (and global maximum). Now, $y = 8 - \left(\frac{2}{3}\sqrt{6}\right)^2 = \frac{16}{3}$. The largest area is equal to $2 \cdot \frac{2}{3}\sqrt{6} \cdot \frac{16}{3} = \frac{64\sqrt{6}}{9}$.

Problem 20.
Let x denote the increase in rental above \$10. Then the income for car rental agency $I(x) = (10+x)(24-x) = -x^2 + 14x + 240$. Now, $I'(x) = -2x + 14$ and $I''(x) = -2$, hence, $x = 7$ is a global maximum of function $I(x)$. The agency should charge \$17 per day.

Problem 21.
(a) $f'(x) = 8\frac{4-x^2}{(x^2+4)^2}$. $f''(x) = \frac{16x(x^2-12)}{(4+x^2)}$. Critical points and local extrema: $x = -2$ and $x = 2$. Points of inflection: $x = -2\sqrt{3}$, $x = 0$, $x = 2\sqrt{3}$. Horizontal asymptotes: $y = 0$. Vertical asymptotes: none.

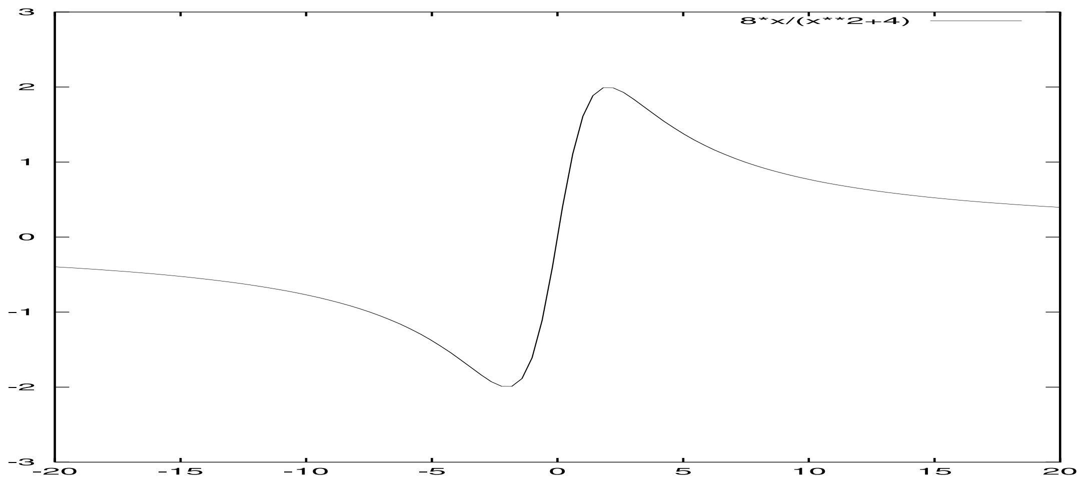


FIGURE 4. Graph of the function in Problem 21(a)

(b) $f'(x) = \frac{2}{3}\frac{1}{x^{1/3}}$. $f''(x) = -\frac{2}{9}\frac{1}{x^{4/3}}$. Critical points and local extrema: $x = 0$. Points of inflection: none. Horizontal asymptotes: none. Vertical asymptotes: none.

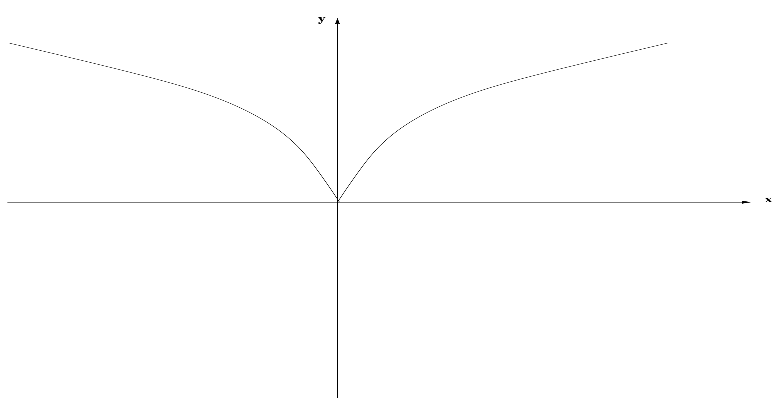


FIGURE 5. Graph of the function in Problem 21(b)

(c) $f'(x) = \frac{1}{3}(x-2)^{-2/3}$. $f''(x) = -\frac{2}{9}(x-2)^{-5/9}$. Critical points: $x = 2$. Local extrema: none. Points of inflection: $x = 2$. Horizontal asymptotes: none. Vertical asymptotes: none.

(d) $f'(x) = \frac{-5x^4}{(x^5+1)^2}$. $f''(x) = 10\frac{x^3(3x^5-2)}{(1+x^5)^3}$. Critical points: $x = 0$. Local extrema: none. Points of inflection: $x = 0$, $x = \left(\frac{2}{3}\right)^{1/5}$. Horizontal asymptotes: $y = 0$. Vertical asymptotes: $x = -1$.

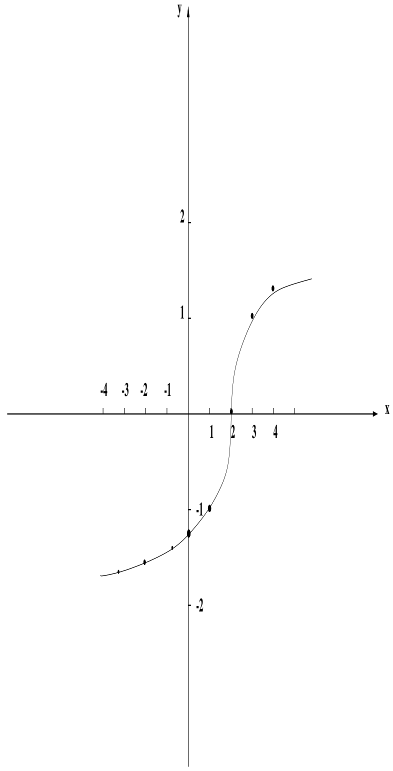


FIGURE 6. Graph of the function in Problem 21(c)

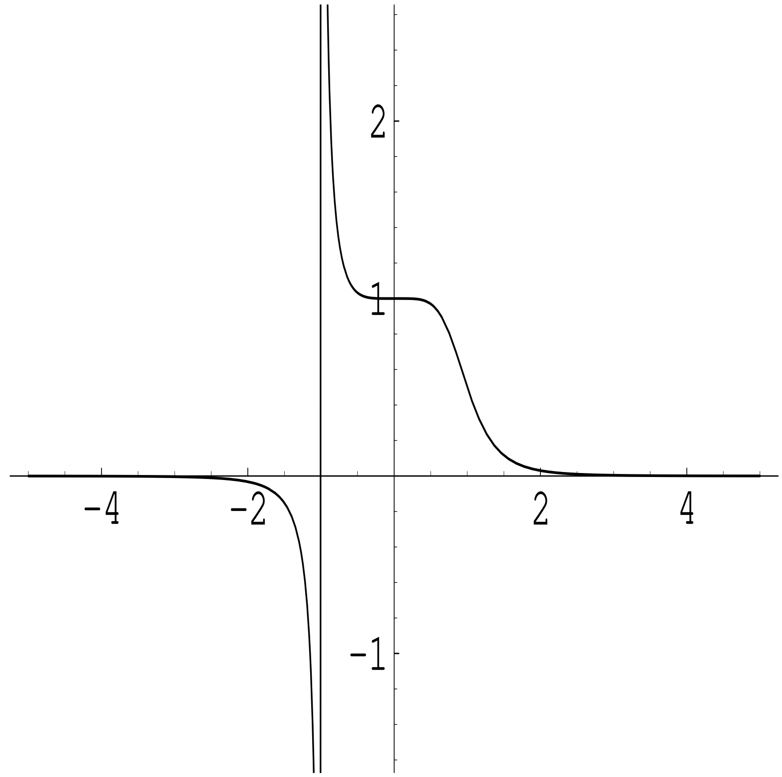


FIGURE 7. Graph of the function in Problem 21(d)

Problem 22.
(a) $f(x) = x^{\frac{5}{3}}$ is continuous for all $x \in \mathbb{R}$, thus, in particular, for $x \in [-1, 1]$. Its derivative is $f'(x) = \frac{5}{3}x^{\frac{2}{3}}$, which is also continuous on $(-1, 1)$. Thus the assumptions of the mean Value Theorem are satisfied and there exists at least one $c \in (-1, 1)$ such that

$$\frac{f(1) - f(-1)}{1 - (-1)} = f'(c) \quad \Longrightarrow \quad 1 = \frac{5}{3}c^{\frac{2}{3}} \quad \Longrightarrow \quad c = \pm\left(\frac{3}{5}\right)^{\frac{3}{2}} \approx \pm 0.46.$$

Note: *Both* $\pm\left(\frac{3}{5}\right)^{\frac{3}{2}} \in [-1, 1]$.

(b) The function $g(x) = |x|$ is continuous for all $x \in \mathbb{R}$, in particular, for $x \in [-2, 2]$. However,

$$g'(x) = \begin{cases} -1, & \text{if } x < 0, \\ 1, & \text{if } x > 0, \end{cases}$$

and $g'(0)$ does **NOT** exist. Thus, $g'(x)$ does not exist for all $x \in (-2, 2)$. The assumptions of the Mean Value Theorem are not satisfied.

Problem 23.
Let $f(x) = \sqrt{x}$ so $f'(x) = \dfrac{1}{2\sqrt{x}}$. Apply the Mean Value Theorem to f on the interval $[x, x + 2]$ for $x > 0$. (*Do you know why the assumptions of the Mean Value Theorem are satisfied ?*) Thus

$$\sqrt{x+2} - \sqrt{x} = \frac{1}{2\sqrt{c}}(x + 2 - x) = \frac{1}{\sqrt{c}} \quad \text{for some } c \in (x, x + 2).$$

Next observe that

$$\frac{1}{\sqrt{x+2}} < \frac{1}{\sqrt{c}} < \frac{1}{\sqrt{x}}.$$

Thus as $x \to \infty$, $\dfrac{1}{\sqrt{c}} \to 0$. Therefore

$$\lim_{x\to\infty}\left(\sqrt{x+2} - \sqrt{x}\right) = \lim_{x\to\infty}\frac{1}{\sqrt{c}} = 0.$$

Problem 24.
(a)

$$\text{(a)} \quad \int \left(x - \frac{1}{x}\right)^2 dx = \int \left(x^2 - 2 + \frac{1}{x}\right) dx = \frac{1}{3}x^3 - 2x - \frac{1}{x} + C.$$

$$\text{(b)} \quad \int \frac{1}{\sqrt{3x+2}}\, dx = \begin{Bmatrix} u = 3x+2 \\ du = 3dx \end{Bmatrix} = \frac{1}{3}\int u^{-1/2}\, du = \frac{2}{3}u^{1/2} + C = \frac{2}{3}(3x+2)^{1/2} + C.$$

$$\text{(c)} \quad \int x(1-x^2)^{1/4}\, dx = \begin{Bmatrix} u = 1-x^2 \\ du = -2xdx \end{Bmatrix} = -\frac{1}{2}\int u^{1/4}\, du = -\frac{2}{5}u^{5/4} + C = -\frac{2}{5}(1-x^2)^{5/4} + C.$$

$$\text{(d)} \quad \int \frac{(1+\tan(x))^{1/3}}{\cos^2(x)}\, dx = \begin{Bmatrix} u = 1+\tan x \\ du = (\cos^2 x)^{-2} dx \end{Bmatrix} = \int u^{1/3}\, du = \frac{3}{4}u^{4/3} + C = \frac{3}{4}(1+\tan x)^{4/3} + C.$$

Problem 25.

$$\text{(a)} \quad \sum_{i=1}^{5}\left[(3i+4)^{10} - (3i+1)^{10}\right] = \sum_{i=1}^{5}\left\{[3(i+1)+1]^{10} - [3i+1]^{10}\right\}.$$

This is a collapsing sum, and thus,

$$\sum_{i=1}^{5}\left[(3i+4)^{10} - (3i+1)^{10}\right] = 19^{10} - 4^{10}.$$

$$\text{(b)} \quad \sum_{k=1}^{n}(3k^2 - 2k + 1) = 3\frac{n(n+1)(2n+1)}{6} - 2\frac{n(n+1)}{2} + n = \frac{1}{2}n(2n^2 + n + 1).$$

Problem 26.
$A_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x = \sum_{i=1}^{n} f(x_i)\Delta x$. Here, $f(x) = x^3$, $\Delta x = \frac{2}{n}$ and $x_i = \frac{2i}{n}$. Thus,

$$\sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n}\left(\frac{2i}{n}\right)^3 \frac{2}{n} = \frac{16}{n^4}\sum_{i=1}^{n} i^3 = \frac{16}{n^4}\left[\frac{n(n+1)}{2}\right]^2 \underset{n\to\infty}{\rightarrow} 4.$$

Problem 27.
$\int_{-1}^{2}(2+4x)\, dx = -A + B = -\frac{1}{2}\cdot 2 \cdot \frac{1}{2} + \frac{1}{2}\cdot\frac{5}{2}\cdot 10 = 12$, where A is the area of the right triangle with vertices $(-1,-2)$, $(-1,0)$, and $(-\frac{1}{2},0)$, and B is the area of the right triangle with vertices $(-\frac{1}{2},0)$, $(2,0)$, and $(2,10)$. For details see the picture below.

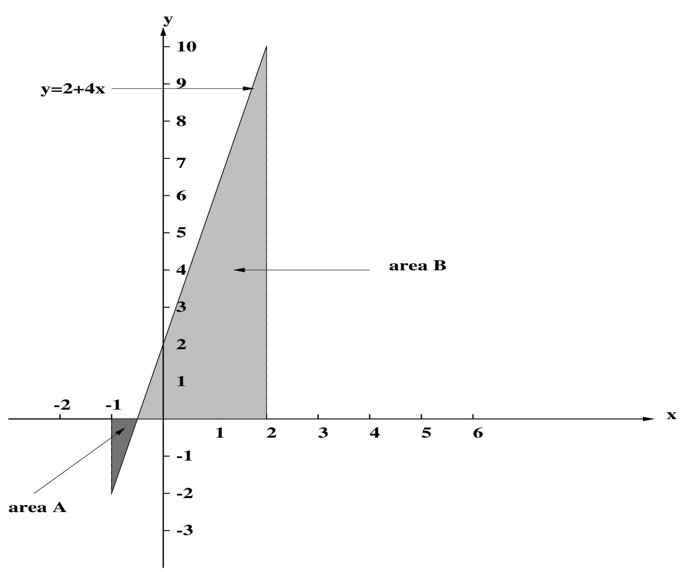


FIGURE 8. Graph of $y = 2 + 4x$ between $[-1, 2]$ with the corresponding regions below and above the graph.

Problem 28.
Partition the interval $[2,6]$ into n equal subintervals, each of length $\Delta x = 4/n$. In each subinterval $[x_{i-1}, x_i]$ use $\bar{x}_i = x_i = 2 + \dfrac{4i}{n}$. Then

$$R_P = \sum_{i=1}^{n} f(\bar{x}_i)\Delta x = \sum_{i=1}^{n} \left[(2 + \frac{4i}{n})^2 + 6\right] \frac{4}{n} = \sum_{i=1}^{n} \left[10 + \frac{16i}{n} + \frac{16i^2}{n^2}\right] \frac{4}{n} \underset{n\to\infty}{\longrightarrow} 40 + 32 + \frac{64}{3} = \frac{280}{3}.$$