Math. 150a, Spring 2024

Solutions to Various Practice Problems I

Problem 1.

(a) $x_{1,2} = \dfrac{-2 \pm \sqrt{4 + 32 \cdot 3}}{16} = \dfrac{-1 \pm 5}{8} = \dfrac{1}{2}, -\dfrac{3}{4}$

(b) $x_{1,2} = \dfrac{2 \pm \sqrt{4 + 20}}{2} = 1 \pm \sqrt{6}$ $\quad (\approx 3.44948, \approx -1.44948)$

Problem 2.

(a) We have, $x^2 + 2x - 15 = (x+5)(x-3)$. Using the sign pattern method, the solution set of the inequality $x^2 + 2x - 15 \geq 0$ is the set $(-\infty, -5] \cup [3, +\infty)$.

(b) Note that the quadratic equation $x^2 + x + 1 = 0$ has no real solutions. Since the coefficient in front of x^2 is positive, the graph of $y = x^2 + x + 1$ lies above the x-axis (**CHECK IT !**).
Another way to look at this case is to notice that after completion of square, $x^2 + x + 1 = (x + 1/2)^2 + 3/4 > 0$. In other words, $x^2 + x + 1 > 0$ for all x. Thus, there is no solution to the inequality $x^2 + x + 1 \leq 0$, or equivalently, the solution set is the empty set.

(c) The inequality $\dfrac{2x+1}{2-x} \leq 1$ is equivalent to $\dfrac{3x-1}{2-x} \leq 0$. Indeed,

$$\frac{2x+1}{2-x} \leq 1 \iff \frac{2x+1}{2-x} - 1 \leq 0 \iff \frac{2x+1-(2-x)}{2-x} \leq 0 \iff \frac{3x-1}{2-x} \leq 0 \iff (3x-1)(2-x) \leq 0 \text{ and } x$$

The solution set of the last inequality is the set $(-\infty, \frac{1}{3}] \cup (2, +\infty)$.

Note: Solving the inequality $\dfrac{2x+1}{2-x} \leq 1$ by reducing it to $2x + 1 \leq 2 - x$ is **incorrect** and leads to a **wrong** solution ! Check it !

Problem 3.

$$(f \circ g)(x) = f(g(x)) = f\left(\frac{1}{x-1}\right) = \sqrt{\frac{1}{x-1} + 1} = \sqrt{\frac{x}{x-1}}$$

Note that $\sqrt{\dfrac{x}{x-1}} \neq \dfrac{\sqrt{x}}{\sqrt{x-1}}$. **Do you know why ?**

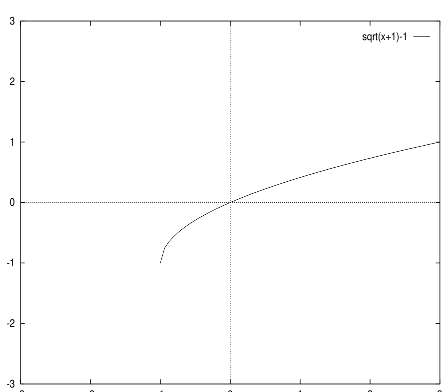
$$(g \circ f)(x) = g(f(x)) = g\left(\sqrt{x+1}\right) = \frac{1}{\sqrt{x+1} - 1}$$

Problem 4.

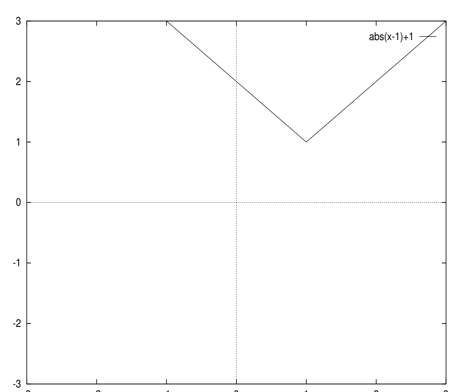
$$\frac{f(-1+h) - f(-1)}{h} = \frac{\dfrac{1}{-1+h} - \dfrac{1}{-1}}{h} = \frac{\dfrac{-1-(-1)-h}{(-1)(-1+h)}}{h} = \frac{1}{(-1+h)}$$

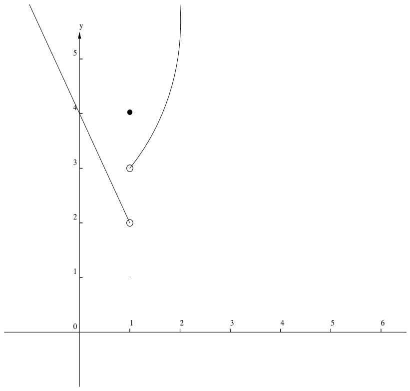
Problem 5.

(a)

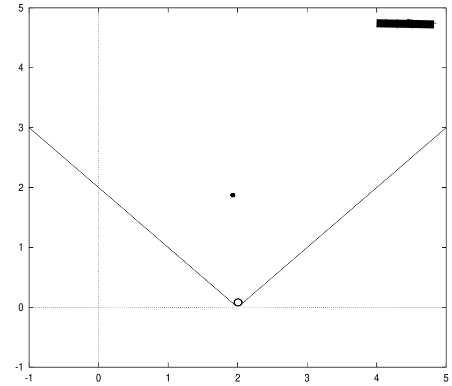


(b)





(c)



(d)

Problem 6.
The slope, m, of the line $4x + 5y + 16 = 0$ is $m = -\dfrac{4}{5}$. Note, the equation $4x + 5y + 16 = 0$ is equivalent to the equation $y = -\dfrac{4}{5}x - \dfrac{16}{5}$. The slope of the line perpendicular to $4x + 5y + 16 = 0$ is equal to $\dfrac{5}{4}$. Therefore, the equation of the line perpendicular to $5x + 4y + 16 = 0$ and passing through $(-1, 0)$ is given by $y - 0 = \dfrac{5}{4}(x + 1)$, or equivalently $y = \dfrac{5}{4}x + \dfrac{5}{4}$. Another equivalent form is $4y - 5x - 5 = 0$.

Problem 7.

$$(f \circ g)(x) = f(g(x)) = f\left(\frac{x}{3x-8}\right) = \frac{\dfrac{2x}{3x-8}}{\dfrac{x}{3x-8} + 5} = \frac{\dfrac{2x}{3x-8}}{\dfrac{x + 15x - 40}{3x-8}} = \frac{2x}{16x - 40} = \frac{x}{8x - 20}$$

$$(g \circ f)(x) = g(f(x)) = g\left(\frac{2x}{x+5}\right) = \frac{\dfrac{2x}{x+5}}{3\dfrac{2x}{x+5} - 8} = \frac{\dfrac{2x}{x+5}}{\dfrac{6x - 8x - 40}{x+5}} = \frac{2x}{-2x - 40} = -\frac{x}{x+20}$$

Problem 8.

$$\text{(a)} \quad \lim_{x\to -1} \frac{3x^2 + 4x + 1}{x+1} = \lim_{x\to -1} \frac{(3x+1)(x+1)}{x+1} = \lim_{x\to -1}(3x+1) = \lim_{x\to -1} 3x + \lim_{x\to -1} 1 = -3 + 1 = -2$$

$$\text{(b)} \quad \lim_{x\to -\infty} \frac{-2x^4 + 3x^3 - 7x - 10}{3x^4 + 6x^2 - x + 100} = \lim_{x\to -\infty} \frac{-2 + \dfrac{3}{x} - \dfrac{7}{x^3} - \dfrac{10}{x^4}}{3 + \dfrac{6}{x^2} - \dfrac{1}{x^3} + \dfrac{100}{x^4}} = \frac{-2 + \lim\limits_{x\to -\infty} \dfrac{3}{x} - \lim\limits_{x\to -\infty} \dfrac{7}{x^3} - \lim\limits_{x\to -\infty} \dfrac{10}{x^4}}{3 + \lim\limits_{x\to -\infty} \dfrac{6}{x^2} - \lim\limits_{x\to -\infty} \dfrac{1}{x^3} + \lim\limits_{x\to -\infty} \dfrac{100}{x^4}} = -\frac{2}{3}$$

Problem 9.

$$\text{(a)} \quad \lim_{x\to -2} \frac{x^2 - 4}{x+2} = \lim_{x\to -2} \frac{(x-2)(x+2)}{x+2} = \lim_{x\to -2}(x-2) = \lim_{x\to -2} x - \lim_{x\to -2} 2 = -2 - 2 = -4$$

$$\text{(b)} \quad \lim_{x\to 4} \frac{x-4}{\sqrt{x} - 2} = \lim_{x\to 4} \frac{(\sqrt{x}-2)(\sqrt{x}+2)}{\sqrt{x}-2} = \lim_{x\to 4}(\sqrt{x} + 2) = \lim_{x\to 4} \sqrt{x} + \lim_{x\to 4} 2 = 2 + 2 = 4$$

$$(c) \quad \lim_{x\to 1} \frac{\sqrt{x+3} - 2}{x - 1} = \lim_{x\to 1} \left[\frac{(\sqrt{x+3} - 2)(\sqrt{x+3} + 2)}{(x-1)(\sqrt{x+3} + 2)}\right] = \lim_{x\to 1} \frac{1}{\sqrt{x+3} + 2} = \frac{1}{\sqrt{\lim\limits_{x\to 1} x + 3} + 2} = \frac{1}{4}$$

$$\text{(d)} \quad \lim_{x\to 0} \frac{1 - \cos(2x)}{3x^2} = \lim_{x\to 0} \frac{1 - (\cos^2 x - \sin^2 x)}{3x^2} = \lim_{x\to 0} \frac{2\sin^2 x}{3x^2} = \frac{2}{3}\left(\lim_{x\to 0} \frac{\sin x}{x}\right)^2 = \frac{2}{3} \cdot 1^2 = \frac{2}{3}$$

Problem 10.
The only point (**Why ?**) where the function may be discontinuous is $x = -1$. Now $f(x)$ is defined at $x = -1$, $f(-1) = 1$. Next, we check whether the limit $\lim\limits_{x\to -1} f(x)$ exists. We have,

$\lim_{x\to-1} f(x) = \lim_{x\to-1} \frac{x^2-1}{x+1} = \lim_{x\to-1} \frac{(x-1)(x+1)}{x+1} = \lim_{x\to-1}(x-1) = \lim_{x\to-1} x - \lim_{x\to-1} 1 = -1-1 = -2$. So the limit $\lim_{x\to-1} f(x)$ indeed exists and is equal to -2.

Finally, we observe that the third condition of continuity is not satisfied: the limit $\lim_{x\to-1} f(x)$ is not equal to the value of the function at $x=-1$. Indeed, $\lim_{x\to-1} f(x) = -2$ but $f(-1) = 1$. Therefore, $f(x)$ is not continuous at $x=-1$ because the third condition of continuity is not satisfied.

Problem 11.

We need to check which of the three conditions of continuity at $x=-1$ is not satisfied. Since $f(-1) = -1$, the function $f(x)$ is defined at $x=-1$ and the condition (1) of continuity is satisfied. Next, we check if the condition (2) is satisfied, i.e., whether the limit $\lim_{x\to-1} f(x)$ exists. The expression $2x-1$ approaches -5 whenever x approaches -1 and $x<-1$. On the other hand, the expression x^2-4 approaches -3 whenever x approaches -1 and $x>-1$. Therefore, $\lim_{x\to-1} f(x)$ does not exist. The second condition of continuity is not satisfied.

Problem 12.

(a) The average rate of change from $x=2$ to $x=3$ is given by

$$\frac{f(3)-f(2)}{3-2} = \frac{-9-(-4)}{3-2} = -5.$$

(b) The instantaneous rate of change at $x=3$ is $f'(3)$:

$$f'(3) = \lim_{h\to0} \frac{f(3+h)-f(3)}{h} = \lim_{h\to0} \frac{-(3+h)^2-(-9)}{h} = \lim_{h\to0} \frac{-h^2-6h}{h} = \lim_{h\to0}(-h-6) = -6.$$

Problem 13.

(a) $g'(s) = 4s + \frac{4}{s^2} - \frac{1}{\sqrt{s^3}}$ (b) $h'(x) = 5\left(x + \frac{1}{x} + \frac{1}{x^2}\right)^4 \left[1 - \frac{1}{x^2} - \frac{2}{x^3}\right]$

(c) $F'(x) = \frac{1}{2}\left(\frac{x^2+1}{x^4+2} + 10\right)^{-1/2} \left[\frac{2x(x^4+2)-(x^2+1)4x^3}{(x^4+2)^2}\right]$

Problem 14.

Since $g'(t) = \left(\sqrt{2t^2+3}\right)' = \frac{2t}{\sqrt{2t^2+3}}$, we have

$$g''(t) = \left(\frac{2t}{\sqrt{2t^2+3}}\right)' = \frac{6}{(2t^2+3)^{3/2}} \quad \text{and} \quad g'''(t) = \left[\frac{6}{(2t^2+3)^{3/2}}\right]' = -\frac{36t}{(2t^2+3)^{5/2}}.$$

Problem 15.

(a) $h'(t) = \frac{(2t-3)(t+1)-(t^2-3t+1)\cdot 1}{(t+1)^2} = \frac{t^2+2t-4}{(t+1)^2}$

(b) $f'(x) = \frac{2c^2x}{2\sqrt{c^2x^2+2}} = \frac{c^2x}{\sqrt{c^2x^2+2}}$

Problem 16.

(a) For $x \neq 0$, $x^3\sin(\frac{1}{x}) = \left|x^3\sin(\frac{1}{x})\right|$ and $\left|\sin(\frac{1}{x})\right| \le 1$. Thus,

$$0 \le \left|x^3\sin(\frac{1}{x})\right| \le |x|^3.$$

Since, $\lim_{x\to0} |x|^3 = 0$, the Squeeze Theorem (page 68) implies that

$$\lim_{x\to0} x^3\sin(\frac{1}{x}) = 0.$$

(b) $\lim_{x\to1} \frac{\sqrt{x+3}-2}{x-1} = \lim_{x\to1}\left[\frac{(\sqrt{x+3}-2)(\sqrt{x+3}+2)}{(x-1)(\sqrt{x+3}+2)}\right] = \lim_{x\to1} \frac{1}{\sqrt{x+3}+2} = \frac{1}{4}.$

(c) $\lim_{w\to-2} \frac{(w+2)(w^2-w-6)}{w^2+4w+4} = \lim_{w\to-2} \frac{(w+2)(w-3)(w+2)}{(w+2)^2} = \lim_{w\to-2}(w-3) = -5.$

Problem 17.

$$\text{(a)} \quad \frac{[f(x)-f(2)]}{x-2} = \frac{3x^2-5-(3\cdot 2^2-5)}{x-2} = \frac{3x^2-12}{x-2}$$

$$\text{(b)} \quad \lim_{x\to 2}\frac{[f(x)-f(2)]}{x-2} = \lim_{x\to 2}\frac{3x^2-5-7}{x-2} = \lim_{x\to 2} 3\frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2} 3(x+2) = 12.$$

Problem 18.

Let $\epsilon > 0$. We must find $\delta > 0$ such that

$$0 < |x-5| < \delta \quad \Longrightarrow \quad |\sqrt{x-1}-2| < \epsilon.$$

Now, for $0 < |x-5| < \delta$ and $0 < \delta \le 4$ $\;x \in D_{\sqrt{x-1}} = \{x : x \ge 1\}$ (the domain of the function $f(x) = \sqrt{x-1}$, **CHECK IT !**), and

$$|\sqrt{x-1}-2| = \left|\frac{(\sqrt{x-1}-2)(\sqrt{x-1}+2)}{\sqrt{x-1}+2}\right| = \frac{|x-5|}{\sqrt{x-1}+2} \le \frac{|x-5|}{2},$$

where we used the fact that $\dfrac{1}{\sqrt{x-1}+2} \le \dfrac{1}{2}$. If we choose $\delta = \min\{4, 2\epsilon\}$, then

$$|\sqrt{x-1}-2| = \left|\frac{(\sqrt{x-1}-2)(\sqrt{x-1}+2)}{\sqrt{x-1}+2}\right| = \frac{|x-5|}{\sqrt{x-1}+2} \le \frac{|x-5|}{2} < \frac{\delta}{2} \le \frac{2\epsilon}{2} = \epsilon.$$

This completes the proof.

Problem 19.

For each $\epsilon > 0$ we need to find $\delta > 0$ (that depends of ϵ) such that the following implication is true:

$$0 < |x+1| < \delta \quad \Longrightarrow \quad |(x^2-2x-1)-2| < \epsilon.$$

We have

$$|(x^2-2x-1)-2| = |x^2-2x-3| = |x+1||x-3|,$$

thus to bound $|x-3|$, we assume that $0 < \delta \le 1$ (*Please carefully explain why we can assume that* $0 < \delta \le 1$). Now,

$$|x+1| < \delta \quad \Longrightarrow \quad |x-3| = |x+1-4| \le |x+1| + |-4| < 1+4 = 5,$$

and choose $\delta = \min\{1, \epsilon/5\}$. For such δ

$$|(x^2-2x-1)-2| = |x^2-2x-3| = |x+1||x-3| < 5\cdot\frac{\epsilon}{5} = \epsilon.$$

Problem 20.

Let $f(x) = x^5 + 4x^3 - 7x + 14$. Since $f(x)$ is a polynomial, it is continuous on any closed interval $[a,b] \subset \mathbb{R}$. We have $f(0) = 14$ and $f(-2) = -32 - 32 + 14 + 14 = -36$. Therefore, $0 \in [-36, 14]$, and the Intermediate Value Theorem implies that there exists $c \in (-2, 0)$ such that $f(c) = 0$.

Problem 21.

The slope of the tangent line to $y = \dfrac{2}{x-2}$ at $(0,-1)$ is given by

$$\mathrm{m}_{\tan} = \lim_{h\to 0}\frac{\frac{2}{0+h-2} - \frac{2}{0-2}}{h} = \lim_{h\to 0}\frac{1}{(h-2)} = -\frac{1}{2}.$$

Hence, the equation of the tangent line is given by $y+1 = -\frac{1}{2}x$.

Problem 22.

$$f'(x) = \lim_{h\to 0}(\sqrt{10})\,\frac{\sqrt{x+h}-\sqrt{x}}{h} = (\sqrt{10})\lim_{h\to 0}\frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})} = \frac{\sqrt{10}}{2\sqrt{x}} = \frac{5}{\sqrt{10x}}.$$

Problem 23.

$$\begin{aligned} D_x[f(x)g(x)h(x)] &= [D_x f(x)][g(x)h(x)] + f(x)D_x[g(x)h(x)] \\ &= [D_x f(x)][g(x)h(x)] + f(x)h(x)[D_x g(x)] + f(x)g(x)[D_x h(x)]. \end{aligned}$$

Problem 24.

$$\frac{dy}{dx} = 4(x^2+1)^3 2x(x^4+1)^3 + (x^2+1)^4 3(x^4+1)^2 4x^3.$$

Now, the slope of the tangent line is equal to $\frac{dy}{dx}(1) = 1280$. Thus, the equation of the tangent line to the graph of the function $y = (x^2+1)^4(x^4+1)^3$ at $(1, 128)$ is $y - 128 = 1280(x-1)$.

Problem 25.

(a)

$$D_x\left[\cos^4\left(\frac{x^2+1}{x+1}\right)\right] = 4\cos^3\left(\frac{x^2+1}{x+1}\right)\left[-\sin\left(\frac{x^2+1}{x+1}\right)\right]\left(\frac{x^2+2x-1}{(x+1)^2}\right).$$

(b)

$$\begin{aligned} D_t\left\{\cos^2[\cos(\cos t)]\right\} &= 2\cos[\cos(\cos t)] \times D_t[\cos(\cos(\cos t))] \\ &= 2\cos[\cos(\cos t)] \times [-\sin(\cos(\cos t))] \times D_t[\cos(\cos t)] \\ &= 2\cos[\cos(\cos t)] \times [-\sin(\cos(\cos t))] \times [-\sin(\cos t)] \times [-\sin t]. \end{aligned}$$

Problem 26.

(a) $D_x^n(x^n) = n!$.

(b) $D_x^n\left(\frac{1}{x}\right) = (-1)(-2)(-3)\cdots(-n)x^{-n-1} = (-1)^n\, n!\, x^{-n-1}$.