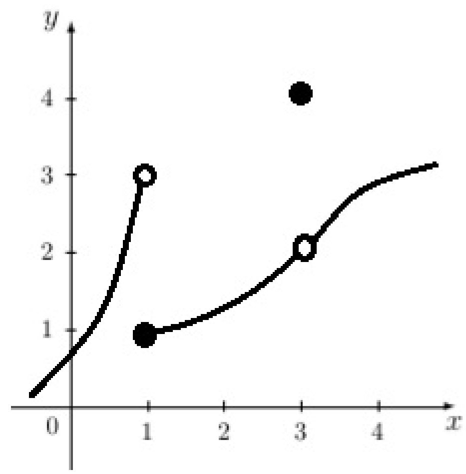
Math 150A. Final Review Questions, Spring 2024.

- Definition of Limits. 2.2) 2, 3-6, 15, 17-18, 19-23, 25-26, 34, 45-49, 50-52.

  1. Find the values, or state they do not exist.



  (a) $\lim_{x\to 1^-} f(x)$ (b) $\lim_{x\to 1^+} f(x)$ (c) $\lim_{x\to 1} f(x)$ (d) $f(1)$ (e) $\lim_{x\to 3^-} f(x)$ (f) $\lim_{x\to 3^+} f(x)$ (g) $\lim_{x\to 3} f(x)$ (h) $f(3)$

  2. Let $f(x) = \begin{cases} \frac{2x}{|x|} & x \neq 0 \\ 1 & x = 0 \end{cases}$.
     Find the limits (or state if they do not exist):
     $\lim_{x\to 0^-} f(x)$, $\lim_{x\to 0^+} f(x)$, $\lim_{x\to 0} f(x)$, $f(0)$, and $f(1)$.

- Computing Limits. 2.3) 7-12, 15-16, 19-41, 45-51, 53, 55-56, 59-60, 62, 71, 81-83, 85, 87-89, 90.

  1. Find the limits (Show work.)

  (a) $\lim_{x\to 3} \frac{x-3}{x^2-x-6}$ (b) $\lim_{x\to 2} \frac{x^3-8}{x-2}$ (c) $\lim_{x\to 4} \frac{2-\sqrt{x}}{4-x}$

  2. Show that $\lim_{x\to 3} f(x)$ exists. $f(x) = \begin{cases} \sqrt{25-x^2} & x < 3 \\ 2 & x = 3 \\ 2x-2 & x > 3 \end{cases}$

- Infinite Limits. 2.4) 6-10, 17-18, 19-20, 21-28, 31-36, 39-42, 47-50.

  1. Find the limits (Show work).

(a) $\lim_{x \to 1^+} \frac{2}{1-x}$

(b) $\lim_{x \to 1^+} \frac{3-x}{x-1}$

(c) $\lim_{x \to \frac{\pi}{2}^-} \tan(x)$

(d) $\lim_{x \to \frac{\pi}{2}^+} \sec(x)$.

- Limits at Infinity. 2.5) 3-10, 17-19, 25-31, 33-34, 35, 37-44, 46, 48, 74-75.

  1. Find the limits (Show work).

     (a) $\lim_{x \to \infty} \frac{x^2+3x}{x^3+5}$

     (b) $\lim_{x \to \infty} \frac{6x^2-5x}{2x^2+1}$

     (c) $\lim_{x \to \infty} \frac{1-x^2}{3x+1}$

     (d) $\lim_{x \to \infty} \frac{2x}{\sqrt{1+x^2}}$

- Continuity. 2.6) 5-8, 21-24, 25-30, 31-38, 61-64, 65-68, 81, 82-83.

  1. Find A such that $f(x)$ is continuous.

$$f(x) = \begin{cases} \frac{\sqrt{x^2+5}-3}{x-2} & x \neq 2 \\ A & x = 2 \end{cases}$$

  2. For $f(x) = x^3 - x$, prove that there is $c \in [1, 2]$ such that $f(c) = 3$. Use complete sentences.

- Precise Definition of Limit. 2.7) 5, 9-12, 15-16, 19-24, 27-30, 32-35.

  1. Prove using the formal definition of limit.

$$\lim_{x \to 2} 2x - 1 = 3.$$

- Introducing the Derivative. 3.1) 6-10, 15, 17, 21-31, 33-42, 43-46, 56-61.

  1. Find $f'(a)$ from the limit definition of derivative, and the tangent line through $(a, f(a))$.

     (a) $f(x) = x^2 + 3x, \quad a = 1.$

     (b) $f(x) = 4\sqrt{x+1}, \quad a = 3.$

  2. Find $f(x)$ and a such that the given limit is $f'(a)$.

     (a) $\lim_{x \to 3} \frac{\sqrt{x+1}-2}{x-3}$

     (b) $\lim_{h \to 0} \frac{\cos(h)-1}{h}$

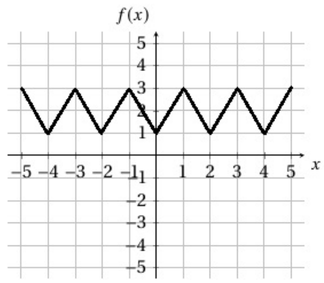
- The Derivative as a Function. 3.2) 5-10, 17-20, 21-29, 35-40, 45-46, 48-51, 55-56.

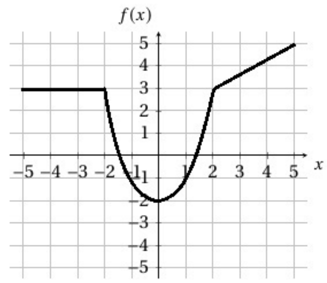
  1. Find $f'(x)$ from the limit definition of derivative.

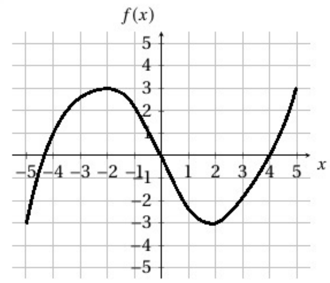
     (a) $f(x) = x^3 - x$

     (b) $f(x) = \frac{3}{x+1}$

     (c) Graph $f'(x)$ from the graphs of $f(x)$ below (next page).
     Where is $f(x)$ not differentiable?







- Basic Rules of Differentiation. 3.3) 7-8, 19-36, 44-52, 55-58, 59-63, 64-68, 69, 78-81.

  1. Find the derivatives.

     (a) $f(x) = \frac{2}{\sqrt{x}} - \frac{1}{x^3}$ (b) $f(x) = \pi^5$

  2. Find the limit by evaluating a derivative at an appropriate value of x.

     (a) $\lim_{h \to 0} \frac{(2+h)^4 - 16}{h}$ (b) $\lim_{x \to 2} \frac{x^5 - 32}{x - 2}$. (c) $\lim_{x \to 1} \frac{\sqrt{x} - 1}{x - 1}$

- The product and quotient rules. 3.4) 7-12, 16, 18, 19-43, 47-51, 57-60, 66-69.

  1. Find $f'(x)$.

     (a) $f(x) = (1 + 2\sqrt{x})(x^5 - 2 + \frac{3}{x})$ (b) $f(x) = \frac{x^2 - 2x}{x^3 + 1}$

  2. Find $f''(x)$. $f(x) = \frac{x^2}{x+3}$.

- Derivatives of Trigonometric Functions. 3.5) 11-18, 21-22, 23-49, 52-54, 57-64, 67-70, 72-75, 83.

  1. Find the limits. Show work.

     (a) $\lim_{x \to 0} \frac{\sin(6x)}{3x}$ (c) $\lim_{x \to 0} \frac{x^2}{1 - \cos^2(x)}$

     (b) $\lim_{x \to 0} \frac{x}{\tan x}$ (d) $\lim_{x \to 0} \frac{1 - \cos(x)}{\sin(x)}$

  2. Find $f''(x)$.

     (a) $f(x) = x^3 \sin(x)$. (b) $f(x) = \tan(x)$.

  3. Find the tangent line to $f(x) = \sec(x)$ at $x = \frac{\pi}{3}$.

- Derivatives and Rates of Change. 3.6) 4, 15-20, 21-27, 36-37.

  1. A ball is thrown into the air and has vertical position $s(t) = -16t^2 + 16t + 4\,ft$.

     (a) How high does the ball go?

(b) What is the downward velocity when it hits the ground?

- The Chain Rule. 3.7) 1-5, 8-11, 13-22, 25-51, 55-62, 74-77, 78, 81-84.

1. Find the derivatives.

(a) $f(x) = \sqrt{x^3 - 9}$

(b) $f(x) = (x^2 + 1)\cos(5x)$

(c) $f(x) = \left(\frac{x+1}{2x-1}\right)^5$

(d) $f(x) = \sin^3(\sqrt{x})$

- Implicit Differentiation. 3.8) 5-8, 13-24, 27-33, 36-37, 45-49.

1. Find $\frac{dy}{dx}$

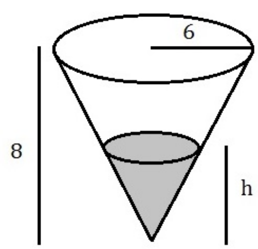
(a) $xy = \cos(x) + \sin(y)$

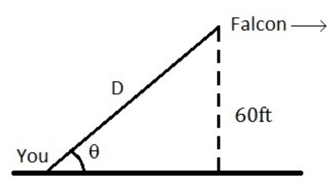
(b) $\frac{x-y}{x+y} = y$

2. Find the tangent lines at the given points.

(a) $4y^2 + x^2y = x^3$ at $(2, 1)$.

(b) $\ln(y - x) = \sin(2x - y)$ at $(1, 2)$.





- Related Rates. 3.9) 4-10, 11-27, 29, 31-33, 35-37, 39, 41-44, 46, 50.

1. An inverted cone has height 8-in, with a base radius of 6-in. Water leaks out of the cone at a constant rate of 2-in^3/min. How fast is the height of the water changing when the height is 4 in?
2. A peregrine falcon flies horizontally at an altitude of 60 feet and a speed of 80 feet per second. Suppose you are on the ground and the falcon went overhead 1 second ago.

(a) How fast is the distance D between you and the falcon changing at this time?

(b) How fast is the angle θ changing at this time?

- Maxima and Minima. 4.1) 11-18, 19-22, 23-40, 41-55, 59-62.

1. Find the absolute maximum and absolute minimum on the given interval.

(a) $f(x) = x^3 - 3x^2 + 1$ on $[1, 3]$.

(b) $f(x) = \frac{x}{x^2+1}$ on $[-1, 2]$.

- The Mean Value Theorem. 4.2) 1-4, 8-10, 11-18, 19-20, 21-22, 24-26, 29-32, 36, 41-43, 47-49.

1. State the mean value theorem for $f : [1,4] \to \mathbb{R}$. Then find $c \in (1,4)$ that satisfies the conclusions of the mean value theorem, if $f(x) = x^2$.
2. For $f(x) = |x|$ on $[-1,2]$, show that there is no $x \in (-1,2)$ such that $f'(x) = (f(2) - f(-1))/3$. Why does $f(x)$ violate the mean value theorem?
3. Show that $\sec^2(x)$ and $\tan^2(x)$ have the same derivative. Then explain why $\sec^2(x) - \tan^2(x) = 1$.

- What Derivatives Tell Us. 4.3) 9-12, 19-39, 41-48, 53-56, 57-64, 66-68, 69-75, 79-84.

1. Use the first derivative test to find all relative maxima and minima.

   (a) $f(x) = 3x^{\frac{2}{3}} - x$ (b) $f(x) = x\sqrt{8 - x^2}$ on $[-\sqrt{8}, \sqrt{8}]$.

2. Find the inflection points. Use the second derivative test to find any relative maxima or minima.

   (a) $f(x) = x^3 - 3x^2 + 1$ (b) $f(x) = x^2 e^{-\frac{1}{2}x}$.

3. Sketch a continuous function f with f decreasing on $(-\infty, 0) \bigcup (3, \infty)$, f increasing on $(0,3)$, f concave up on $(-\infty, -2)$, f concave down on $(-2, \infty)$. List any local maxima, local minima, and inflection points.

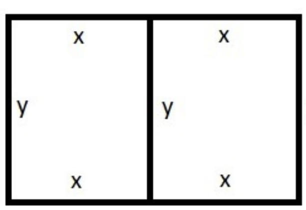
- Graphing. 4.4) 7-12, 15-42, 44, 46.

1. Sketch the graphs by finding all relative extrema, inflection points, asymptotes, and concavity.

   (a) $f(x) = x^3 - 6x^2$ (b) $f(x) = \frac{1}{x^2 - 2x}$

- Optimization. 4.5) 5-10, 11-22, 25, 32-35, 37-38, 40-41.

1. For each question, find the extremum and **prove that it is a local extremum using the first derivative test**.



   (a) Two rectangular pens of the same dimensions are to be constructed out of fencing as in the picture above. If fence costs \$ 3 per foot, find the minimum cost to build the fence if each pen has area 1200 square feet.

(b) A box has a wooden base, which is square and costs 5$ per square foot. The rectangular sides are made of paper and cost 1$ per square foot. The box has no top. If you have 60 dollars to spend, what is the largest volume box you can make?

- Linear Approximations and Differentials. 4.6) 19-24, 25-36, 37-46, 55-58.

  1. Use a linear approximation to estimate $\sqrt{103}$.
  2. The volume of a balloon is $V = \frac{4}{3}\pi R^3$. Find error in the volume, if the radius is measured to be $3 \pm .2$ inches.

- Antiderivatives. 4.9) 2-5, 11-22, 23-37, 39-43, 45-47, 55-60, 61-66, 73-78, 79-84, 89-92, 94-96, 98, 101-104.

  1. Find the antiderivatives.

     (a) $\int \frac{4}{\sqrt{x}} - \frac{1}{x^2}\, dx$ (b) $\int 2x(1+3x)^2\, dx$ (c) $\int \sec(2x)\tan(2x)\, dx$

  2. Solve the initial value problems.

     (a) $\frac{dy}{dx} = \sec^2(x), \quad y(\frac{\pi}{4}) = 3.$ (b) $y'(x) = \frac{4x^3+1}{x^2}, \quad y(-1) = 0.$

- Approximating Area. 5.1) 23-24, 25-30, 48-49, 50i, 61.

  1. For the given definite integral, find the left approximation by Riemann sum with $N = 4$ and the right approximation by Riemann sum with $N = 4$.

     (a) $\int_0^\pi \cos(x)\, dx$ (b) $\int_3^5 \ln(x)\, dx$

- Definite Integrals. 5.2) 7-11, 25-28, 29-32, 33-36, 37-43, 50-51, 57-64, 75-81, 82-83.

  1. Find the area by using a right-handed Riemann sum.

$$\int_1^4 x^2 - 1\, dx.$$

  2. Use **geometry** to find the values of the definite integrals.

     (a) $\int_0^3 |2-x|\, dx$ (b) $\int_{-1}^2 2 - |x|\, dx$ (c) $\int_{-5}^0 \sqrt{25-x^2}\, dx$

- Fundamental Theorem of Calculus. 5.3) 7-11, 25-28, 29-32, 33-36, 37-43, 50-51, 57-64, 75-81, 82-83.

  1. Find the derivatives.

     (a) $\frac{d}{dx}\int_{-3}^{\sin(x)}\sqrt{1+t^4}\,dt.$ (b) $\frac{d}{dx}\int_{3x}^{x^2}\cos(t^2)\,dt.$

  2. Find the definite integrals.

     (a) $\int_1^4 \frac{x-1}{\sqrt{x}}\,dx$ (b) $\int_0^{\frac{\pi}{6}}\sin(3x)\,dx$ (c) $\int_0^4 \frac{x}{\sqrt{25-x^2}}\,dx$

- Working with Integrals. 5.4) 9, 13-22, 25-28, 29-54, 71-84, 101-103.

  1. Evaluate the integrals using symmetry.

     (a) $\int_{-\pi}^{\pi}\sin(|x|)\,dx.$ (b) $\int_{-1}^{1}\tan(x^3)\,dx.$

  2. Find the average value of the function on the interval. Then verify the Mean Value Theorem for Integrals.

     (a) $f(x) = 8 - 2x$ on $[-1, 5]$. (b) $f(x) = \sqrt{2x-2}$ on $[1, 3]$.

- Substitution. 5.5) 7-10, 11-14, 17-26, 28-32, 34-38, 41-66, 74-82.

  1. Evaluate the integrals.

     (a) $\int\sqrt{3-2x}\,dx$
     (b) $\int\sec^2(x)\tan(x)\,dx$
     (c) $\int\frac{\sin\left(\frac{1}{x}\right)}{x^2}\,dx$
     (d) $\int\frac{x+1}{(x^2+2x)^3}\,dx$
     (e) $\int\frac{x}{(2-x)^3}\,dx$
     (f) $\int(5x+1)\sqrt{x+2}\,dx$

- Regions Between Curves. 6.2) 9-32, 39-43, 45-48, 55-58.

  1. Sketch the region between the two curves. Then find the area of the region. $x = y^2 - 2y$, $x = 4 - y^2$.
  2. Sketch the region between the two curves, then find the total area using two integrals. $y = x^3 - x$, $y = 3x$.

- Volume by slicing. 6.3) 11-16, 17-38, 45-50, 62, 65-66.

  1. A pyramid has cross-sections that are rectangles of side-lengths x meters and $2x$ meters. Find the volume of the pyramid, if x ranges from 0 to 3.
  2. Prove that the volume of a sphere of radius R is $\frac{4}{3}\pi R^3$.
  3. Let R be the region between $y = 4 - x^2$ and $y = 4 - 2x$.

     (a) Find the volume given by revolving R around the x-axis.

(b) Find the volume given by revolving R around the line $y = 4$.

- Volume by Shells. 6.4) 5-8, 9-21, 35-40, 45, 48, 49-58, 60-63.

  1. Let R be the region bounded by $y = \frac{1}{\sqrt{x+1}}$, $y = 0$, $x = 0$, and $x = 3$. Find the volume given by revolving R around the y-axis.
  2. Let R be the region between $y = 2x$ and $y = x^2$.
     (a) Find the volume given by revolving R around the y-axis.
     (b) Find the volume given by revolving R around the line $x = 3$.
     (c) Find the volume given by revolving R around the line $x = -2$.

- Length of Curves. 6.5) 3-6, 7-8, 9-16.

  1. Find the arclength of $y = \frac{1}{4}x^2 - \frac{1}{2}\ln(x)$ from $x = 1$ to $x = e$.

- Surface Area. 6.6) 7-13, 15-16, 18, 21-22, 32-33.

  1. Find the surface area of $y = x^3$ when revolved around the x-axis, from $x = 0$ to $x = 1$.
  2. Find the surface area of $y = \frac{4}{3}x^{\frac{3}{2}}$ when revolved around the y-axis, from $x = 0$ to $x = 2$.

- Physical Applications. 6.7) 21-25, 27-29, 31-34.

  1. (a) A metal ball is attached to a spring. It takes 15 Newtons of force to hold the ball at a distance of 3 meters from equilibrium. How much work does it take the ball to move from equilibrium to 2 meters from equilibrium?
     (b) Another metal ball is attached to a different spring. For this ball it takes 18 N-m of work to move the ball from equilibrium to 3 meters from equilibrium. How much work does it take to move the ball another two meters from equilibrium? (From $x = 3$ to $x = 5$.)
  2. A box of mass 20-kg hangs on a rope of mass .5 kg per meter, which is attached to a pulley on a tall building. If the rope is 10 meters long, calculate the work that the pulley must exert to lift the rope and box to the top. Use that the force of gravity is 9.8 N/kg.

- Inverse Functions. 7.1) 39-44, 45-48, 49-52, 53-54.

  1. Find the derivative of $f^{-1}(x)$ at $x = 5$. $f(x) = x^3 - x - 1$. (Hint: You might need to check $f(x)$ for some small values of x.)
  2. Find the derivative of $f^{-1}(x)$ at $x = 0$. $f(x) = \int_2^x \sqrt{1 + t^3}\, dt$. (Hint: When is $f(x) = 0$? Then use FTC.)

- Logarithms and Exponentials. 7.2) 17-38, 39-40, 41-46, 48, 51-60, 62, 63-70, 73, 76-91, 93-94, 96, 98-99.

1. Find $f'(x)$.

   (a) $f(x) = \ln(x\sin(x))$

   (b) $f(x) = e^{x^3}\ln(\sec(x))$

   (c) $f(x) = \ln\left(\frac{\sqrt{x-1}}{(x+1)^3}\right)$

   (d) $f(x) = \frac{e^{2x}+3}{e^{3x}+5}$

2. Use logarithmic differentiation to find $f'(x)$.

   (a) $f(x) = \frac{x^3}{\sqrt{x^2+1}}$.

   (b) $f(x) = (3x+1)^{2x}$.

3. Find the integrals.

   (a) $\int_1^3 \frac{1}{2x-1}\,dx$

   (b) $\int \frac{x^2}{x^3+1}\,dx$

   (c) $\int \frac{\sec^2(\ln(x))}{x}\,dx$

   (d) $\int_{e^2}^{e^3} \frac{1}{x(\ln(x))^2}\,dx$

   (e) $\int (2e^{3x}+1)^2\,dx$

   (f) $\int \frac{e^x-e^{-x}}{e^x+e^{-x}}\,dx$

   (g) $\int_0^{\ln(4)} e^x\sqrt{5-e^x}\,dx$

   (h) $\int_0^{\frac{\pi}{2}} \cos(x)e^{\sin(x)}\,dx$

4. Find the limits

   (a) $\lim\limits_{x\to\infty} \left(\frac{2x+1}{x-3}\right)e^{-x}$.

   (b) $\lim\limits_{x\to\infty} \frac{5e^x+1}{2e^x-1}$.

5. (a) Find the volume given by revolving the region between $y = e^x$, $y = -1$, $x = 0$, and $x = 1$ around the line $y = -1$.

   (b) Find the volume given by revolving the region between $y = \frac{\ln(x)}{x^2}$, $y = 0$, $x = 1$, and $x = e$ around the y-axis.