# Exponential Functions and Half-Lives



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Photo by C. C. Plummer

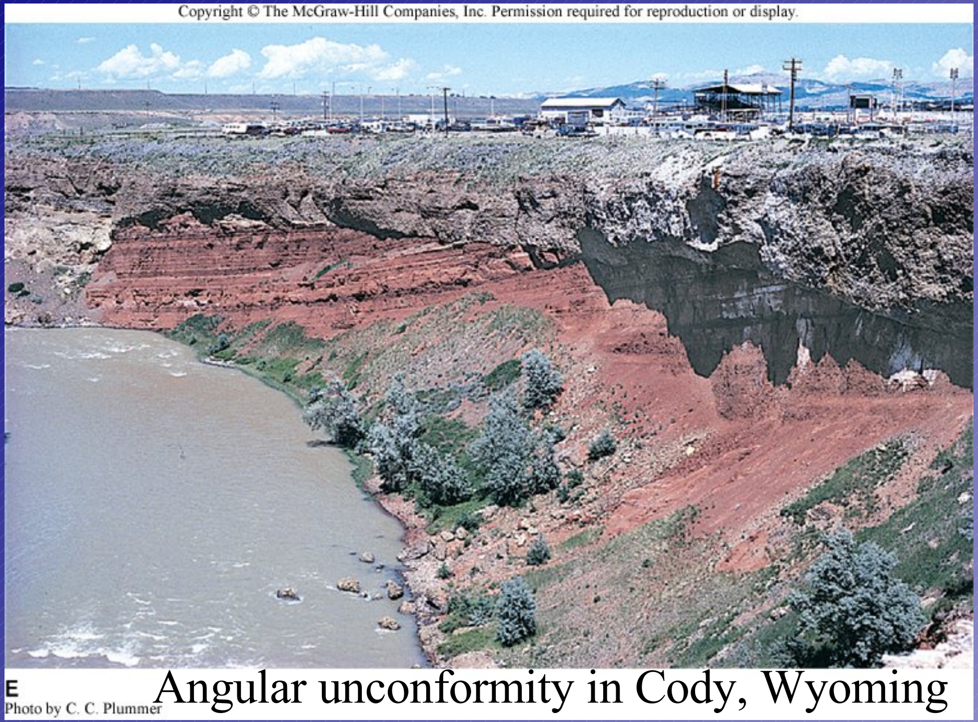
Angular unconformity in Cody, Wyoming

NE Hudson Bay Canada, 4.28 Ba

- Radioactive isotopes and geochronology methods use *exponential functions* to date rock samples.

- You're on an outcrop wondering what is the age of this stratigraphic section.....Do you know your exponents ?

# Exponential Functions and Half-Lives

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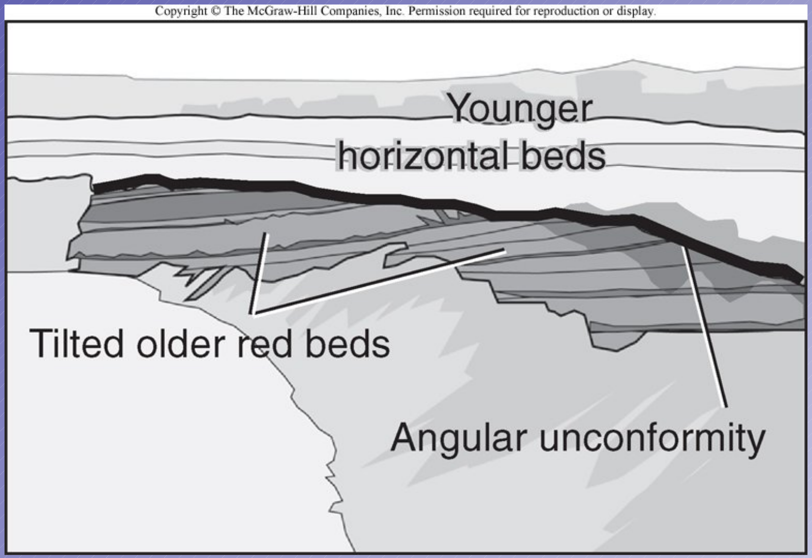



- While structural principles of superposition and fossil identification can give you relative age dates
- Absolute ages are obtained by studying radioactive decay

# Exponential Functions and Half-Lives

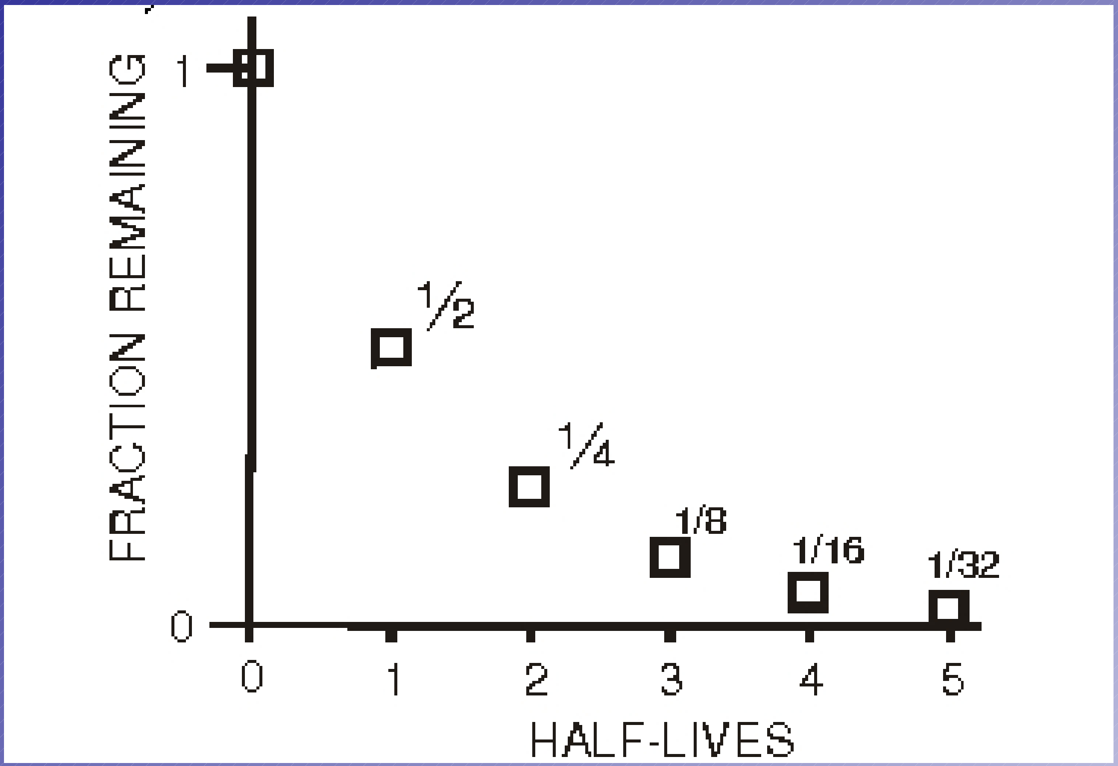
Hudson Bay amphibole with abundant garnet

- What is a *half-life* ?

- If you start with eight million atoms of a parent isotope (P), how many P isotopes will you have after decay of P to D (daughter isotopes) in one *half-life* of 1000 yrs ?

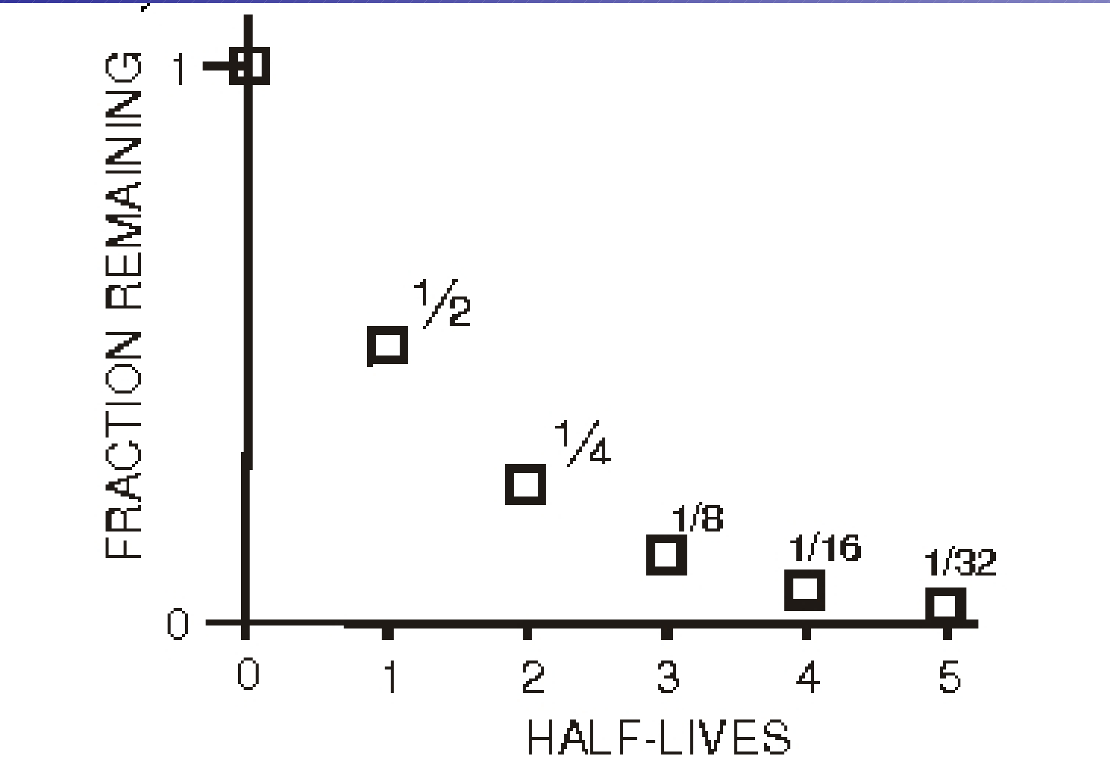
- After 2000 yrs, how many parent isotopes will you have ?

# Exponential Functions and Half-Lives




- After 3000 years, you have 1 million parent isotopes
- This is 1/8 of the original amount
- After 9000 years, you have 15625 atoms, 1/1024 (or 0.1%)
- The succession of fractions are shown above.

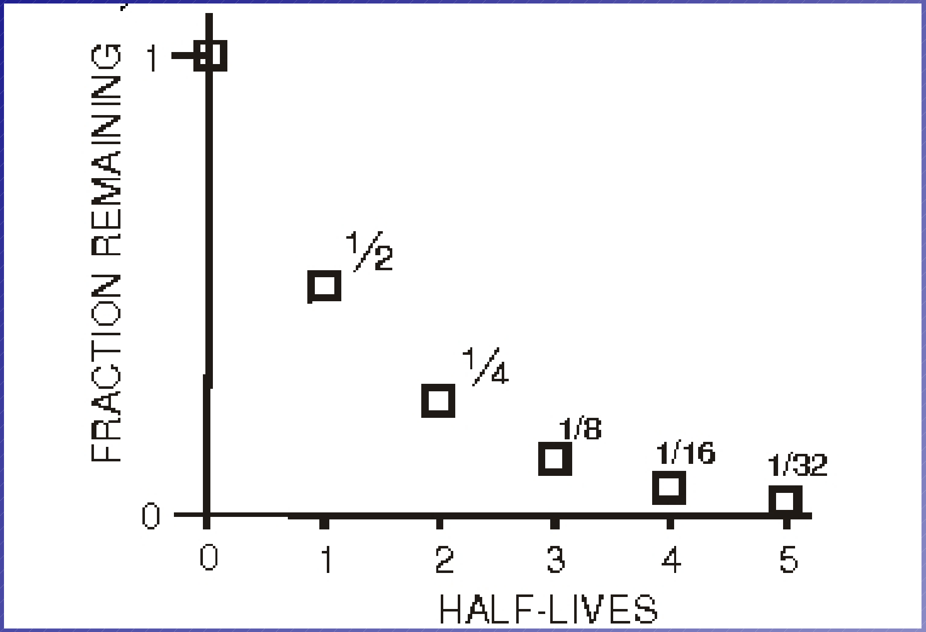
# Exponential Functions and Half-Lives




- What is the equation that corresponds to this graph ?

$$P = P_o\,(1/2)^{\frac{t}{t_{1/2}}}$$

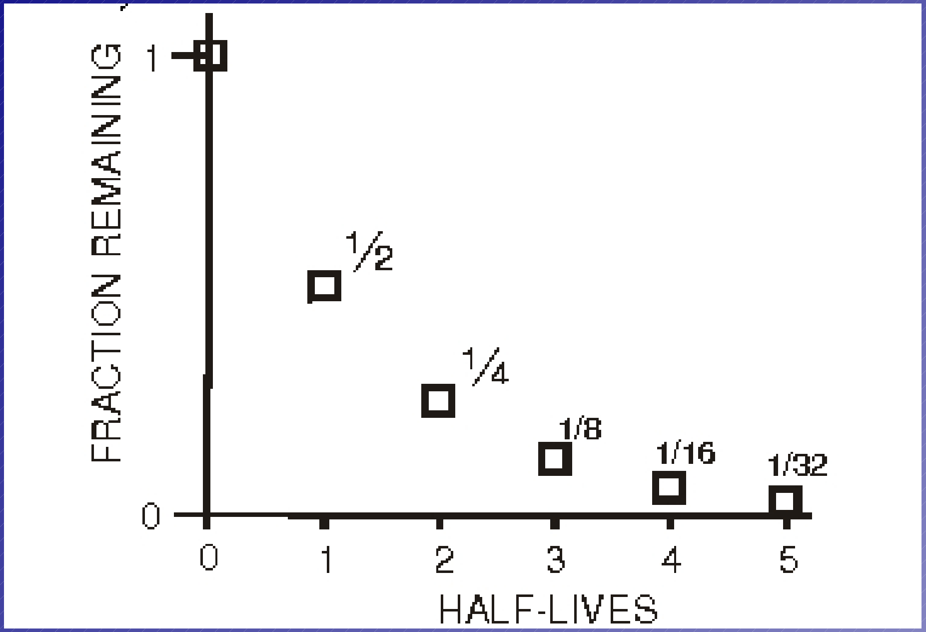
# Exponential Functions and Half-Lives




$$P = P_o\,(1/2)^{\frac{t}{t_{1/2}}}$$

- ***P*** is the number of parent atoms remaining
- After elapsed time, ***t***
- $P_o$ is the number of parent atoms at the start
- $t_{1/2}$ is the half-life for a particular isotope

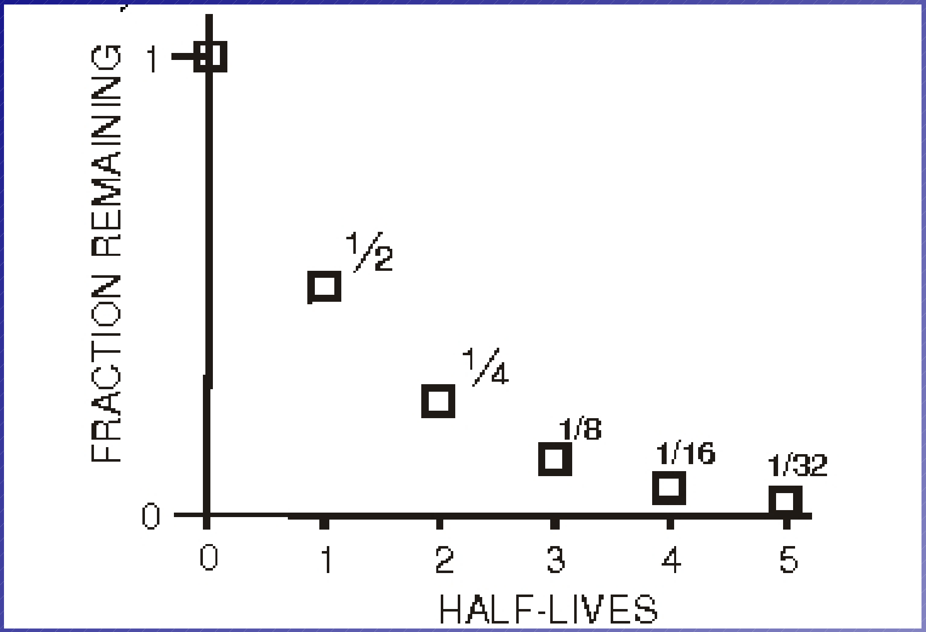
# Exponential Functions and Half-Lives




$$P = P_o (1/2)^{\frac{t}{t_{1/2}}}$$

- What is in the exponent, $\frac{t}{t_{1/2}}$ ?
- This gives the number of half-lives which have ellapsed during the time, ***t***.
- So for a half-life of 1000 yrs, after 1000 yrs have passed: $\frac{t}{t_{1/2}} = 1$, which is one half-life.
- What is this exponent after 3000 years ?

# Exponential Functions and Half-Lives




$$P = P_o\,(1/2)^{\frac{t}{t_{1/2}}}$$

- The **(1/2)** in the parenthesis – represents "***half-lives***".
- If we wanted to know when a *third* of the initial population of atoms decayed to a daughter atom, then this would be **(1/3).**
- In this case, the exponent would be: $\frac{t}{t_{1/3}}$
- If you rearrange, ***P/Po*** is the remaining parents after one ***half-life***.

# The Exponential Function

$$P = P_o\,(1/2)^{\frac{t}{t_{1/2}}}$$

- If we take the logarithm of both sides of this equation :

$$\ln P = \ln P_o + \frac{t}{t_{1/2}}\ln(1/2)$$

$$\ln P = \ln P_o - \frac{t}{t_{1/2}}\ln(2)$$

# The Exponential Function

$$\ln \boldsymbol{P} = \ln \boldsymbol{P}_o - \frac{\boldsymbol{t}}{\boldsymbol{t}_{1/2}} \ln (2)$$

- What in this equation is a constant ?
- The constants are: ln (2) and $\boldsymbol{t}_{1/2}$
- Let's describe this ratio with one letter, $\boldsymbol{\lambda} = \ln(2) / \boldsymbol{t}_{1/2}$
- What are the units of $\boldsymbol{\lambda}$ ?
- The dimensions of $\boldsymbol{\lambda}$ are $\boldsymbol{time}^{-1}$ .

# The Exponential Function

- Then substituting $\lambda$, we get

$$\ln P = \ln P_o - \lambda t \,.$$

- Can we now get rid of the ***ln*** ?

- Raise the left and right side the power of ***e*** :

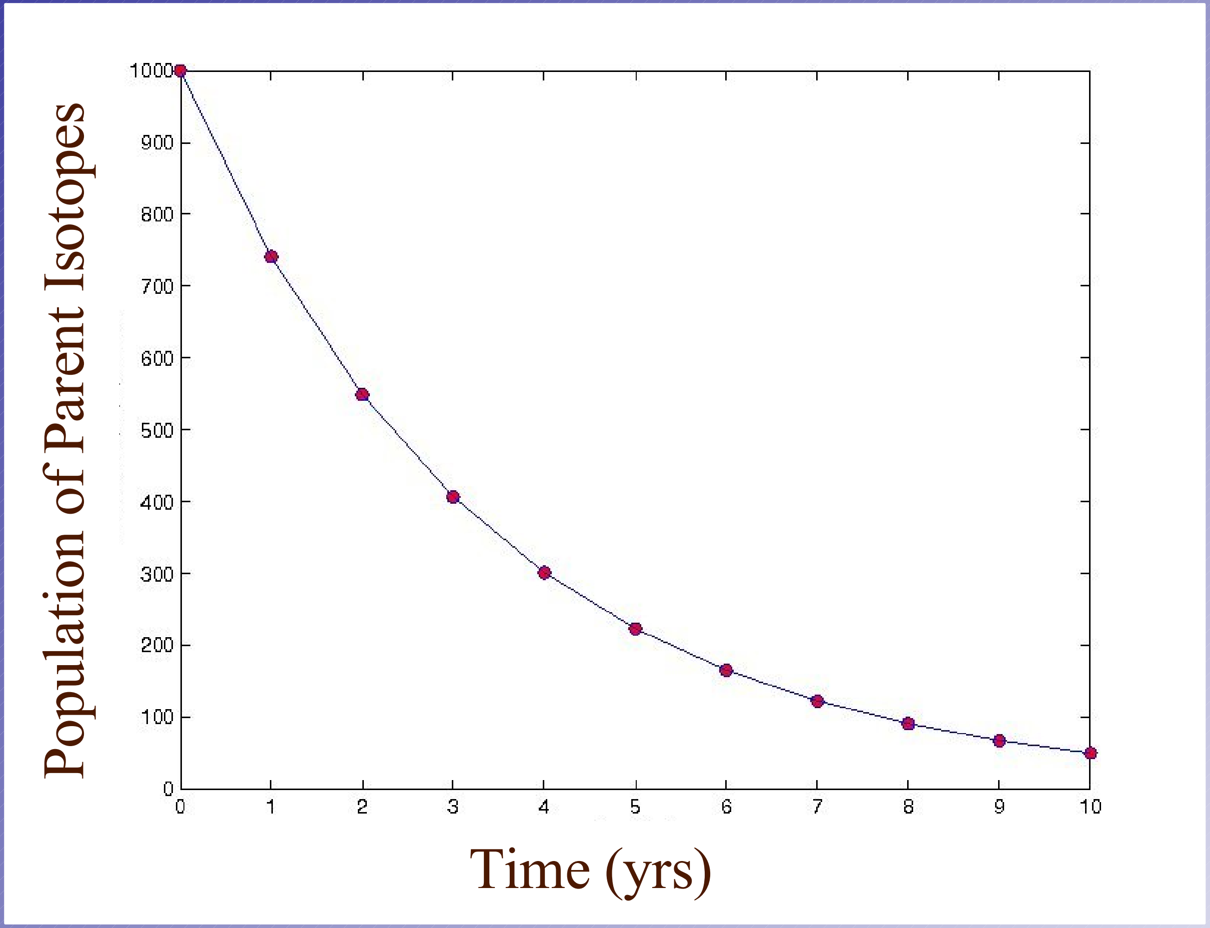
$$P = P_o \, e^{-\lambda t}$$

- Does this look more familiar ?
  This is the traditional expression for exponential decay.

# The Exponential Decay

$$P = P_o e^{-\lambda t}$$

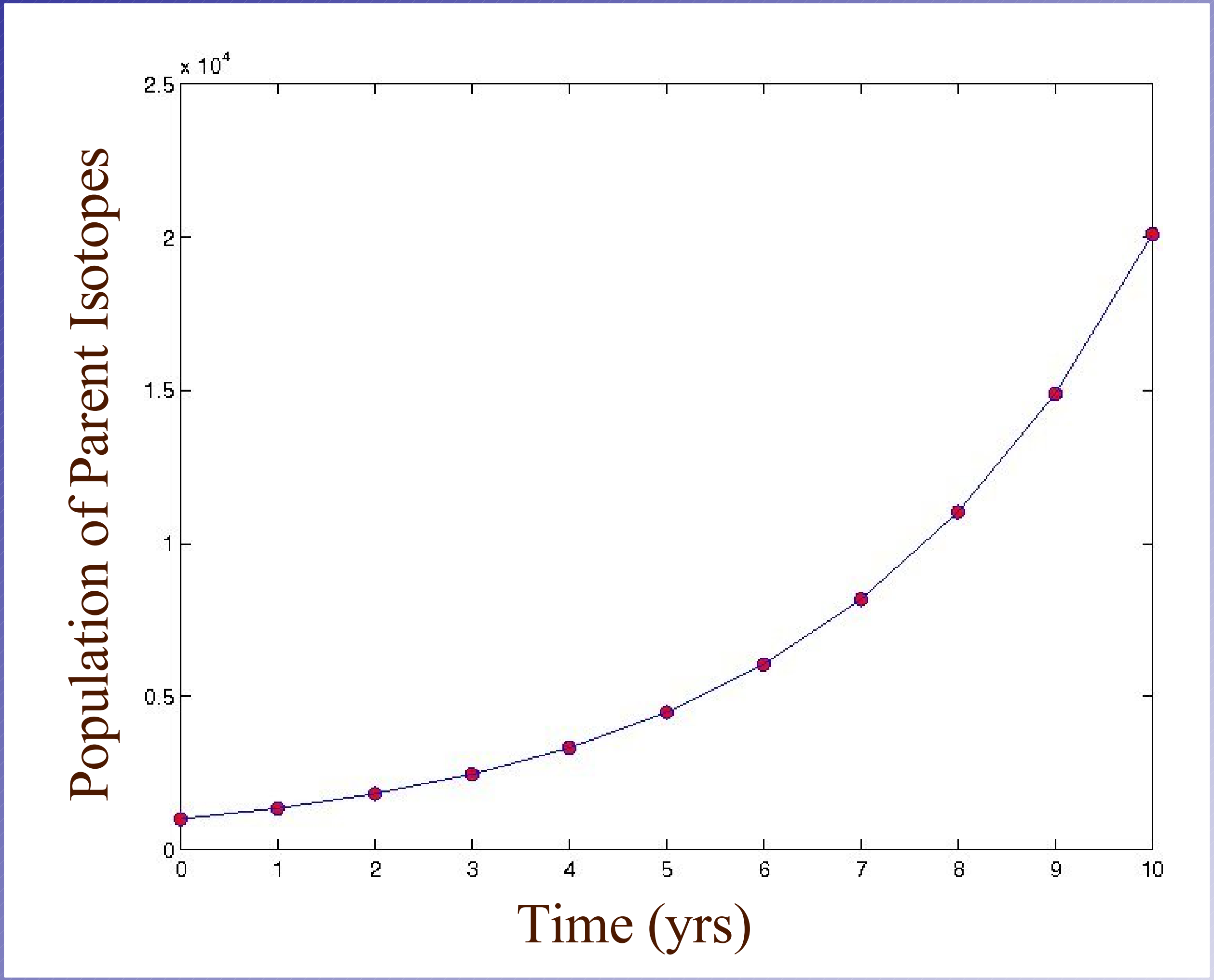
- What about this equation describes decay ?



# The Exponential Growth

- How could we describe exponential growth ?

$$P = P_o e^{\lambda t}$$



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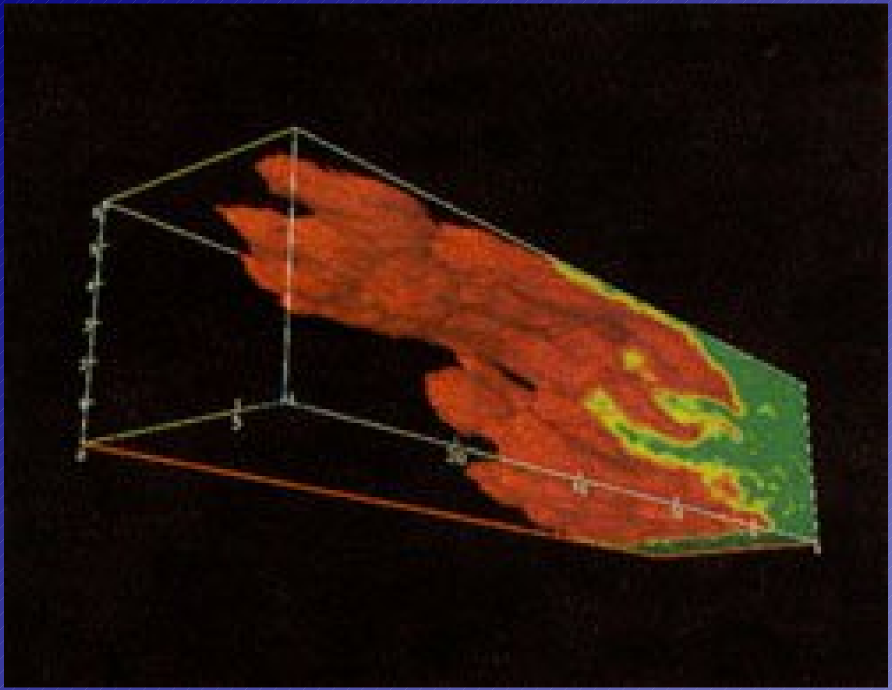
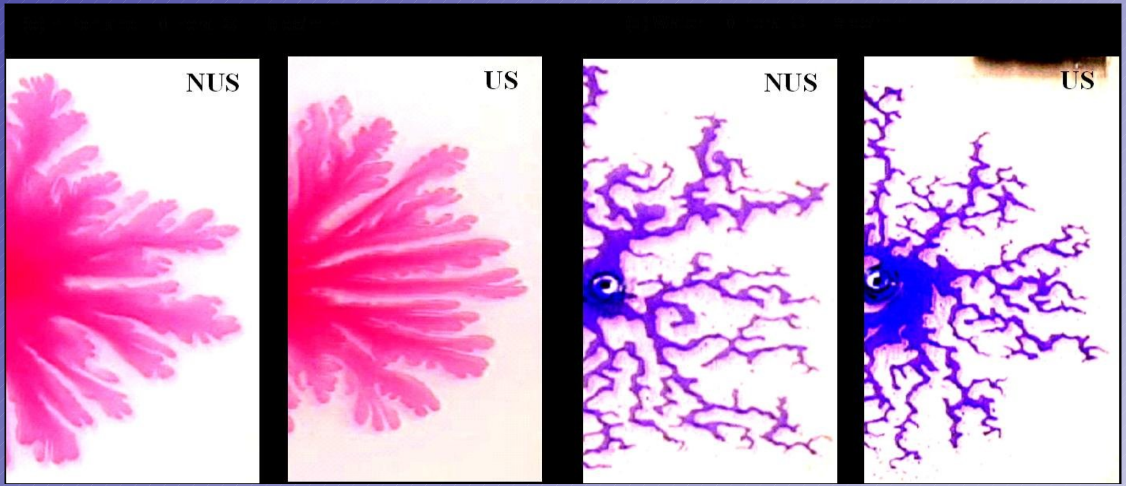
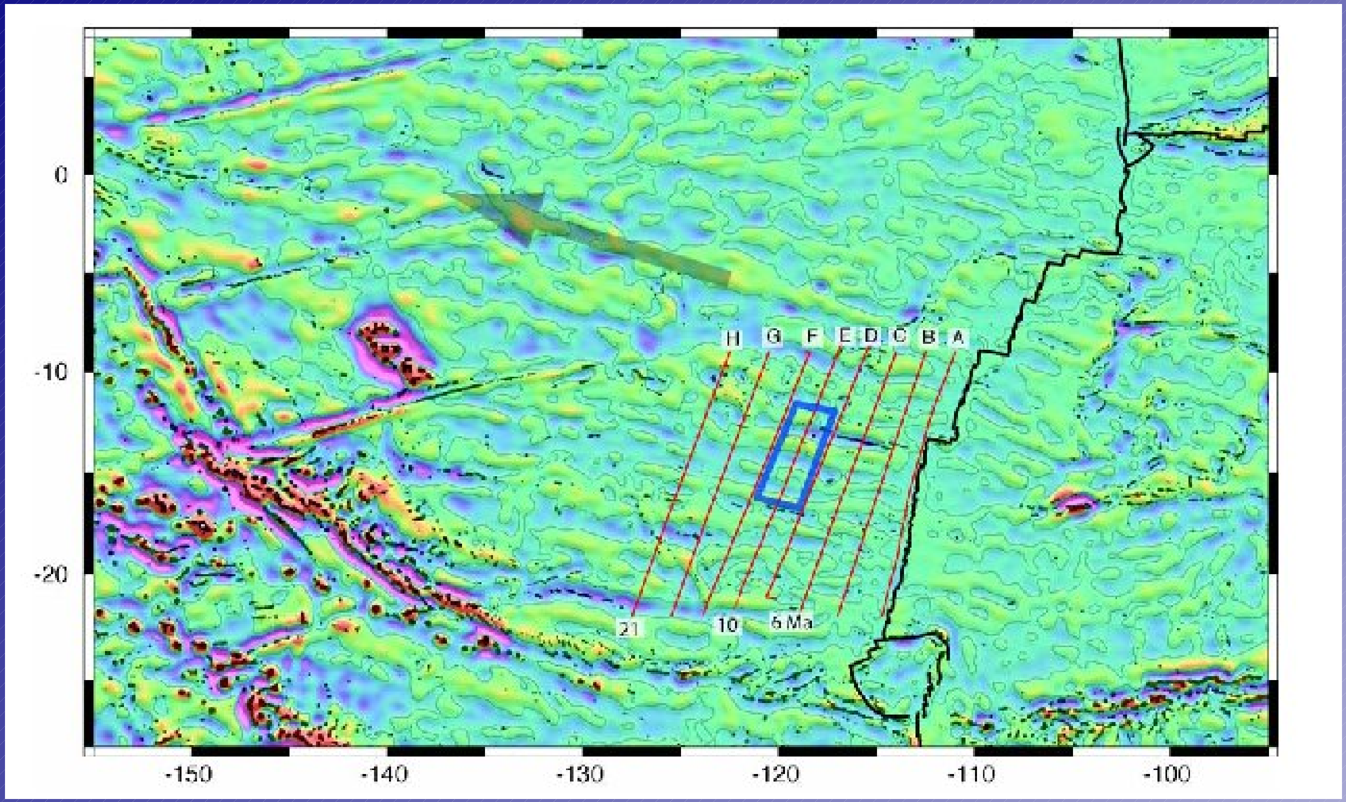




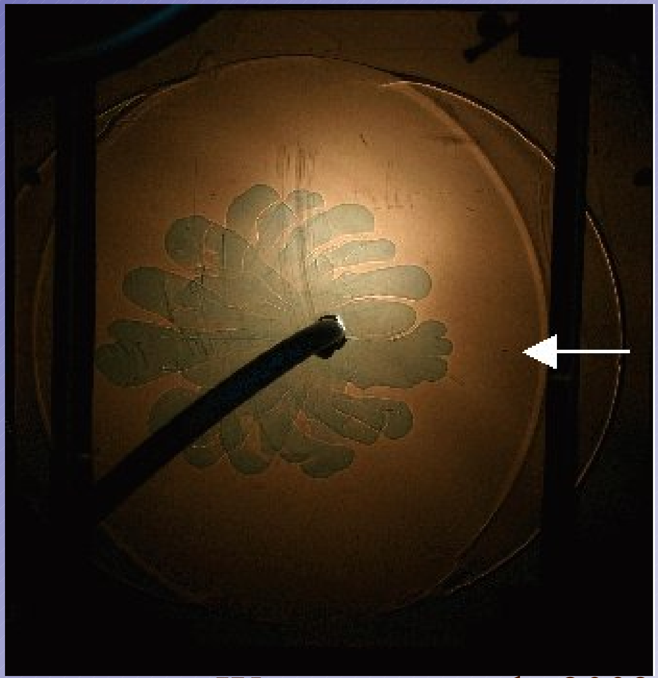

Displacement of mineral oil by water

- Oil is usually pumped by natural pressure or water pressure
- More than 50% of known oil reserves cannot be recovered by these conventional means because water does not pump oil efficiently.
- Low viscosity water “fingers” through high viscosity oil fluids
- The theory of “*viscous fingering*” predicts *exponential* growth of fingering instabilities – and and can become chaotic

# The Exponential Growth in the Earth's Mantle




Sandwell, 2008



Weeraratne et al., 2003

- Linear gravity anomalies observed in the Pacific ocean
- Have been investigated as viscous fingers of plume material traveling through the asthenosphere.

# Leonard Euler (1707 - 1783)


- Leonard Euler was most prolific mathematician of all time (lived during the time of Benjamin Franklin (1706-1790)
- Born in Basel, Switzerland, completed university at age 15
- By 1771, Euler was nearly completely blind and dictated his calculations to a note taker and published 70 volumes!
- Euler established the branch of mathematics known as “analysis” (e.g. calculus, complex variables, potential theory)

# **Leonard Euler (1707 - 1783)**


- In three Latin texts and famous *Introductio*, he introduced the concept of a ***function***.
- As well as the modern concept of a ***logarithm***, and ***exponential function***

# Euler's $e$ and $e^x$

- Where does $e$ come from ?
- Let $x$ be an infinitely large number, $\varepsilon$ be infinitely small
- Then $N = x / \varepsilon$ is an infinitely large number.

$$x = N\varepsilon$$

- We know that $\mathbf{a}^{o} = 1$, then is $\varepsilon$ is just a number then,

$$a^{\varepsilon} = 1 + k\varepsilon$$

- We can expand and use binomial series to get

$$a^{x} = 1 + k^{x}/1! + k^{2}x^{2}/2! + k^{3}x^{3}/3! + \ldots$$

# Euler's *e* and $e^x$

$$a^x = 1 + k^x/1! + k^2x^2/2! + k^3x^3/3! + \ldots$$

| | $kx = 1$ | $kx = 2$ | $kx = 2.3026$ |
|---|---|---|---|
| $(kx)^0/0! =$ | 1 | 1 | 1 |
| $(kx)^1/1! =$ | 1 | 2 | 2.3026 |
| $(kx)^2/2! =$ | 0.5 | 2 | 2.650983 |
| $(kx)^3/3! =$ | 0.166666667 | 1.333333 | 2.034718 |
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| sum= | 2.718281526 | 7.388713 | 9.998697 |

Table 1. First ten terms of Equation 19 for three choices of *kx*.

- Summing 10 terms for “kx = 1” gives 2.718281526 = “*e*”
- Coined by Euler himself

# Euler's $e$ and $e^x$

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- If we let $a = e$, when $k = 1$ we get:

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- What happens if we take the derivative of this function ?

$$d/dx\,(e^x) = 1 + x/1! + x^2/2! + x^3/3!$$

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$$d/dx\,(e^x) = 1 + x/1! + x^2/2! + x^3/3!$$

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$$d/dx\,(e^x) = e^x$$

- The derivative of the exponential becomes the exponential

# Euler's $e$ and $e^x$

- What does this mean to us ?

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- If $y = Ae^{\alpha x}$

- Then dy/dx $= \alpha Ae^{\alpha x}$ or $dy/dx = \alpha y$

- This means that the rate of change of y is proportional to the amount y that is present

# The Exponential Function

geometric

$$P = P_o e^{\lambda t}$$

arithmetic

- Exponential growth is also a *geometric progression*
- These equations pair *geometric* and *arithmetic* progressions
- The independent variable, ***t***, increases arithmetically
- While the dependent variable, ***P***, increases geometrically
- Changes in ***P*** are controlled by incremental increase in ***t*** and the rate constant, $\lambda$ .

# The Exponential Function

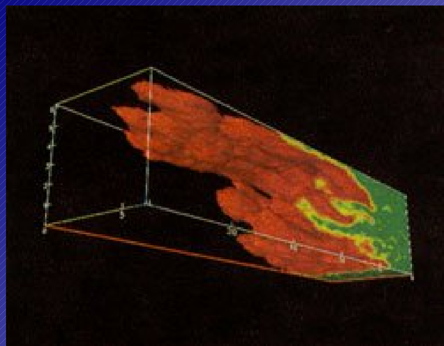
$$P = P_o e^{\lambda t}$$

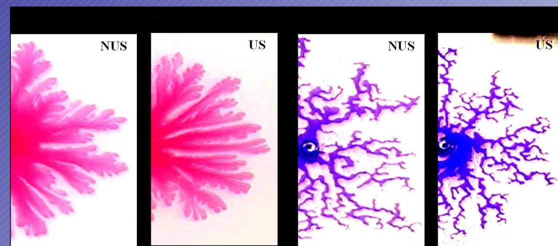
- These equations are variations of the general form

$$y = e^x$$

- This equation is *unique* among functions !
- The derivative of $e^x$ returns itself, $e^x$ .
- How does this work ?

# The Exponential Growth





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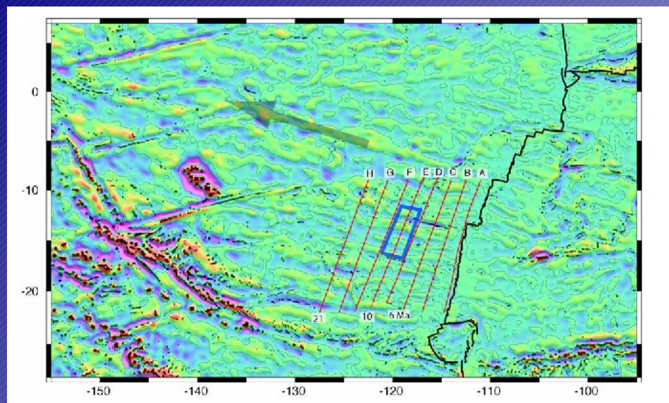


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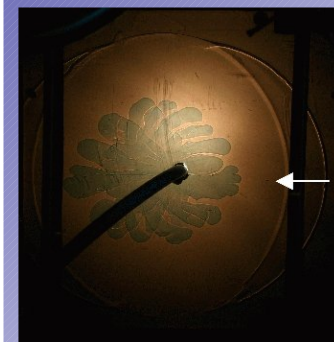

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