4 HELLY-TYPE THEOREMS AND GEOMETRIC TRANSVERSALS

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INTRODUCTION

Let $\mathcal{F}$ be a family of convex sets in $\mathbb{R}^d$. A *geometric transversal* is an affine subspace that intersects every member of $\mathcal{F}$. More specifically, for a given integer $0 \leq k < d$, a k-dimensional affine subspace that intersects every member of $\mathcal{F}$ is called a *k-transversal* to $\mathcal{F}$. Typically, we are interested in necessary and sufficient conditions that guarantee the existence of a k-transversal to a family of convex sets in $\mathbb{R}^d$, and furthermore, to describe the space of all k-transversals to the given family. Not much is known for general k and d, and results deal mostly with the cases $k = 0$, 1, or $d - 1$.

Helly's theorem gives necessary and sufficient conditions for the members of a family of convex sets to have a point in common, or in other words, a 0-transversal. Section 4.1 is devoted to some of the generalizations and variations related to Helly's theorem. In the study of k-transversals, there is a clear distinction between the cases $k = 0$ and $k > 0$, and Section 4.2 is devoted to results and questions dealing with the latter case.

NOTATION

Most of the notation and terminology is quite standard. This chapter deals mainly with families of subsets of the d-dimensional real linear space $\mathbb{R}^d$. In certain cases we consider $\mathbb{R}^d$ to be equipped with a metric, in which case it is the usual Euclidean metric. The convex hull of a set $X \subset \mathbb{R}^d$ is denoted by $\operatorname{conv} X$. The cardinality of a set X is denoted by $|X|$. If $\mathcal{F}$ is a family of sets, then $\bigcap \mathcal{F}$ denotes the intersection and $\bigcup \mathcal{F}$ the union of all members of $\mathcal{F}$. If $\mathcal{F}$ is a family of subsets of $\mathbb{R}^d$, then $\operatorname{conv} \mathcal{F}$ denotes the convex hull of $\bigcup \mathcal{F}$.

4.1 HELLY'S THEOREM AND ITS VARIATIONS

One of the most fundamental results in combinatorial geometry is Helly's classical theorem on the intersection of convex sets.

THEOREM 4.1.1 *Helly's Theorem* [Hel23]

Let $\mathcal{F}$ be a family of convex sets in $\mathbb{R}^d$, and suppose that $\mathcal{F}$ is finite or at least one member of $\mathcal{F}$ is compact. If every $d + 1$ or fewer members of $\mathcal{F}$ have a point in common, then there is a point in common to every member of $\mathcal{F}$.

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Helly's theorem has given rise to numerous generalizations and variations. A typical *"Helly-type"* theorem has the form:

> If every m or fewer members of a family of objects have property $\mathcal{P}$, then the entire family has property $\mathcal{P}$.

In many cases the extension to infinite families can be dealt with by standard compactness arguments, so for simplicity we deal only with finite families. Under this assumption it is usually not any restriction to assume that every member of the family is open, or that every member of the family is closed, so we will usually choose the form which is the most convenient for the statement at hand. The reader should also be aware that results of this kind are by no means restricted to combinatorial geometry, and Helly-type theorems appear also in graph theory, combinatorics, and related areas; but here we will focus mainly on geometric results.

Let us briefly describe the kind of variations of Helly's theorem we will be treating. The first variation deals with *replacing the convex sets by other objects.* For instance, we may replace convex sets by a more general class of subsets of $\mathbb{R}^d$. In the other direction, one might obtain more structure by specializing the convex sets to homothets, or translates, of a fixed convex set.

The second variation deals with *changing the local condition.* Instead of asking for every $d+1$ or fewer members of $\mathcal{F}$ to have a point in common, one may ask for something less. As an example, we may suppose that among any $d+2$ members of $\mathcal{F}$ *some* $d+1$ have a point in common. This direction has uncovered several deep results concerning the intersection patterns of convex sets. Another direction would be to strengthen the local condition: What if we assume that any $d+1$ or fewer members have an intersection that is at least 1-dimensional, or has volume at least 1? Do any of these assumptions lead to a stronger conclusion? If not, what if we assume the same for the intersection of any $100d$ or fewer members?

This brings us to the third and final variation, which deals with *changing the conclusion.* For instance, our goal may not only be that the intersection is nonempty, but that its diameter is at least some $\delta > 0$. Or that the intersection contains a point with integer coordinates, or more generally, a point from some given set M.

In the last decade we have seen an enormous activity in this area and we have tried to emphasize some of the recent developments. Unfortunately, it is not possible to include every result related to Helly's theorem, and it has been necessary to make some subjective choices. For instance, we have chosen not to discuss Helly-type results related to spherical convexity, and we do not treat abstract convexity spaces in any detail.

GLOSSARY

Helly-number: The Helly-number of a family of sets $\mathcal{C}$ is the smallest integer h such that if $\mathcal{F}$ is a finite subfamily of $\mathcal{C}$ and any h or fewer members of $\mathcal{F}$ have nonempty intersection, then $\bigcap \mathcal{F} \neq \emptyset$. In the case when such a number does not exist, we say that the Helly-number is unbounded, or simply write $h(\mathcal{C}) = \infty$.

Fractional Helly-number: A family of sets $\mathcal{F}$ has fractional Helly-number k if for every $\alpha > 0$ there exists a $\beta > 0$ such that for any finite subfamily $\mathcal{G} \subset \mathcal{F}$ where at least $\alpha\binom{|\mathcal{G}|}{k}$ of the k-member subfamilies of $\mathcal{G}$ have nonempty intersection, there exists a point common to at least $\beta|\mathcal{G}|$ members of $\mathcal{G}$.

Piercing number: The piercing number of a family of sets $\mathcal{F}$ is the smallest integer k for which it is possible to partition $\mathcal{F}$ into subfamilies $\mathcal{F}_1, \ldots, \mathcal{F}_k$ such that $\bigcap \mathcal{F}_i \neq \emptyset$ for every $1 \leq i \leq k$.

Homology cell: A topological space X is a homology cell if it is nonempty and its (singular reduced) homology groups vanish in every dimension. In particular, nonempty convex sets are homology cells.

Good cover: A family of open subsets of $\mathbb{R}^d$ is a good cover if any finite intersection of its members is empty or a homology cell.

Convex lattice set: A convex lattice set in $\mathbb{R}^d$ is a subset $S \subset \mathbb{R}^d$ such that $S = K \cap \mathbb{Z}^d$ for some convex set $K \subset \mathbb{R}^d$.

Diameter: The diameter of a point set $X \subset \mathbb{R}^d$ is the supremum of the distances between pairs of points in X.

Width: The width of a closed convex set $X \subset \mathbb{R}^d$ is the smallest distance between parallel supporting hyperplanes of X.

GENERALIZATIONS TO NONCONVEX SETS

Helly's theorem asserts that the Helly-number of the family of all convex sets in $\mathbb{R}^d$ equals $d+1$. One of the basic problems related to Helly's theorem has been to understand to which extent the assumption of convexity can be weakened.

Early generalizations of Helly's theorem involve families of homology cells in $\mathbb{R}^d$ such that the intersection of any $d+1$ or fewer members is also a homology cell [Hel30, AH35], or that the intersection of any $1 \leq j \leq d$ is j-acyclic [Deb70] (which means that all the homology groups up to dimension j vanish). Montejano showed that these conditions can be relaxed even further.

THEOREM 4.1.2 [Mon14]

Let $\mathcal{F}$ be a family of open subsets of $\mathbb{R}^d$ such that the intersection of any j members of $\mathcal{F}$ has vanishing (singular reduced) homology in dimension $d-j$, for every $1 \leq j \leq d$. Then $h(\mathcal{F}) \leq d+1$.

In fact, Theorem 4.1.2 holds for families of open subsets of any topological space X for which the homology of any open subset vanishes in dimensions greater or equal to d.

In 1961, Grünbaum and Motzkin [GM61] conjectured a Helly-type theorem for families of *disjoint unions* of sets. Their conjecture (now a theorem) was formulated in a rather abstract combinatorial setting, and here we discuss some of the related results in increasing order of generality. The basic idea is that if $\mathcal{C}$ is a family of sets with a bounded Helly-number, and $\mathcal{F}$ is a finite family of sets such that the intersection of any subfamily of $\mathcal{F}$ can be expressed as a disjoint union of a bounded number of members of $\mathcal{C}$, then the Helly-number of $\mathcal{F}$ is also bounded.

THEOREM 4.1.3 [Ame96]

Let $\mathcal{C}$ be the family of compact convex sets in $\mathbb{R}^d$, and let $\mathcal{F}$ be a finite family of sets in $\mathbb{R}^d$ such that the intersection of any subfamily of $\mathcal{F}$ can be expressed as the union of at most k pairwise disjoint members of $\mathcal{C}$. Then $h(\mathcal{F}) \leq k(d+1)$.

Note that Theorem 4.1.3 reduces to Helly's theorem when $k = 1$. Amenta gave a short and elegant proof of a more general result set in the abstract combinatorial framework of "Generalized Linear Programming," from which Theorem 4.1.3 is a simple consequence (see Chapter 49). Examples show that the upper bound $k(d+1)$ cannot be reduced.

Recently, Kalai and Meshulam obtained a topological generalization of Theorem 4.1.3. A family $\mathcal{C}$ of open subsets of $\mathbb{R}^d$ is called a *good cover* if any finite intersection of members of $\mathcal{C}$ is empty or a homology cell. Note that the family of all open convex subsets of $\mathbb{R}^d$ is a good cover.

THEOREM 4.1.4 [KM08]

Let $\mathcal{C}$ be a good cover in $\mathbb{R}^d$, and let $\mathcal{F}$ be a finite family of sets in $\mathbb{R}^d$ such the intersection of any subfamily of $\mathcal{F}$ can be expressed as a union of at most k pairwise disjoint members of $\mathcal{C}$. Then $h(\mathcal{F}) \leq k(d+1)$.

Let us make a remark concerning "topological Helly-type theorems." Let $\mathcal{F}$ be a finite family of subsets of some ground set. The *nerve* $N(\mathcal{F})$ of $\mathcal{F}$ is the simplicial complex whose vertex set is $\mathcal{F}$ and whose simplices are all subfamilies $\mathcal{G} \subset \mathcal{F}$ such that $\bigcap \mathcal{G} \neq \emptyset$. If $\mathcal{F}$ is a family of convex sets in $\mathbb{R}^d$, then the nerve $N(\mathcal{F})$ is *d-collapsible*. This implies that every induced subcomplex of $N(\mathcal{F})$ has vanishing homology in all dimensions greater or equal to d, a property often called *d-Leray*. In the case when $\mathcal{F}$ is a good cover in $\mathbb{R}^d$, the *Nerve theorem* from algebraic topology implies that $\bigcup \mathcal{F}$ is topologically equivalent to $N(\mathcal{F})$ (on the level of homology), which again implies that $N(\mathcal{F})$ is d-Leray. These important tools allow us to transfer combinatorial properties regarding the intersection patterns of convex sets, or good covers, in $\mathbb{R}^d$ into properties of simplicial complexes. The proof of Theorem 4.1.4 is based on simplicial homology and the computation of certain spectral sequences. In fact, Kalai and Meshulam established a more general topological result concerning the Leray-numbers of "dimension preserving" projections of simplicial complexes, from which Theorem 4.1.4 is deduced via the Nerve theorem. See the survey [Tan13] for more information regarding simplicial complexes arising from the intersection patterns of geometric objects.

A remarkable feature of the proofs of Theorems 4.1.3 and 4.1.4 is that they are set in quite abstract frameworks, and as we remarked earlier, the original motivating conjecture of Grünbaum and Motzkin [GM61] was formulated in the general combinatorial setting of *abstract convexity spaces*, which includes Theorems 4.1.3 and 4.1.4 as special cases. The general conjecture was treated by Morris [Mor73], but his proof is extremely involved, and the correctness of some of his arguments have been open to debate [EN09]. However, the skepticism has been put to rest after Eckhoff and Nischke [EN09] revisited Morris's approach and provided a complete proof of the Grünbam–Motzkin conjecture.

To describe this abstract setting we must introduce a few definitions. Let $\mathcal{C}$ be a family of subsets of some ground set. We say that $\mathcal{C}$ is *intersectional* if for any finite subfamily $\mathcal{F} \subset \mathcal{C}$, the intersection $\bigcap \mathcal{F}$ is empty or belongs to $\mathcal{C}$. Furthermore, we say that $\mathcal{C}$ is *nonadditive* if for any finite subfamily $\mathcal{F} \subset \mathcal{C}$ consisting of at least two nonempty pairwise disjoint members of $\mathcal{C}$, the union $\bigcup \mathcal{F}$ does not belong to $\mathcal{C}$. For example, the family of all compact convex sets in $\mathbb{R}^d$ is an intersectional and nonadditive family with Helly-number $d+1$. The same is also true for any good cover in $\mathbb{R}^d$ if we include all possible intersections as well. Thus the following theorem, conjectured by Grünbaum and Motzkin, is a common generalization of both Theorems 4.1.3 and 4.1.4.

THEOREM 4.1.5 [EN09]

Let $\mathcal{C}$ be an intersectional and nonadditive family with Helly-number $h(\mathcal{C})$, and let $\mathcal{F}$ be a family of sets such that the intersection of any k or fewer members of $\mathcal{F}$ can be expressed as a union of at most k pairwise disjoint members of $\mathcal{C}$. Then $h(\mathcal{F}) \leq k \cdot h(\mathcal{C})$.

Before ending the discussion of the Grünbaum–Motzkin conjecture let us mention a related result. For a locally arcwise connected topological space Γ, let d_Γ denote the smallest integer such that every open subset of Γ has trivial rational homology in dimension d_Γ and higher. A family $\mathcal{F}$ of open subsets of Γ is called *acyclic* if for any nonempty subfamily $\mathcal{G} \subset \mathcal{F}$, the intersection of the members of $\mathcal{G}$ has trivial rational homology in dimensions greater than zero. This means that the intersections of members of $\mathcal{F}$ need not be connected, but each connected component of such an intersection is a homology cell.

THEOREM 4.1.6 [CVGG14]

Let $\mathcal{F}$ be a finite acyclic family of open subsets of a locally arcwise connected topological space Γ, and suppose that for any subfamily $\mathcal{G} \subset \mathcal{F}$, the intersection $\bigcap \mathcal{G}$ has at most r connected components. Then $h(\mathcal{F}) \leq r(d_\Gamma + 1)$.

It should be noted that Theorem 4.1.6 includes both Theorems 4.1.3 and 4.1.4, but does not seem to follow from Theorem 4.1.5 since it does not require $\mathcal{F}$ to be intersectional. Thus, Theorems 4.1.5 and 4.1.6 appear to be distinct generalizations of Theorem 4.1.4, and their proofs differ significantly. While the proof of Theorem 4.1.5 is combinatorial and uses elementary methods, the proof of Theorem 4.1.6 is based on the homology of simplicial posets and introduces the concept of the *multinerve* (which generalizes the nerve complex). It would be interesting to find a common generalization of Theorems 4.1.5 and 4.1.6. A generalization of Theorem 4.1.6 is also given in [CVGG14] which implies bounds on Helly-numbers for higher dimensional transversals.

The results related to the Grünbaum–Motzkin conjecture deal with families of sets that are built up as *disjoint* unions of a bounded number of members from a sufficiently "nice" class of sets for which the Helly-number is known (with an exception of Theorem 4.1.6). If we do not require the unions to be disjoint, it is still possible to bound the Helly-number.

THEOREM 4.1.7 [AK95, Mat97]

For any integers $k \geq 1$ and $d \geq 1$ there exists an integer $c = c(k, d)$ such that the following holds. Let $\mathcal{F}$ be a finite family of subsets of $\mathbb{R}^d$ such that the intersection of any subfamily of $\mathcal{F}$ can be expressed as the union of at most k convex sets (not necessarily disjoint). Then $h(\mathcal{F}) \leq c$.

Alon and Kalai [AK95] obtain Theorem 4.1.7 as a consequence of a more general theorem. On the other hand, their method is more complicated than Matoušek's proof [Mat97], which is based mostly on elementary methods. Both proof methods yield rather poor numerical estimates on the Helly-numbers $c(k, d)$. For instance, for unions of convex sets in the plane, Matoušek's proof gives $c(2, 2) \leq 20$, $c(2, 3) \leq 90$, $c(2, 4) \leq 231$, etc. It would be interesting to obtain sharper bounds.

We conclude this discussion with a recent interesting Helly-type result due to Goaoc et al., which generalizes Theorem 4.1.7. For a topological space X, let $\tilde{\beta}_i(X)$ denote the i-th reduced Betti-number of X with coefficients in $\mathbb{Z}_2$.

THEOREM 4.1.8 [GP+15]

For any integers $b \geq 0$ and $d \geq 1$ there exists an integer $c = c(b,d)$ such that the following holds. Let $\mathcal{F}$ be a finite family of subsets of $\mathbb{R}^d$ such that for any proper subfamily $\mathcal{G} \subset \mathcal{F}$ we have $\tilde{\beta}_i(\bigcap G) \leq b$ for all $0 \leq i \leq \lceil d/2 \rceil - 1$. Then $h(F) \leq c$.

The proof of Theorem 4.1.8 relies on a general principle for obtaining Helly-type theorems from nonembeddability results for certain simplicial complexes, combined with an application of Ramsey's theorem. This generalizes previous work by Matoušek [Mat97] who established Theorem 4.1.8 in the special case $b = 0$ (using homotopy instead of homology). As a result, the bounds obtained on $c(b,d)$ are enormous and probably very far from the truth. However, examples show that Theorem 4.1.8 is sharp in the sense that *all* the (reduced) Betti numbers $\tilde{\beta}_i$ with $0 \leq i \leq \lceil d/2 \rceil - 1$ must be bounded in order to obtain a finite Helly-number.

INTERSECTIONS IN MORE THAN A POINT

Here we discuss some generalizations of Helly's theorem that apply to families of convex sets, but strengthen both the hypothesis and the conclusion of the theorem. The typical goal here is to guarantee that the sets intersect in more than a single point. The first such result is due to Vincensini [Vin39] and Klee [Kle53] and is a direct consequence of Helly's theorem.

THEOREM 4.1.9 [Vin39, Kle53]

Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$ and let B be some convex set in $\mathbb{R}^d$. If for any $d+1$ or fewer members of $\mathcal{F}$ there exists a translate of B contained in their common intersection, then there exists a translate of B contained in $\bigcap \mathcal{F}$.

Note that we obtain Helly's theorem when K is a single point. In Theorem 4.1.9, one may also replace the words "contained in" by "containing" or "intersecting." The following related result was obtained by De Santis [San57], who derived it from a generalization of Radon's Theorem. Note that it reduces to Helly's theorem for $k = 0$.

THEOREM 4.1.10 [San57]

Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$. If any $d-k+1$ or fewer members of $\mathcal{F}$ contain a k-flat in common, then there exists a k-flat contained in $\bigcap \mathcal{F}$.

One may also interpret the conclusion of Helly's theorem as saying that the intersection of the sets is at least 0-dimensional. Similarly, if $\mathcal{F}$ is a finite family of convex sets in $\mathbb{R}^d$, then Theorem 4.1.9 implies that if the intersection of every $d+1$ or fewer members is d-dimensional then $\bigcap \mathcal{F}$ is d-dimensional. For arbitrary $1 \leq k < d$, Grünbaum [Grü62] showed that if the intersection of every $2n-k$ or fewer sets is at least k-dimensional, then $\bigcap \mathcal{F}$ is at least k-dimensional. It turns out that the threshold $2n-k$ is not optimal, and the exact Helly-numbers for the dimension were obtained by Katchalski around a decade later.

THEOREM 4.1.11 [Kat71]

Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$. Let $\psi(0,d) = d+1$ and $\psi(k,d) = \max(d+1, 2(d-k+1))$ for $1 \leq k \leq d$. If the intersection of any $\psi(k,d)$ or fewer members of $\mathcal{F}$ has dimension at least k, then $\bigcap \mathcal{F}$ has dimension at least k.

Another direction in which Helly's theorem has been generalized is with a "quantitative" conclusion in mind. Loosely speaking, this means that we want the intersection of a family of convex sets to be "large" in some metrical sense. The first such result was noted by Buchman and Valentine [BV82]. For a nonempty closed convex set K in $\mathbb{R}^d$, the *width* of K is defined as the smallest possible distance between two supporting hyperplanes of K.

THEOREM 4.1.12 [BV82]

Let $\mathcal{F}$ be a finite family of closed convex sets in $\mathbb{R}^d$. If the intersection of any $d+1$ or fewer members of $\mathcal{F}$ has width at least w, then $\bigcap \mathcal{F}$ has width at least w.

Notice that Theorem 4.1.12 follows directly from Theorem 4.1.9 by observing that a convex set has width at least ω if and only if it contains a translate of every segment of length ω. Replacing "width" by "diameter" we get the following result due to Bárány Katchalski, and Pach.

THEOREM 4.1.13 [BKP84]

Let $\mathcal{F}$ be a finite family convex sets in $\mathbb{R}^d$. If the intersection of any $2d$ or fewer members of $\mathcal{F}$ has diameter at least 1, then $\bigcap \mathcal{F}$ has diameter at least $d^{-2d}/2$.

The proof of Theorem 4.1.13 is less obvious than the proof of Theorem 4.1.12 outlined above, and relies on a quantitative version of Steinitz' theorem. Examples show that the Helly-number $2d$ in Theorem 4.1.13 is best possible, but it is conjectured that the bound on the diameter can be improved.

CONJECTURE 4.1.14 [BKP84]

Let $\mathcal{F}$ be a finite family convex sets in $\mathbb{R}^d$. If the intersection of any $2d$ or fewer members of $\mathcal{F}$ has diameter at least 1, then $\bigcap \mathcal{F}$ has diameter at least $c_1 d^{-1/2}$ for some absolute constant $c_1 > 0$.

Bárány et al. [BKP84] also proved a Helly-type theorem for the volume, and conjectured a lower bound for the volume of the intersection given that the intersection of any $2d$ or fewer members has volume at least 1. A proof of this conjecture, using John's decomposition of the identity and the Dvoretzky–Rogers lemma, was recently given by Naszódi.

THEOREM 4.1.15 [Nas16]

Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$. If the intersection of any $2d$ or fewer members of $\mathcal{F}$ has volume at least 1, then $\bigcap \mathcal{F}$ has volume at least d^{-cd} for some absolute constant $c > 0$.

The current best estimate for the constant c in Theorem 4.1.15 is given in [Bra17].

Before ending the discussion on quantitative Helly theorems, we mention some extensions of Theorems 4.1.13 and 4.1.15 that answer a question by Kalai and Linial.

THEOREM 4.1.16 [DLL$^+$15a]

For every $d \in \mathbb{N}$ and $\epsilon > 0$ there exists integers $n_1 = n_1(d, \epsilon)$ and $n_2 = n_2(d, \epsilon)$ such that the following hold. Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$.

1. *If the intersection of any $n_1 d$ or fewer members of $\mathcal{F}$ has diameter at least 1,*

then $\bigcap \mathcal{F}$ has diameter at least $1-\epsilon$.

2. *If the intersection of any $n_2 d$ or fewer members of $\mathcal{F}$ has volume at least 1, then $\bigcap \mathcal{F}$ has volume at least $1-\epsilon$.*

These results witness a trade-off between the Helly-numbers and the lower bounds on the diameter and volume of $\bigcap \mathcal{F}$. The numbers $n_1(d,\epsilon)$ and $n_2(d,\epsilon)$ are related to the minimal number of facets needed to approximate an arbitrary convex body in $\mathbb{R}^d$ by a polytope. Moreover, for any fixed d, the numbers $n_1(d,\epsilon)$ and $n_2(d,\epsilon)$ are in $\Theta(\epsilon^{-(d-1)/2})$ [DLL+15a, Sob16]. See also [LS09] for other quantitative versions of Helly's theorem.

WEAKENING THE LOCAL CONDITION

For a finite family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$, let $f_k(\mathcal{F})$ denote the number of subfamilies of $\mathcal{F}$ of size $k+1$ with nonempty intersection. In particular, $f_0(\mathcal{F})$ denotes the number of (nonempty) members of $\mathcal{F}$. Helly's theorem states that if $f_d(\mathcal{F})$ equals $\binom{|\mathcal{F}|}{d+1}$, then there is a point in common to all the members of $\mathcal{F}$. What can be said if $f_d(\mathcal{F})$ is some value less than $\binom{|\mathcal{F}|}{d+1}$? This question was considered by Katchalski and Liu [KL79] who showed that if "almost every" subfamily of size $d+1$ have nonempty intersection, then there is a point in common to "almost every" member of $\mathcal{F}$. The precise meaning of "almost every" can be viewed as a consequence of "the upper bound theorem" for families of convex sets, and gives us the following.

THEOREM 4.1.17 [Kal84]

Let $\mathcal{F}$ be a finite family of $n \geq d+1$ convex sets in $\mathbb{R}^d$. For any $0 \leq \beta \leq 1$, if $f_d(\mathcal{F}) > (1-(1-\beta)^{d+1})\binom{n}{d+1}$, then some $\lfloor \beta n \rfloor + 1$ members of $\mathcal{F}$ have a point in common.

This result is commonly known as the *fractional Helly theorem* and has many important applications in discrete geometry (most notably its role in the (p,q)-problem discussed below). The upper bound theorem from which it is derived was discovered by Kalai [Kal84], and independently by Eckhoff [Eck85], and gives optimal upper bounds for the numbers $f_d(\mathcal{K}), \ldots, f_{d+r-1}(\mathcal{K})$ in terms of $f_0(\mathcal{F})$, provided $f_{d+r}(\mathcal{K}) = 0$. Here we state the particular instance that implies Theorem 4.1.17.

THEOREM 4.1.18 [Kal84, Eck85]

Let $\mathcal{F}$ be a finite family of $n \geq d+1$ convex sets in $\mathbb{R}^d$. For any $0 \leq r \leq n-d-1$, if $f_d(\mathcal{F}) > \binom{n}{d+1} - \binom{n-r}{d+1}$, then some $d+r+1$ members of $\mathcal{F}$ have a point in common.

The lower bound on $f_d(\mathcal{F})$ given in Theorem 4.1.18 cannot be reduced, which can be seen by considering r copies of $\mathbb{R}^d$ and $n-r$ hyperplanes in general position. The two proofs of Theorem 4.1.18 are quite different, but they both use the Nerve theorem and deal with the more general setting of *d-collapsible* simplicial complexes. It is known that the result extends to general *d-Leray* complexes as well [AKM+02]. Yet another proof of Theorem 4.1.18 was given by Alon and Kalai [AK85].

Due to its importance, several generalizations of the fractional Helly theorem have been considered. Let $\mathcal{F}$ be an arbitrary set system. We say that $\mathcal{F}$ has *fractional Helly-number* k if for every $\alpha > 0$ there exists a $\beta > 0$ such that for any finite subfamily $\mathcal{G} \subset \mathcal{F}$ where at least $\alpha\binom{|\mathcal{G}|}{k}$ of the k-member subfamilies of $\mathcal{G}$

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have nonempty intersection, there exists a point common to at least $\beta|\mathcal{G}|$ members of $\mathcal{G}$. We say that $\mathcal{F}$ has the *fractional Helly-property* if it has a finite fractional Helly-number. An important remark is that a set system may have the fractional Helly-property even though its Helly-number is unbounded.

Matoušek [Mat04] showed that any set system with bounded VC-dimension has the fractional Helly-property. More precisely, he shows that if the dual shatter function of $\mathcal{F}$ is bounded by $o(m^k)$, then $\mathcal{F}$ has fractional Helly-number k. For related geometric variants see also [Pin15]. Another noteworthy example is given by Rolnick and Soberón [RS17], who develop a framework to show that families of convex sets in $\mathbb{R}^d$ have the fractional Helly-property with respect to an abstract quantitative function defined on the family of all convex sets in $\mathbb{R}^d$. As a consequence, they obtain fractional analogues of Theorem 4.1.16 with respect to volume and surface area. A similar result has also been shown for the diameter [Sob16].

Another direction that has received a great deal of attention involves *piercing problems* for families of convex sets. Let $\mathcal{F}$ be a family of sets and suppose $\mathcal{F}$ can be partitioned into subfamilies $\mathcal{F}_1, \ldots, \mathcal{F}_k$ such that $\bigcap \mathcal{F}_i \neq \emptyset$ for every $1 \leq i \leq k$. The smallest number k for which this is possible is the *piercing number* of $\mathcal{F}$ and will be denoted by $\tau(\mathcal{F})$.

One early generalization of Helly's theorem involving the piercing number was considered by Hadwiger and Debrunner [HD57], where the hypothesis that every $d+1$ members of $\mathcal{F}$ have a point in common was replaced by the hypothesis that among any p members of $\mathcal{F}$ some q have a point in common, where $p \geq q \geq d+1$. For certain values of p and q, they obtained the following result.

THEOREM 4.1.19 [HD57]

Let $\mathcal{F}$ be a finite family of at least p convex sets in $\mathbb{R}^d$. If among any p members of $\mathcal{F}$ some q have a point in common, where $p \geq q \geq d+1$ and $p(d-1) < (q-1)d$, then $\tau(\mathcal{F}) \leq p - q + 1$.

Examples show that the value $p - q + 1$ is tight. Notice that Theorem 4.1.19 reduces to Helly's theorem when $p = q = d + 1$. For general values of p and q, not covered by Theorem 4.1.19, even the existence of a bounded piercing number remained unknown for a long time, and this became known as the *Hadwiger-Debrunner (p, q)-problem.* (Notice that repeated applications of Theorem 4.1.17 only give an upper bound of $\log |\mathcal{F}|$.) The (p, q)-problem was finally resolved by Alon and Kleitman, and is now referred to as the *(p, q)-theorem.*

THEOREM 4.1.20 [AK92]

For any integers $p \geq q \geq d+1$, there exists an integer $c = c(p, q, d)$ such that the following holds. Let $\mathcal{F}$ be a finite family of at least p convex sets in $\mathbb{R}^d$. If among any p members of $\mathcal{F}$ some q have a point in common, then $\tau(\mathcal{F}) \leq c$.

The proof of the (p, q)-theorem combines several tools from discrete geometry, and the most prominent roles are played by the fractional Helly theorem (Theorem 4.1.17) and the weak ϵ-net theorem for convex sets. It is of considerable interest to obtain better bounds on $c(p, q, d)$ (the only exact values known are the ones covered by Theorem 4.1.19). For the first open case of convex sets in the plane, the best known bounds are $3 \leq c(4, 3, 2) \leq 13$ [KGT99]. See [Eck03] for a survey on the (p, q)-problem, and also [KT08] and [Mül13].

It should be mentioned that the (p, q)-theorem holds in much more general

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settings. For instance, it holds for families of sets that are unions of convex sets as well as for good covers in $\mathbb{R}^d$ [AK95, AKM$^+$02]. In fact, a very general combinatorial framework was established in [AKM$^+$02] that provides bounds on the piercing number for general set systems that have the fractional Helly-property.

Recently, several other variations of the (p,q)-theorem have been considered. This includes a fractional version [BFM$^+$14] as well as quantitative versions [RS17]. As a sample of these results we mention the following theorem due to Soberón.

THEOREM 4.1.21 [Sob16]

For any integers $p \geq q \geq 2d > 1$ and real $0 < \epsilon < 1$ there exists an integer $c = c(p,q,d,\epsilon)$ such that the following holds. Let $\mathcal{F}$ be a finite family of at least p convex sets in $\mathbb{R}^d$, each of diameter at least 1. If among any p members of $\mathcal{F}$ some q have intersection of diameter at least 1, then there are c segments $S_1, \ldots, S_c$ each of length at least $1-\epsilon$ such that every member of $\mathcal{F}$ contains at least one of the S_i.

Piercing problems have also been studied for restricted classes of convex sets, in which case more precise (or even exact) bounds are known. For the special case of homothets, the intersection of every two members of $\mathcal{F}$ suffices to guarantee a bounded piercing number.

THEOREM 4.1.22 [Grü59]

For any integer $d \geq 1$ there exists an integer $c = c(d)$ such that the following holds. If $\mathcal{F}$ is a finite family of homothets of a convex set in $\mathbb{R}^d$ and any two members of $\mathcal{F}$ intersect, then $\tau(\mathcal{F}) \leq c$.

The special case when $\mathcal{F}$ is a family of circular disks in $\mathbb{R}^2$ was a question raised by Gallai, and answered by Danzer in 1956 (but not published until 1986).

THEOREM 4.1.23 [Dan86]

Let $\mathcal{F}$ be a finite family of circular disks in $\mathbb{R}^2$. If any two members of $\mathcal{F}$ intersect, then $\tau(\mathcal{F}) \leq 4$.

An example consisting of 10 disks shows that the number 4 cannot be reduced, and it is known that for 9 disks in the plane that pairwise intersect the piercing number is at most 3. For the even more restricted case when $\mathcal{F}$ is a family of pairwise intersecting *unit* disks in $\mathbb{R}^2$ it is known that $\tau(\mathcal{F}) \leq 3$. This is a special case of a conjecture of Grünbaum that stated that $\tau(\mathcal{F}) \leq 3$ for any family $\mathcal{F}$ of pairwise intersecting translates of a compact convex set $K \subset \mathbb{R}^2$. The conjecture was confirmed in full generality by Karasev.

THEOREM 4.1.24 [Kar00]

Let $\mathcal{F}$ be a finite family of translates of a compact convex set $K \subset \mathbb{R}^2$. If any two members of $\mathcal{F}$ intersect, then $\tau(\mathcal{F}) \leq 3$.

Karasev [Kar08] extended the methods from the proof of Theorem 4.1.24 to obtain higher dimensional analogues, where "pairwise intersecting" is replaced by the property that any d members have a point in common. For any family $\mathcal{F}$ of Euclidean balls in $\mathbb{R}^d$ whose radii differ by no more than a factor of d, he shows that if every d members of $\mathcal{F}$ have a point in common, then $\tau(F) \leq d+1$. Using this result he gives an upper bound on the piercing number for families of Euclidean balls without any size constraint.

THEOREM 4.1.25 [Kar08]

Let $\mathcal{F}$ be a family of Euclidean balls in $\mathbb{R}^d$ such that any d members of $\mathcal{F}$ have a point in common. Then $\tau(\mathcal{F}) \leq 4(d+1)$ for $d \leq 4$ and $\tau(\mathcal{F}) \leq 3(d+1)$ for $d \geq 5$.

The bounds in Theorem 4.1.25 are probably not tight, for instance when $d = 2$ it only gives $\tau(\mathcal{F}) \leq 12$ (whereas Theorem 4.1.23 shows that $\tau(\mathcal{F}) \leq 4$), and it would be interesting to obtain sharper bounds. By the same method Karasev also shows that for families $\mathcal{F}$ of positive homothets of a simplex where every d members of $\mathcal{F}$ have a point in common, we have $\tau(\mathcal{F}) \leq d + 1$ (which is not known to be tight for $d > 2$).

We conclude this discussion with a conjecture due to Katchalski and Nashtir.

CONJECTURE 4.1.26 [KN99]

There exists a constant k_0 such that $\tau(\mathcal{F}) \leq 3$ for every finite family $\mathcal{F}$ of convex sets in $\mathbb{R}^2$ where $\tau(\mathcal{G}) = 2$ for every subfamily $\mathcal{G} \subset \mathcal{F}$ with $|\mathcal{G}| \leq k_0$.

COLORFUL VERSIONS

The following remarkable generalization of Helly's theorem was discovered by Lovász and described by Bárány.

THEOREM 4.1.27 [Bár82]

Let $\mathcal{F}_1, \ldots, \mathcal{F}_{d+1}$ be finite families of convex sets in $\mathbb{R}^d$. If $\bigcap_{i=1}^{d+1} K_i \neq \emptyset$ for each choice of $K_i \in \mathcal{F}_i$, then $\bigcap \mathcal{F}_i \neq \emptyset$ for some $1 \leq i \leq d+1$.

This is commonly known as the *colorful Helly theorem.* Notice that it reduces to Helly's theorem by setting $\mathcal{F}_1 = \cdots = \mathcal{F}_{d+1}$. The dual version, known as the *Colorful Carathéodory theorem*, was discovered by Bárány, and has several important applications in discrete geometry. For example, it plays key roles in Sarkaria's proof of Tverberg's theorem [Sar92, BO97] and in the proof of the existence of weak ϵ-nets for the family of convex sets in $\mathbb{R}^d$ [ABFK92].

Recently, Kalai and Meshulam gave a far-reaching topological generalization of Theorem 4.1.27. Recall that a *matroid* $\mathcal{M}$ defined on a finite set E is uniquely determined by its *rank function* ρ, and that a subset $A \subset E$ is *independent* in $\mathcal{M}$ if and only if $\rho(A) = |A|$.

THEOREM 4.1.28 [KM05]

Let $\mathcal{F}$ be a good cover in $\mathbb{R}^d$ and let $\mathcal{M}$ be a matroid with rank function ρ defined on the members of $\mathcal{F}$. If every subfamily of $\mathcal{F}$ that is independent in $\mathcal{M}$ has a point in common, then there exists a subfamily $\mathcal{G} \subset \mathcal{F}$ such that $\rho(\mathcal{F} \setminus \mathcal{G}) \leq d$ and $\bigcap \mathcal{G} \neq \emptyset$.

This reduces to Theorem 4.1.27 in the case when $\mathcal{F} = \mathcal{F}_1 \cup \ldots \cup \mathcal{F}_{d+1}$ is a family of convex sets in $\mathbb{R}^d$ and $\mathcal{M}$ is the partition matroid induced by the $\mathcal{F}_i$. The proof of Theorem 4.1.28 is based on simplicial homology and, in fact, Kalai and Meshulam prove an even more general result concerning arbitrary d-Leray complexes. Further extensions of Theorem 4.1.28 were obtained in [Hol16]. See also [Flø11] for an algebraic formulation of Theorem 4.1.27.

Quantitative versions of the colorful Helly theorem have also been obtained.

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For instance, a colorful version of Theorem 4.1.16 for the volume was proved in [DLL+15a], with an essentially different proof given in [RS17]. Here is a colorful version of Theorem 4.1.16 for the diameter.

THEOREM 4.1.29 [Sob16]

For every integer $d \geq 1$ and real $0 < \epsilon < 1$ there exists an integer $n = n(d, \epsilon)$ such that the following holds. Let $\mathcal{F}_1, \ldots, \mathcal{F}_n$ be finite families of convex sets in $\mathbb{R}^d$, and suppose $\bigcap_{i=1}^n K_i$ has diameter 1 for every choice of $K_i \in \mathcal{F}_i$. Then $\bigcap \mathcal{F}_i$ has diameter at least $1 - \epsilon$ for some $1 \leq i \leq n$. Moreover, for any fixed d, $n(d, \epsilon) = \Theta(\epsilon^{-(d-1)/2})$.

There are also colorful versions of the (p, q)-theorem. One such theorem was proved by Bárány and Matoušek [BM03], while another variation was obtained in [BFM+14].

THEOREM 4.1.30 [BFM+14]

For any integers $p \geq q \geq d + 1$ there exists an integer $c = c(p, q, d)$ such that the following holds. Let $\mathcal{F}_1, \ldots, \mathcal{F}_p$ be finite families of convex sets in $\mathbb{R}^d$. Suppose for any choice $K_1 \in \mathcal{F}_1, \ldots, K_p \in \mathcal{F}_p$ some q of the K_i have a point in common. Then there are at least $q - d$ of the families for which $\tau(\mathcal{F}_i) \leq c$.

We conclude with a conjecture regarding a colorful generalization of Theorem 4.1.24.

CONJECTURE 4.1.31 [JMS15]

Let K be a compact convex set in the plane and let $\mathcal{F}_1, \ldots, \mathcal{F}_n$ be nonempty families of translates of K, with $n \geq 2$. Suppose $K_i \cap K_j \neq \emptyset$ for every choice $K_i \in \mathcal{F}_i$, $K_j \in \mathcal{F}_j$, and $i \neq j$. Then there exists $m \in \{1, 2, \ldots, n\}$ such that $\tau(\bigcup_{i \neq m} \mathcal{F}_i) \leq 3$.

HELLY THEOREMS FOR CONVEX LATTICE SETS

Let $\mathbb{Z}^d$ denote the integer lattice in $\mathbb{R}^d$. A *convex lattice set* in $\mathbb{R}^d$ is a subset $S \subset \mathbb{R}^d$ such that $S = K \cap \mathbb{Z}^d$ for some convex set $K \subset \mathbb{R}^d$. Note that the convex hull operator in $\mathbb{R}^d$ equips $\mathbb{Z}^d$ with the structure of a general *convexity space*, but the specific structure of convex lattice sets has drawn particular interest due to its connection with integer linear programming, geometry of numbers, and crystallographic lattices. The family of all subsets of $\{0, 1\}^d$ of size $2^d - 1$ shows that the Helly-number for convex lattice sets in $\mathbb{R}^d$ is at least 2^d, and the following theorem, due to Doignon [Doi73], shows that this is indeed the correct Helly-number.

THEOREM 4.1.32 [Doi73]

Let $\mathcal{C}_{\mathbb{Z}^d}$ be the family of convex lattice sets in $\mathbb{R}^d$. Then $h(\mathcal{C}_{\mathbb{Z}^d}) = 2^d$.

This theorem was rediscovered independently by Bell [Bel77] and Scarf [Sca77]. Recently, Aliev et al. [ABD+16, ADL14] obtained the following quantitative version of Theorem 4.1.32.

THEOREM 4.1.33 [ADL14]

For any integers $d \geq 1$ and $k \geq 1$ there exists a constant $c = c(d, k)$ such that the following holds. Let $\mathcal{F}$ be a finite family of convex lattice sets in $\mathbb{R}^d$. If the

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intersection of every c or fewer members of $\mathcal{F}$ contains at least k points, then $\bigcap \mathcal{F}$ contains at least k points.

This result was initially proved in [ABD$^+$16]. The bounds on $c(d,k)$ were improved in [ADL14] and in subsequent work [CHZ15, AGMP$^+$17] it was shown that $c(d,k)$ is in $\Theta(k^{(d-1)/(d+1)})$.

The fractional version of Theorem 4.1.32 has also been considered and it was shown in [AKM$^+$02] that the family of convex lattice sets in $\mathbb{R}^d$ has the fractional Helly-property with the fractional Helly-number 2^d. This was later improved by Bárány and Matoušek.

THEOREM 4.1.34 [BM03]

For any integer $d \geq 1$ and real number $0 < \alpha < 1$ there exists a real number $\beta = \beta(d,\alpha) > 0$ such that the following holds. Let $\mathcal{F}$ be a finite family of convex lattice sets in $\mathbb{R}^d$ and suppose there are at least $\alpha\binom{|\mathcal{F}|}{d+1}$ subfamilies of size $d+1$ that have nonempty intersection. Then there is a point contained in at least $\beta|\mathcal{F}|$ members of $\mathcal{F}$.

This result shows that the large Helly-number of Theorem 4.1.32 can be regarded as a "local anomaly" and that the relevant number for other, more global Helly-type properties is only $d+1$. For instance, by the tools developed in [AKM$^+$02], it can be shown that Theorem 4.1.34 implies a (p,q)-theorem for convex lattice sets [BM03]. The proof of Theorem 4.1.34 uses surprisingly little geometry of $\mathbb{Z}^d$ and relies mostly on tools from extremal combinatorics. These methods also yield interesting colorful Helly-type theorems for convex lattice sets [AW12, BM03].

We now describe a common generalization of Theorem 4.1.32 and Helly's theorem, which was stated by Hoffman [Hof79], and rediscovered by Averkov and Weismantel [AW12]. Let M be a subset of $\mathbb{R}^d$. A subset $S \subset M$ is called *M-convex* if $S = M \cap C$ for some convex set $C \subset \mathbb{R}^d$. Thus, the convex hull operator in $\mathbb{R}^d$ equips M with the structure of a general convexity space.

Averkov and Weismantel proved the following Helly-type theorem for mixed integer spaces, that is, sets of the form $M = \mathbb{R}^m \times \mathbb{Z}^n$.

THEOREM 4.1.35 [Hof79, AW12]

Let $\mathcal{C}_{\mathbb{R}^m\times\mathbb{Z}^n}$ be the family of all $(\mathbb{R}^m\times\mathbb{Z}^n)$-convex sets in $\mathbb{R}^{m+n}$. Then $h(\mathcal{C}_{\mathbb{R}^m\times\mathbb{Z}^n}) = (m+1)2^n$.

In fact, Averkov and Weismantel deduce Theorem 4.1.35 from more general inequalities concerning the Helly-numbers of spaces of the form $\mathbb{R}^m \times M$ and $M \times \mathbb{Z}^n$. They also establish fractional and colorful Helly-type theorems for mixed integer spaces, and they obtain the following generalization of Theorem 4.1.34.

THEOREM 4.1.36 [AW12]

Let M be a nonempty closed subset of $\mathbb{R}^d$ and let $\mathcal{C}_M$ be the family of all M-convex sets in $\mathbb{R}^d$. If $h(\mathcal{C}_M)$ is finite, then $\mathcal{C}_M$ has the fractional Helly-property with fractional Helly-number $d+1$.

Further generalizations and variations of Helly's theorem for M-convex sets have been established in [Ave13, Hal09, DLL$^+$15b].

PROBLEM 4.1.37 [DLL+15b]

Let $\mathbb{P}$ denote the set of prime numbers, and let $\mathcal{C}_{\mathbb{P}\times\mathbb{P}}$ denote the family of all $(\mathbb{P}\times\mathbb{P})$-convex sets in $\mathbb{R}^2$. Does $\mathcal{C}_{\mathbb{P}\times\mathbb{P}}$ have a finite Helly-number ?

It is known that $h(\mathcal{C}_{\mathbb{P}\times\mathbb{P}}) \geq 14$, and it is conjectured that the Helly-number is unbounded [DLL+15b].

4.2 GEOMETRIC TRANSVERSALS

GLOSSARY

Transversal: A k-transversal to a family $\mathcal{F}$ of convex sets is an affine subspace of dimension k that intersects every member of $\mathcal{F}$.

Line transversal: A 1-transversal to a family of convex sets in $\mathbb{R}^d$.

Hyperplane transversal: A $(d-1)$-transversal to a family of convex sets in $\mathbb{R}^d$.

Separated: A family $\mathcal{F}$ of convex sets is k-separated if no $k+2$ members of $\mathcal{F}$ have a k-transversal.

Ordering: A k-ordering of a family $\mathcal{F} = \{K_1, \dots, K_n\}$ of convex sets is a family of orientations of $(k+1)$-tuples of $\mathcal{F}$ defined by a mapping $\chi : \mathcal{A}^{k+1} \to \{-1, 0, 1\}$ corresponding to the orientations of some family of points $X = \{x_1, \dots, x_n\}$ in $\mathbb{R}^k$. The orientation of $(a_{i_0}, a_{i_1}, \dots, a_{i_k})$ is the orientation of the corresponding points $(x_{i_0}, x_{i_1}, \dots, x_{i_k})$, i.e.,

$$\text{sgn}\ \det \begin{pmatrix} 1 & x_{i_0}^1 & \cdots & x_{i_0}^k \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{i_k}^1 & \cdots & x_{i_k}^k \end{pmatrix}.$$

Geometric permutation: A geometric permutation of a $(k-1)$-separated family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$ is the pair of k-orderings induced by some k-transversal of $\mathcal{F}$.

Perhaps due to the fact that the space of all affine k-flats in $\mathbb{R}^d$ is no longer contractible when $k > 0$, there is a clear distinction between the cases $k = 0$ and $k > 0$ in the study of k-transversals. In 1935, Vincensini asked whether Helly's theorem can be generalized to k-transversals for arbitrary $k < d$. In other words, is there a number $m = m(k, d)$ such that for any family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$, if every m or fewer members of $\mathcal{F}$ have a k-transversal, then $\mathcal{F}$ has a k-transversal? The answer to Vincensini's question is no, and as Santaló pointed out, even the number $m(1, 2)$ does not exists in general, that is, there is no Helly-type theorem for *line transversals* to convex sets in the plane.

It is evident that to get a "Helly-type theorem with transversals" one needs to impose additional conditions on the shapes and/or the relative positions of the sets of the family. Helly-type theorems for transversals to families of a restricted class of convex sets in $\mathbb{R}^d$ are closely related to the combinatorial complexity of the *space of transversals*, and a bound on the Helly-number can often be deduced from the topological Helly theorem or one of its variants.

As before, we mainly restrict our attention to finite families of compact convex sets, unless stated otherwise. In most cases, this causes no loss in generality and replacing compact sets by open sets is usually straightforward.

HADWIGER-TYPE THEOREMS

In 1957, Hadwiger [Had57] gave the first necessary and sufficient conditions for the existence of a line transversal to a finite family $\mathcal{F}$ of pairwise disjoint convex sets in $\mathbb{R}^2$. The basic observation is that if a line L intersects every member of $\mathcal{F}$, then L induces a linear ordering of $\mathcal{F}$; this is simply the order in which L meets the members of $\mathcal{F}$ (as it is traversed in one of its two opposite directions). In particular, this implies that there exists a linear ordering of $\mathcal{F}$ such that any *three* members of $\mathcal{F}$ can be intersected by a directed line consistently with the ordering. Hadwiger's transversal theorem asserts that this necessary condition is also sufficient.

THEOREM 4.2.1 *Hadwiger's Transversal Theorem* [Had57]

Let $\mathcal{F}$ be a finite family of pairwise disjoint convex sets in $\mathbb{R}^2$. If there exists a linear ordering of $\mathcal{F}$ such that every three members of $\mathcal{F}$ can be intersected by a directed line in the given order, then $\mathcal{F}$ has a line transversal.

An interesting remark is that even though the conditions of Theorem 4.2.1 guarantee the existence of a line transversal to $\mathcal{F}$, we are not guaranteed a line that intersects the members of $\mathcal{F}$ in the given order. To obtain this stronger conclusion we need to impose the condition that every *four* members can be intersected by a directed line in the given order, which is a theorem discovered by Wenger [Wen90c], and independently by Tverberg [Tve91].

Hadwiger's transversal theorem has been generalized to higher dimensions and gives necessary and sufficient conditions for the existence of a *hyperplane* transversal to a family of compact convex sets in $\mathbb{R}^d$. Partial results in this direction were obtained by Katchalski [Kat80], Goodman and Pollack [GP88], and Wenger [Wen90c], before Pollack and Wenger [PW90] found a common generalization of these previous results.

Let $\mathcal{F}$ be a finite family of compact convex sets in $\mathbb{R}^d$ and let P be a subset of $\mathbb{R}^k$. We say that $\mathcal{F}$ *separates consistently* with P if there exists a map $\varphi : \mathcal{F} \to P$ such that for any two of subfamilies $\mathcal{F}_1$ and $\mathcal{F}_2$ of $\mathcal{F}$ we have

$$\operatorname{conv} \mathcal{F}_1 \cap \operatorname{conv} \mathcal{F}_2 = \emptyset \quad \Rightarrow \quad \operatorname{conv} \varphi(\mathcal{F}_1) \cap \operatorname{conv} \varphi(\mathcal{F}_2) = \emptyset.$$

Note that, if $k < d$ and $\mathcal{F}$ separates consistently with a set $P \subset \mathbb{R}^k$, then every $k+2$ members of $\mathcal{F}$ have a k-transversal. (This follows from Radon's theorem.) Moreover, if $\mathcal{F}$ has a hyperplane transversal, then $\mathcal{F}$ separates consistently with a set $P \subset \mathbb{R}^{d-1}$; simply choose one point from each member of $\mathcal{F}$ contained in the hyperplane transversal.

THEOREM 4.2.2 [PW90]

A family $\mathcal{F}$ of compact convex sets in $\mathbb{R}^d$ has a hyperplane transversal if and only if $\mathcal{F}$ separates consistently with a set $P \subset \mathbb{R}^{d-1}$.

When $\mathcal{F}$ is a family of pairwise disjoint sets in the plane, the separation condition of Theorem 4.2.2 is equivalent to the ordering condition in Hadwiger's theorem. The proof of Theorem 4.2.2 uses the Borsuk-Ulam theorem and Kirchberger's the-

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orem, and it was generalized by Anderson and Wenger [AW96] who showed that the point set P can be replaced by an acyclic oriented matroid of rank at most d.

Theorem 4.2.2 was strengthened further by Arocha et al. [ABM+02] to include a description of the topological structure of the space of hyperplane transversals [ABM+02].

Let $\mathcal{G}_{d-1}^d$ denote the space of all affine hyperplanes in $\mathbb{R}^d$. Note that $\mathcal{G}_{d-1}^d$ is retractible to the space of hyperplanes passing through the origin, and therefore homotopy equivalent to $\mathbb{RP}^{d-1}$. For a family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$, let $\mathcal{T}_{d-1}^d(\mathcal{F})$ denote the subspace of $\mathcal{G}_{d-1}^d$ of all hyperplane transversals to $\mathcal{F}$. We say that $\mathcal{F}$ has a *virtual k-transversal* if the homomorphism induced by the inclusion, $H_{d-1-k}(\mathcal{T}_{d-1}^d(\mathcal{F})) \to H_{d-1-k}(\mathcal{G}_{d-1}^d)$, is nonzero. In particular, if L is a k-transversal to $\mathcal{F}$, then the set of all hyperplanes containing L shows that $\mathcal{F}$ has a virtual k-transversal. Thus, the property of having a virtual k-transversal can be interpreted as saying that there are "as many" hyperplane transversals as if there exists a k-transversal.

THEOREM 4.2.3 [ABM+02]

Let $\mathcal{F}$ be a finite family of compact convex sets in $\mathbb{R}^d$ and let P be a set of points in $\mathbb{R}^k$, for some $0 \leq k < d$. If $\mathcal{F}$ separates consistently with P, then $\mathcal{F}$ has a virtual k-transversal.

The proof of Theorem 4.2.3 follows the same ideas as the proof of Theorem 4.2.2, and uses Alexander duality to obtain the stronger conclusion.

There are currently no known conditions, similar in spirit of Hadwiger's transversal theorem, which guarantee the existence of a k-transversal to a family of compact convex sets in $\mathbb{R}^d$ for $0 < k < d-1$. Already in the simplest case, line transversals in $\mathbb{R}^3$, examples show that there is no direct analogue of Hadwiger's theorem. In particular, for any integer $n \geq 3$ there is a family $\mathcal{F}$ of pairwise disjoint convex sets in $\mathbb{R}^3$ and a linear ordering of $\mathcal{F}$ such that any $n-2$ members of $\mathcal{F}$ are met by a directed line consistent with the ordering, yet $\mathcal{F}$ has no line transversal [GPW93]. It is even possible to restrict the members of $\mathcal{F}$ to be pairwise disjoint translates of a fixed compact convex set in $\mathbb{R}^3$ [HM04]. In view of these examples, the following result is all the more remarkable.

THEOREM 4.2.4 [BGP08]

Let $\mathcal{F}$ be a finite family of pairwise disjoint Euclidean balls in $\mathbb{R}^d$. If there exists a linear ordering of $\mathcal{F}$ such that every $2d$ or fewer members of $\mathcal{F}$ can be intersected by a directed line in the given order, then $\mathcal{F}$ has a line transversal.

The special case when the members of $\mathcal{F}$ are congruent was established in [CGH+08], and it was shown by Cheong et al. [CGH12] that the number $2d$ in Theorem 4.2.4 is nearly optimal, in particular it cannot be reduced to $2d-2$, and it is conjectured that the correct number should be $2d-1$. The proof of Theorem 4.2.4 uses a homotopy argument due to Klee, which was also used by Hadwiger in his proof of Theorem 4.2.1. Essentially, one continuously contracts the members of $\mathcal{F}$ until the hypothesis is about to fail. In the planar case it is easily seen that the "limiting configuration" consists of precisely three members of $\mathcal{F}$ that "pin" a unique line, but in dimensions greater than two the situation is more difficult to analyze.

We conclude our discussion of Hadwiger's transversal theorem with a colorful generalization due to Arocha et al.

THEOREM 4.2.5 [ABM08]

Let $\mathcal{F}_1$, $\mathcal{F}_2$, and $\mathcal{F}_3$ be nonempty finite families of compact convex sets in the plane. Suppose $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$ admits a linear ordering $\prec$ such that $B \cap \mathrm{conv}(A \cup C) \neq \emptyset$ for any choice $A \prec B \prec C$ of sets belonging to distinct $\mathcal{F}_i$. Then one of the $\mathcal{F}_i$ has a line transversal.

Note that when $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3$ we recover the planar case of Theorem 4.2.2. The proof of Theorem 4.2.5 combines the topological arguments from [Wen90c] with some additional combinatorial arguments. Arocha et al. conjectured that Theorem 4.2.5 holds in every dimension, and some partial results (requiring roughly d^2, rather than $d+1$, color classes) were established in [HR16], but the general conjecture remains open.

THE SPACE OF TRANSVERSALS

Given a family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$, let $\mathcal{T}_k^d(\mathcal{F})$ denote the space of all k-transversals of $\mathcal{F}$. The space of *point* transversals, that is $\mathcal{T}_0^d$, has a relatively simple structure, since the intersection of convex sets is convex. For $k \geq 1$, the space $\mathcal{T}_k^d(\mathcal{F})$ can be much more complicated; it need not be connected, and each connected component may have nontrivial topology. The "combinatorial complexity" of $\mathcal{T}_k^d(\mathcal{F})$ can be measured in various ways. These give rise to problems that are interesting in their own right, but are also closely related to Helly-type transversal theorems. It is usual to witness a drop in the complexity of $\mathcal{T}_k^d(\mathcal{F})$ as one restricts the family $\mathcal{F}$ to more specialized classes of convex sets, and typically, one gets a Helly-type theorem when the complexity of $\mathcal{T}_k^d(\mathcal{F})$ is universally bounded over all families $\mathcal{F}$ within the given class.

If $\mathcal{F}$ is a family of pairwise disjoint convex sets, then a directed line that intersects every member of $\mathcal{F}$ induces a well-defined order on $\mathcal{F}$. Thus an undirected line transversal to $\mathcal{F}$ induces a pair of opposite linear orderings or "permutations" on $\mathcal{F}$. More generally, a family $\mathcal{F}$ of convex sets is $(k-1)$*-separated* if no $k+1$ members have a $(k-1)$-transversal. An oriented k-transversal H intersects a $(k-1)$-separated family $\mathcal{F} = \{K_1, \ldots, K_n\}$ of convex sets in a well-defined k-ordering. The orientation of $(K_{i_0}, K_{i_1}, \ldots, K_{i_k})$ is the orientation in H of any corresponding set of points $(x_{i_0}, x_{i_1}, \ldots, x_{i_k})$, where $x_{i_j} \in K_{i_j} \cap H$. An unoriented k-transversal to a $(k-1)$-separated family $\mathcal{F}$ of convex sets induces a pair of k-orderings on $\mathcal{F}$, consisting of the two k-orderings on $\mathcal{F}$ induced by the two orientations of the k-transversal. We identify each such pair of k-orderings and call this a *geometric permutation* of $\mathcal{F}$.

If $\mathcal{F}$ is $(k-1)$-separated, then two k-transversals that induce different geometric permutations on $\mathcal{F}$ must necessarily belong to different connected components of $\mathcal{T}_k^d(\mathcal{F})$. The converse is also true for hyperplane transversals.

THEOREM 4.2.6 [Wen90b]

Let $\mathcal{F}$ be a $(d-2)$-separated family of compact convex sets in $\mathbb{R}^d$. Two hyperplane transversals induce the same geometric permutation on $\mathcal{F}$ if and only if they lie in the same connected component of $\mathcal{T}_{d-1}^d(\mathcal{F})$.

It is not hard to show that if $|\mathcal{F}| \geq d$, then each connected component of $\mathcal{T}_{d-1}^d(\mathcal{F})$ is contractible. For k-transversals with $0 < k < d-1$, the situation is not so pleasant. There are constructions of families $\mathcal{F}$ of pairwise disjoint translates in

$\mathbb{R}^3$ where $\mathcal{T}_1^3(\mathcal{F})$ has arbitrarily many connected components, each corresponding to the same geometric permutation [HM04]. However, for the specific case of disjoint balls an analogous result holds for line transversals.

THEOREM 4.2.7 [BGP08]

Let $\mathcal{F}$ be a family of pairwise disjoint Euclidean balls in $\mathbb{R}^d$. Two line transversals induce the same geometric permutation on $\mathcal{F}$ if and only if they lie in the same connected component of $\mathcal{T}_1^d(\mathcal{F})$.

In fact, it can also be shown that each connected component of $\mathcal{T}_1^d(\mathcal{F})$ is contractible when $|\mathcal{F}| \geq 2$. It would be interesting to know if similar results hold for families of Euclidean balls in $\mathbb{R}^d$ with respect to k-transversals for $1 < k < d-1$.

A natural problem that arises is to bound the number of distinct geometric permutations that $\mathcal{T}_k^d(\mathcal{F})$ induces on $\mathcal{F}$ in terms of the size of $\mathcal{F}$. Let $g_k^d(n)$ denote the maximum number of geometric permutations over all $(k-1)$-separated families of n convex sets in $\mathbb{R}^d$. It is not hard to show that $g_1^2(n)$ is linear in n, and in this case the precise bound is known. In general, the following is known about $g_k^d(n)$:

THEOREM 4.2.8

1. $g_1^2(n) = 2n-2$ [ES90].
2. $g_1^d(n) = \Omega(n^{d-1})$ [KLL92].
3. $g_{d-1}^d(n) = O(n^{d-1})$ [Wen90a].
4. $g_k^d(n) = O(n^{k(k+1)(d-k)})$ for fixed k and d [GPW96].

One of the longest-standing conjectures concerning geometric permutations is that the lower bound in part 2 of Theorem 4.2.8 is tight, that is, $g_1^d(n) = \Theta(n^{d-1})$. For nearly 20 years, the best known upper bound was $g_1^d(n) = O(n^{2d-2})$ [Wen90a], until Rubin et al. improved the bound using the Clarkson-Shor probabilistic analysis and a charging scheme technique developed by Tagansky.

THEOREM 4.2.9 [RKS12]

The maximum number of geometric permutations induced by the line transversals to a family of n pairwise disjoint convex sets in $\mathbb{R}^d$ is $O(n^{2d-3}\log n)$.

By further specializing the shape of the convex sets of the family, it is possible to obtain sharper bounds on the number of geometric permutations induced by line transversals. Smorodinsky et al. [SMS00], showed that any family $\mathcal{F}$ of n pairwise disjoint Euclidean balls in $\mathbb{R}^d$ has a "separating set" of size linear in n. In other words, there exists a family $\mathcal{H}$ of hyperplanes in $\mathbb{R}^d$, with $|\mathcal{H}| = O(n)$, such that every pair of members in $\mathcal{F}$ can be separated by some hyperplane in $\mathcal{H}$. This "separation lemma" is specific to families of "fat" convex objects [KV01] and does not generalize to arbitrary families of convex sets. As a consequence we have the following.

THEOREM 4.2.10 [SMS00]

The maximum number of geometric permutations induced by the line transversals to a family of n pairwise disjoint Euclidean balls in $\mathbb{R}^d$ is $\Theta(n^{d-1})$.

This theorem shows that the conjectured bound for $g_1^d(n)$ is tight for the case of families of pairwise disjoint Euclidean balls in $\mathbb{R}^d$. See also [AS05] for other partial results.

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CONJECTURE 4.2.11

The maximum number of geometric permutations induced by the line transversals to a family of n pairwise disjoint convex sets in $\mathbb{R}^d$ *is* $\Theta(n^{d-1})$.

For families of pairwise disjoint translates in the plane, the bounds on the maximum number of geometric permutations can be reduced even further.

THEOREM 4.2.12 [KLL87, KLL92]

The line transversals to a family of pairwise disjoint translates of a compact convex set in $\mathbb{R}^2$ *induce at most three geometric permutations.*

The possible patterns of geometric permutations induced on families of pairwise disjoint translates in the plane have also been studied. This work was initiated by Katchalski [Kat86] in connection with a conjecture of Grünbaum, and a complete characterization is given in [AHK$^+$03]. For families of n pairwise disjoint translates in $\mathbb{R}^3$, it is known that the maximum number of geometric permutations is $\Omega(n)$ [AK05]. Again, the situation changes drastically when we restrict our attention to the Euclidean ball.

THEOREM 4.2.13 [HXC01, KSZ03, CGN05, NCG$^+$16]

The line transversals to a family of pairwise disjoint translates of the Euclidean ball in $\mathbb{R}^d$ *induce at most three distinct geometric permutations. Furthermore, if the family has size at least 7, then there are at most two distinct geometric permutations.*

The planar version of this result was proved by Smorodinsky et al. [SMS00], and improved in [AHK$^+$03] where it was shown that a family of at least *four* pairwise disjoint unit disks in the plane can have at most two distinct geometric permutations.

CONJECTURE 4.2.14 [NCG$^+$16]

The line transversals to a family of at least four disjoint unit balls in $\mathbb{R}^3$ *induce at most two geometric permutations.*

We now discuss a different type of "combinatorial complexity" of the space of transversals. If the members of $\mathcal{F}$ are closed, then the boundary of $\mathcal{T}_k^d(\mathcal{F})$ consists of k-flats that support one or more members of $\mathcal{F}$. This boundary can be partitioned into subspaces of k-flats that support the same subfamily of $\mathcal{F}$. Each of these subspaces can be further partitioned into connected components. The *combinatorial complexity* of $\mathcal{T}_k^d(\mathcal{F})$ is the number of such connected components.

Even in $\mathbb{R}^2$, the boundaries of two convex sets can intersect in an arbitrarily large number of points and may therefore have an arbitrarily large number of common supporting lines. Thus the space of line transversals to two convex sets in $\mathbb{R}^2$ can have arbitrarily large combinatorial complexity. However, if $\mathcal{F}$ consists of pairwise disjoint convex sets in $\mathbb{R}^2$ or, more generally, suitably separated convex sets in $\mathbb{R}^d$, then the complexity is bounded. If the convex sets have constant description complexity, then again the transversal space complexity is bounded. Finally, if the sets are convex polytopes, then the transversal space is bounded by the total number of polytope faces. Table 4.2.1 gives bounds on the transversal space complexity for various families of sets.

The function $\alpha(n)$ is the very slowly growing inverse of the Ackermann function. The function $\lambda_s(n)$ is the maximum length of an (n,s) Davenport-Schinzel sequence,

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TABLE 4.2.1 Bounds on $\mathcal{T}_k^d(\mathcal{F})$.

FAMILY $\mathcal{F}$	k	d	COMPLEXITY OF $\mathcal{T}_k^d(\mathcal{F})$	SOURCE
$(d-2)$-separated family of n compact and strictly convex sets	$d-1$	d	$O(n^{d-1})$	[CGP+94]
n connected sets such that any two sets have at most s common supporting lines	1	2	$O(\lambda_s(n))$	[AB87]
n convex sets with const. description complexity	1	3	$O(n^{3+\epsilon})$ for any $\epsilon > 0$	[KS03]
n convex sets with const. description complexity	2	3	$O(n^{2+\epsilon})$ for any $\epsilon > 0$	[ASS96]
n convex sets with const. description complexity	3	4	$O(n^{3+\epsilon})$ for any $\epsilon > 0$	[KS03]
n line segments	$d-1$	d	$O(n^{d-1})$	[PS89]
Convex polytopes with a total of n_f faces	$d-1$	d	$O(n_f^{d-1}\alpha(n_f))$	[PS89]
Convex polytopes with a total of n_f faces	1	3	$O(n_f^{3+\epsilon})$ for any $\epsilon > 0$	[Aga94]
n $(d-1)$-balls	$d-1$	d	$O(n^{\lceil d/2 \rceil})$	[HII+93]

which equals $n\alpha(n)^{O(\alpha(n)^{s-3})}$. Note that $\lambda_s(n) \in O(n^{1+\epsilon})$ for any $\epsilon > 0$. The asymptotic bounds on the worst case complexity of hyperplane transversals ($k = d-1$) to line segments and convex polytopes are tight. There are examples of families $\mathcal{F}$ of convex polytopes where the complexity of $\mathcal{T}_1^3(\mathcal{F})$ is $\Omega(n_f^3)$.

As may be expected, the time to construct a representation of $\mathcal{T}_k^d(\mathcal{F})$ is directly related to the complexity of $\mathcal{T}_k^d(\mathcal{F})$. Most algorithms use upper and lower envelopes to represent and construct $\mathcal{T}_k^d(\mathcal{F})$. Table 4.2.2 gives known bounds on the worst case time to construct a representation of the space $\mathcal{T}_k^d(\mathcal{F})$ for various families of convex sets. All sets are assumed to be compact. As noted, for $\mathcal{T}_1^3(\mathcal{F})$ and $\mathcal{T}_3^4(\mathcal{F})$, the bound is for expected running time, not worst case time.

TABLE 4.2.2 Algorithms to construct $\mathcal{T}_k^d(\mathcal{F})$.

FAMILY $\mathcal{F}$	k	d	TIME COMPLEXITY	SOURCE
$(d-2)$-separated family of n strictly convex sets with constant description complexity	$d-1$	d	$O(n^{d-1}\log^2(n))$	[CGP+94]
n convex sets with const. description complexity s.t. any two sets have at most s common supporting lines	1	2	$O(\lambda_s(n)\log n)$	[AB87]
n convex sets with const. description complexity	1	3	$O(n^{3+\epsilon})\ \forall\ \epsilon > 0$ (exp'd.)	[KS03]
n convex sets with const. description complexity	2	3	$O(n^{2+\epsilon})\ \forall\ \epsilon > 0$	[ASS96]
n convex sets with const. description complexity	3	4	$O(n^{3+\epsilon})\ \forall\ \epsilon > 0$ (exp'd.)	[KS03]
Convex polygons with a total of n_f faces	1	2	$\Theta(n_f\log(n_f))$	[Her89]
Convex polytopes with a total of n_f faces	1	3	$O(n_f^{3+\epsilon})\ \forall\ \epsilon > 0$	[PS92]
Convex polytopes with a total of n_f faces	2	3	$\Theta(n_f^2\alpha(n_f))$	[EGS89]
Convex polytopes with a total of n_f faces	$d-1$	d	$O(n_f^d)$, $d > 3$	[PS89]
n $(d-1)$-balls	$d-1$	d	$O(n^{\lceil d/2 \rceil + 1})$	[HII+93]
n convex homothets	1	2	$O(n\log(n))$	[Ede85]
n pairwise disjoint translates of a convex set with constant description complexity	1	2	$O(n)$	[EW89]

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The model of computation used in the lower bound for the time to construct $\mathcal{T}_1^2(\mathcal{F})$ is an algebraic decision tree. In the worst case, $\mathcal{T}_2^3(\mathcal{F})$ may have $\Omega(n_f^2\alpha(n_f))$ complexity, which is a lower bound for constructing $\mathcal{T}_2^3(\mathcal{F})$. Similarly, $\mathcal{T}_1^3(\mathcal{F})$ may have $\Omega(n_f^3)$ complexity, giving an $\Omega(n_f^3)$ lower bound for the time to construct it.

HELLY-TYPE THEOREMS

As remarked earlier, there is no Helly-type theorem for general families of convex sets with respect to k-transversals for $k \geq 1$. Therefore, one direction of research in geometric transversal theory has focused on finding restricted classes of families of convex sets for which there is a Helly-type theorem. The earliest such results were established by Santaló.

THEOREM 4.2.15 [San40]

Let $\mathcal{F}$ be a family of parallelotopes in $\mathbb{R}^d$ with edges parallel to the coordinate axes.

1. *If any $2^{d-1}(d+1)$ or fewer members of $\mathcal{F}$ have a hyperplane transversal, then $\mathcal{F}$ has a hyperplane transversal.*
2. *If any $2^{d-1}(2d-1)$ or fewer members of $\mathcal{F}$ have a line transversal, then $\mathcal{F}$ has a line transversal.*

It is known that the Helly-number $2^{d-1}(d+1)$ for hyperplane transversals is tight, but for the case of line transversals the correct Helly-number seems to be unknown. Grünbaum [Grü64] generalized Theorem 4.2.15 to families of polytopes whose vertex cones are "related" to those of a fixed polytope in $\mathbb{R}^d$.

Let us remark that Theorem 4.2.15 can be deduced from Kalai and Meshulam's topological version of Amenta's theorem (Theorem 4.1.4). To see this, note that it is no loss in generality to assume that any subfamily of $\mathcal{F}$ that admits a transversal, admits a transversal that is not orthogonal to any of the coordinate axes. We can then partition the set of all hyperplanes (or lines) that are not orthogonal to any coordinate axis into 2^{d-1} distinct "direction classes." The next step is to show that the space of transversals (to any given subfamily) within a fixed direction class is a contractible set. Now Theorem 4.2.15 follows from Theorem 4.1.4 since the space of hyperplanes in $\mathbb{R}^d$ is d-dimensional and the space of lines in $\mathbb{R}^d$ is $(2d-2)$-dimensional.

In most cases, when a certain class of families of convex sets admits a Helly-type theorem for k-transversals, it is possible to show that the Helly-number is bounded by applying Theorem 4.1.8. This reduces the problem of bounding the Helly-number to the problem of bounding the complexity of the space of transversals. The resulting upper bound on the Helly-number will in general be very large, and often more direct arguments are needed to obtain sharper bounds.

Let us illustrate how Theorem 4.1.8 can be used to prove an upper bound on the Helly-number. A finite family $\mathcal{F}$ of compact convex sets in $\mathbb{R}^d$ is ϵ-*scattered* if, for every $0 < j < d$, any j of the sets can be separated from any other $d-j$ of the sets by a hyperplane whose distance is more than $\epsilon D(\mathcal{F})/2$ away from all d of the sets, where $D(\mathcal{F})$ is the largest diameter of any member of $\mathcal{F}$. It can be shown that for every $\epsilon > 0$ there exists a constant $C_d(\epsilon)$ such that the space of hyperplane transversals of an ϵ-scattered family induces at most $C_d(\epsilon)$ distinct geometric permutations. By Theorem 4.2.6, this implies that the space of hyperplane transversals has at most

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$C_d(\epsilon)$ connected components. Moreover, each connected component is contractible if $|\mathcal{F}| \geq d$, and if $|\mathcal{F}| = k < d$, then the space of hyperplane transversals is homotopy equivalent to $\mathbb{RP}^{d-k}$. For each member $K \in \mathcal{F}$, let $H_K = \mathcal{T}^d_{d-1}(\{K\})$, that is, the set of hyperplanes that intersect K. Note that the family $\{H_K\}_{K\in\mathcal{F}}$ can be parameterized as a family of compact subsets of $\mathbb{R}^{2d}$ and that the Betti-numbers of the intersection of any subfamily can be bounded by some absolute constant. Therefore, Theorem 4.1.8 implies that there is a finite Helly-number (depending on ϵ and d) for hyperplane transversals to ϵ-scattered families of convex sets in $\mathbb{R}^d$.

By a direct geometric argument, the Helly-number can be reduced drastically under the assumption that the family $\mathcal{F}$ is sufficiently large.

THEOREM 4.2.16 [AGPW01]

For any integer $d > 1$ and real $\epsilon > 0$ there exists a constant $N = N_d(\epsilon)$, such that the following holds. If $\mathcal{F}$ is an ϵ-scattered family of at least N compact convex sets in $\mathbb{R}^d$ and every $2d+2$ members of $\mathcal{F}$ have a hyperplane transversal, then $\mathcal{F}$ has a hyperplane transversal.

Goodman and Pollack conjectured that there is a bounded Helly-number for plane transversals to 1-separated families translates of the Euclidean ball in $\mathbb{R}^3$. A counter-example to this conjecture was given in [Hol07], and can be seen as a consequence of the fact that there exists a 1-separated family $\mathcal{F}$ of unit balls in $\mathbb{R}^3$ whose plane transversals induce a linear number of geometric permutations on $\mathcal{F}$. This example illustrates that in order to obtain a bounded Helly-number for hyperplane transversals it is necessary to assume that the family is ϵ-scattered.

From the previous discussion it is clear that there is a bounded Helly-number for *line transversals* to families of disjoint unit balls in $\mathbb{R}^d$. This follows from Theorems 4.2.7, 4.2.13, and 4.1.8 as in the argument above. By a more direct argument based on Theorem 4.2.4, the Helly-number can be reduced even further.

THEOREM 4.2.17 [CGH$^+$08]

Let $\mathcal{F}$ be a family of pairwise disjoint unit balls in $\mathbb{R}^d$. If any $4d-1$ or fewer members of $\mathcal{F}$ have a line transversal, then $\mathcal{F}$ has a line transversal.

The planar case of Theorem 4.2.17 was proven by Danzer [Dan57] with the optimal Helly-number five, who also conjectured that the Helly-number was bounded for arbitrary dimension. For $d \geq 6$ the Helly-number can be further reduced to $4d-2$ [CVGG14]. A lower bound construction [CGH12] shows that the bound is tight up to a factor of two.

Motivated by Danzer's Helly-theorem for line transversals to unit disks in the plane, Grünbaum [Grü58] proved a Helly-theorem for line transversals to families of pairwise disjoint translates of a parallelogram in the plane, again with Helly-number five. The same result for arbitrary families of pairwise disjoint translates was proved by Tverberg.

THEOREM 4.2.18 [Tve89]

Let $\mathcal{F}$ be a family of pairwise disjoint translates of a compact convex set in $\mathbb{R}^2$. If every five or fewer members of $\mathcal{F}$ have a line transversal, then $\mathcal{F}$ has a line transversal.

The fact that there is a bounded Helly-number follows from Theorems 4.2.6, 4.2.12, and 4.1.8, but obtaining the optimal Helly-number is significantly more

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difficult. Tverberg's proof involves a detailed analysis of the patterns of the possible geometric permutations of families of translates, and it would be interesting to find a simpler proof. Several generalizations of Theorems 4.2.18 are known. For instance, the condition that the members of $\mathcal{F}$ are pairwise disjoint can be weakened and Robinson [Rob97] showed that for every $j > 0$ there exists a number $c(j)$ such that if $\mathcal{F}$ is a family of translates of a compact convex set in $\mathbb{R}^2$ such that the intersection of any j members of $\mathcal{F}$ is empty and such that every $c(j)$ or fewer members of $\mathcal{F}$ have a line transversal, then $\mathcal{F}$ has a line transversal. See also [BBC$^+$06] for further variations.

For families of pairwise disjoint translates in $\mathbb{R}^3$ there is no universal bound on the Helly-number with respect to line transversals. For any integer $n > 2$, there exists a family $\mathcal{F}$ of n pairwise disjoint translates of a compact convex set such that every $n-1$ members of $\mathcal{A}$ have a line transversal, but $\mathcal{F}$ does not have a line transversal [HM04]. This specific example shows that there can be arbitrarily many connected components in the space of transversals that all correspond to the same geometric permutation, and again illustrates that there is no Helly-theorem when the complexity of the space of transversals is unbounded.

PARTIAL TRANSVERSALS AND GALLAI-TYPE PROBLEMS

Even though there is no pure Helly-type theorem for line transversals to general families of convex sets in the plane, there is still much structure to a family of convex sets that satisfy the local Helly-property with respect to line transversals. In particular, families of convex sets in the plane admit a so-called *Gallai-type* theorem. In other words, for a family $\mathcal{F}$ of convex sets in the plane, if every k or fewer members of $\mathcal{F}$ have a line transversal, then there exists a small number of lines whose union intersects every member of $\mathcal{F}$. The natural problem is to determine the smallest number of lines that will suffice, and Eckhoff has determined nearly optimal bounds for all values of $k \geq 3$.

THEOREM 4.2.19 [Eck73, Eck93a]

Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^2$.

1. *If every four or fewer members of $\mathcal{F}$ have a line transversal, then there are two lines whose union intersects every member of $\mathcal{F}$.*
2. *If every three or fewer members of $\mathcal{F}$ have a line transversal, then there are four lines whose union intersects every member of $\mathcal{F}$.*

The bound in part 1 is obviously tight, and the lines can actually be chosen to be orthogonal. The proof of part 1 is relatively simple and follows by choosing a minimal element of a suitably chosen partial order on the set of orthogonal pairs of lines in the plane. The proof of part 2 is more difficult and is a refinement of an argument due to Kramer [Kra74]. The idea is to consider a pair of disjoint members of $\mathcal{F}$ that are extremal in the sense that their separating tangents form the smallest angle among all disjoint pairs in the family. Using this pair he defines four candidate lines, and proceeds to show that through a series of rotations and translations of these lines one can reach a position in which their union meets every member of $\mathcal{F}$.

CONJECTURE 4.2.20 [Eck93a]

If every three or fewer members of $\mathcal{F}$ have a line transversal, then there are three lines whose union intersects every member of $\mathcal{F}$.

A different question one can consider is to look for the largest *partial transversal.* This was considered by Katchalski, who in 1978 conjectured that if $\mathcal{F}$ is a finite family of convex sets in the plane such that every three members of $\mathcal{F}$ have a line transversal, then there is a line that intersects at least $\frac{2}{3}|\mathcal{F}|$ members of $\mathcal{F}$. Katchalski and Liu considered the more general problem, when every k or fewer members have a line transversal.

THEOREM 4.2.21 [KL80a]

For any integer $k \geq 3$ there exists a real number $\rho = \rho(k)$ such that the following holds. Let $\mathcal{F}$ be a family of compact convex sets in the plane. If every k or fewer members of $\mathcal{F}$ have a line transversal, then there is a subfamily of $\mathcal{F}$ of size at least $\rho|\mathcal{F}|$ that has a transversal. Moreover, $\rho(k) \to 1$ as $k \to \infty$.

In view of the absence of a Helly-type theorem for line transversals, Theorem 4.2.21 is quite remarkable. It is an interesting problem to determine the optimal values for $\rho(k)$, especially for small values of k. The current best bounds for $\rho(k)$ are given in [Hol10]: the upper bound is $\rho(k) \leq 1 - \frac{1}{k-1}$, disproving Katchalski's conjecture for $\rho(3)$, and a lower bound is of order $1 - \frac{\log k}{k}$. For small values of k, the known bounds are $1/3 \leq \rho(3) \leq 1/2$, $1/2 \leq \rho(4) \leq 2/3$, $1/2 \leq \rho(5) \leq 3/4$, etc.

Alon and Kalai [AK95] showed that the family of all convex sets in $\mathbb{R}^d$ has fractional Helly-number $d+1$ with respect to hyperplane transversals. In particular, for every $\alpha > 0$ there exists a $\beta > 0$ (which depends only on α and d) such that if $\mathcal{F}$ is a finite family of convex sets in $\mathbb{R}^d$ such that $\alpha\binom{|\mathcal{F}|}{d+1}$ of the $(d+1)$-tuples of $\mathcal{F}$ have a hyperplane transversal, then there exists a hyperplane that intersects at least $\beta|\mathcal{F}|$ members of $\mathcal{F}$. By the techniques developed in [AKM$^+$02] this implies the following (p,q)-theorem for hyperplane transversals.

THEOREM 4.2.22 [AK95]

For any integers $p \geq q \geq d+1$ there exists an integer $c = c(p,q,d)$ such that the following holds. If $\mathcal{F}$ is a finite family of at least p convex sets in $\mathbb{R}^d$ and out of every p members of $\mathcal{F}$ there are some q that have a hyperplane transversal, then there are c hyperplanes whose union intersects every member of $\mathcal{F}$.

The theorem above can not be extended to k-transversals for $0 < k < d-1$. In particular, for every $k \geq 3$, Alon et al. [AKM$^+$02] construct a family of convex sets in $\mathbb{R}^3$ in which every k members have a line transversal, but no $k+4$ members have a line transversal.

Problems concerning partial transversals have also been studied for families of pairwise disjoint translates in the plane. Katchalski and Lewis [KL80b] showed that there is a universal constant C such that if $\mathcal{F}$ is a family of pairwise disjoint translates of a compact convex set K in the plane such that every three members of $\mathcal{F}$ have a line transversal, then there is a line that intersects all but at most C members of $\mathcal{F}$. The original upper bound given in [KL80b] was $C \leq 603$, but they conjectured that $C = 2$. Bezdek [Bez94] gave an example of $n \geq 6$ pairwise disjoint congruent disks where every 3 have a line transversal, but no line meets more than $n-2$ of the disks, showing that the Katchalski–Lewis conjecture is best possible.

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Heppes [Hep07] proved that the Katchalski–Lewis conjecture holds for the disk.

THEOREM 4.2.23 [Hep07]

Let $\mathcal{F}$ be a family of pairwise disjoint congruent disks in the plane. If every three members of $\mathcal{F}$ have a line transversal, then there is a line that intersects all but at most two members of $\mathcal{F}$.

It turns out that the conjectured shape-independent upper bound $C = 2$ does not hold in general. It was shown in [Hol03] that, for any $n \geq 12$, there exists a family $\mathcal{F}$ of n pairwise disjoint translates of a parallelogram where every three members of $\mathcal{F}$ have a line transversal, but no line meets more than $n - 4$ of the members of $\mathcal{F}$. For families of disjoint translates of a general convex set K, the best known upper bound is $C \leq 22$ [Hol03].

In view of Theorem 4.2.18 and Theorem 4.2.23 it is natural to ask about families of disjoint translates in the plane where every four or fewer members have a line transversal. This problem was originally investigated by Katchalski and Lewis [KL82] for families of disjoint translates of a parallelogram who showed that there exists a line that intersects all but at most two members of the family. In a series of papers, Bisztriczky et al. [BFO05a, BFO05b, BFO08] investigated the same problem for families of disjoint congruent disks, and obtained the following result.

THEOREM 4.2.24 [BFO08]

Let $\mathcal{F}$ be a family of pairwise disjoint congruent disks in the plane. If every four members of $\mathcal{F}$ have a line transversal, then there is a line that intersects all but at most one member of $\mathcal{F}$.

The result is best possible, as can be seen from a family of five nearly touching unit disks with centers placed at the vertices of a regular pentagon. It is not known whether Theorem 4.2.24 generalizes to families of disjoint translates of a compact convex set in the plane, but combining Theorem 4.2.18 with an elementary combinatorial argument shows that if a counter-example exists, then there exists one consisting of at most 12 translates.

We conclude this section with another transversal problem concerning congruent disks in the plane. Let D denote a closed disk centered at the origin in $\mathbb{R}^2$, let $\mathcal{F} = \{x_i + D\}$ be a finite family of translates of D, and let $\lambda\mathcal{F} = \{x_i + \lambda D\}$ denote the family where each disk has been inflated by a factor of $\lambda > 0$ about its center.

CONJECTURE 4.2.25 *Dol'nikov, Eckhoff*

If every three members of $\mathcal{F} = \{x_i + D\}$ have a line transversal, then the family $\lambda\mathcal{F} = \{x_i + \lambda D\}$ has a line transversal for $\lambda = \frac{1+\sqrt{5}}{2}$.

It is easily seen the factor $\lambda = \frac{1+\sqrt{5}}{2}$ is best possible by centering the disks at the vertices of a regular pentagon. Heppes [Hep05] has shown that $\lambda \leq 1.65$, under the additional assumption that the members of $\mathcal{F}$ are pairwise disjoint. If we instead assume that every *four* members of $\mathcal{F}$ have a line transversal, Jerónimo [Jer07] has shown that $\lambda = \frac{1+\sqrt{5}}{2}$ is the optimal inflation factor.

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4.3 SOURCES AND RELATED MATERIAL

SURVEYS

[DGK63]: The classical survey of Helly's theorem and related results.

[Eck93b]: A survey of Helly's theorem and related results, updating the material in [DGK63].

[GPW93]: A survey of geometric transversal theory.

[SA95]: Contains applications of Davenport-Schinzel sequences and upper and lower envelopes to geometric transversals.

[Mat02]: A textbook covering many aspects of discrete geometry including the fractional Helly theorem and the (p, q)-problem.

[Eck03]: A survey on the Hadwiger–Debrunner (p, q)-problem.

[Tan13]: A survey on intersection patterns of convex sets from the viewpoint of nerve complexes.

[ADLS17]: A recent survey on Helly's theorem focusing on the developments since [Eck93b].

RELATED CHAPTERS

Chapter 2: Packing and covering
Chapter 3: Tilings
Chapter 6: Oriented matroids
Chapter 17: Face numbers of polytopes and complexes
Chapter 21: Topological methods in discrete geometry
Chapter 28: Arrangements
Chapter 41: Ray shooting and lines in space
Chapter 49: Linear programming

REFERENCES

[AB87] M. Atallah and C. Bajaj. Efficient algorithms for common transversals. *Inform. Process. Lett.*, 25:87–91, 1987.

[ABD+16] I. Aliev, R. Bassett, J.A. De Loera, and Q. Louveaux. A quantitative Doignon-Bell-Scarf theorem. *Combinatorica*, online first, 2016.

[ABFK92] N. Alon, I. Bárány, Z. Füredi, and D.J. Kleitman. Point selections and weak e-nets for convex hulls. *Combin. Probab. Comput.*, 1:189–200, 1992.

[ABM08] J.L. Arocha, J. Bracho, and L. Montejano. A colorful theorem on transversal lines to plane convex sets. *Combinatorica*, 28:379–384, 2008.

[ABM+02] J.L. Arocha, J. Bracho, L. Montejano, D. Oliveros, and R. Strausz. Separoids, their categories and a Hadwiger-type theorem for transversals. *Discrete Comput. Geom.*, 27:377–385, 2002.

[ADL14] I. Aliev, J.A. De Loera, and Q. Louveaux. Integer programs with prescribed number of solutions and a weighted version of Doignon-Bell-Scarf's theorem. In *Proc. 17th Conf. Integer Progr. Combin. Opt.*, vol. 8494 of *LNCS*, pages 37–51, Springer, Berlin, 2014.

[ADLS17] N. Amenta, J.A. De Loera, and P. Soberón. Helly's theorem: New variations and applications. in H.A. Harrington, M. Omar, and M. Wright, editors, *Algebraic and Geometric Methods in Discrete Mathematics*, vol. 685 of *Contemp. Math.*, pages 55–96, Amer. Math. Soc., Providence, 2017.

[Aga94] P.K. Agarwal. On stabbing lines for polyhedra in 3D. *Comput. Geom.*, 4:177–189, 1994.

[AGP+17] G. Averkov, B. González Merino, I. Paschke, M. Schymura, and S. Weltge. Tight bounds on discrete quantitative Helly numbers. *Adv. Appl. Math.*, 89:76–101, 2017.

[AGPW01] B. Aronov, J.E. Goodman, R. Pollack, and R. Wenger. A Helly-type theorem for hyperplane transversals to well-separated convex sets. *Discrete Comput. Geom.*, 25:507–517, 2001.

[AH35] P. Alexandroff and H. Hopf. *Topologie I*, vol. 45 of *Grundlehren der Math.* Springer-Verlag, Berlin, 1935.

[AHK+03] A. Asinowski, A. Holmsen, M. Katchalski, and H. Tverberg. Geometric permutations of large families of translates. In: Aronov et al., editors, *Discrete and Computational Geometry: The Goodman-Pollack Festschrift*, vol. 25 of *Algorithms and Combinatorics*, pages 157–176, Spinger, Heidelberg, 2003.

[AK85] N. Alon and G. Kalai. A simple proof of the upper bound theorem. *European J. Combin.* 6:211–214, 1985.

[AK92] N. Alon and D. Kleitman. Piercing convex sets and the Hadwiger–Debrunner (p, q)-problem. *Adv. Math.*, 96:103–112, 1992.

[AK95] N. Alon and G. Kalai. Bounding the piercing number. *Discrete Comput. Geom.*, 13:245–256, 1995.

[AK05] A. Asinowski and M. Katchalski. The maximal number of geometric permutations for n disjoint translates of a convex set in $\mathbb{R}^3$ is $\Omega(n)$. *Discrete Comput. Geom.*, 35:473–480, 2006.

[AKM+02] N. Alon, G. Kalai, J. Matoušek, and R. Meshulam. Transversal numbers for hypergraphs arising in geometry. *Adv. Appl. Math.*, 29:79–101, 2002.

[Ame96] N. Amenta. A short proof of an interesting Helly-type theorem. *Discrete Comput. Geom.*, 15:423–427, 1996.

[AS05] B. Aronov and S. Smorodinsky. On geometric permutations induced by lines transversal through a fixed point. *Discrete Comput. Geom.*, 34:285–294, 2005.

[ASS96] P.K. Agarwal, O. Schwarzkopf, and M. Sharir. The overlay of lower envelopes and its applications. *Discrete Comput. Geom.*, 15:1–13, 1996.

[Ave13] G. Averkov. On maximal S-free sets and the Helly number for the family of S-convex sets. *SIAM J. Discrete Math.*, 27:1610–1624, 2013.

[AW96] L. Anderson and R. Wenger. Oriented matroids and hyperplane transversals. *Adv. Math.*, 119:117–125, 1996.

[AW12] G. Averkov and R. Weismantel. Transversal numbers over subsets of linear spaces. *Adv. Geom.*, 12:19–28, 2012.

[Bár82] I. Bárány. A generalization of Carathéodory's theorem. *Discrete Math.*, 40:141–152, 1982.

[BBC+06] K. Bezdek, T. Bisztriczky, B. Csikós, and A. Heppes. On the transversal Helly numbers of disjoint and overlapping disks. *Arch. Math.*, 87:86–96, 2006.

[Bel77] D.E. Bell. A theorem concerning the integer lattice. *Stud. Appl. Math.*, 56:187–188, 1977.

[Bez94] A. Bezdek. On the transversal conjecture of Katchalski and Lewis. In G. Fejes Tóth, editor, *Intuitive Geometry*, vol. 63 of *Colloquia Math. Soc. János Bolyai*, pages 23–25, North-Holland, Amsterdam, 1991.

[BFM+14] I. Bárány, F. Fodor, L. Montejano, and D. Oliveros. Colourful and fractional (p, q)-theorems. *Discrete Comput. Geom.*, 51:628–642, 2014.

[BFO05a] T. Bisztriczky, F. Fodor, and D. Oliveros. Large transversals to small families of unit disks. *Acta. Math. Hungar.*, 106:273–279, 2005.

[BFO05b] T. Bisztriczky, F. Fodor, and D. Oliveros. A transversal property to families of eight or nine unit disks. *Bol. Soc. Mat. Mex.*, 3:59–73, 2005.

[BFO08] T. Bisztriczky, F. Fodor, and D. Oliveros. The $T(4)$ property of families of unit disks. *Isr. J. Math*, 168:239–252, 2008.

[BGP08] C. Borcea, X. Goaoc, and S. Petitjean. Line transversals to disjoint balls. *Discrete Comput. Geom.*, 39:158–173, 2008.

[BKP84] I. Bárány, M. Katchalski, and J. Pach. Helly's theorem with volumes. *Amer. Math. Monthly*, 91:362–365, 1984.

[BM03] I. Bárány and J. Matoušek. A fractional Helly theorem for convex lattice sets. *Adv. Math.*, 174:227–235, 2003.

[BO97] I. Bárány and S. Onn. Carathéodory's theorem, colorful and applicable. In I. Bárány and K. Böröczky, editors, *Intuitive Geometry*, vol. 6 of *Bolyai Soc. Math. Stud.*, pages 11–21, János Bolyai Math. Soc., Budapest, 1997.

[Bra17] S. Brazitikos. Brascamp-Lieb inequality and quantitative versions of Helly's theorem. *Mathematika.*, 63:272–291, 2017.

[BV82] E.O. Buchman and F.A. Valentine. Any new Helly numbers? *Amer. Math. Monthly*, 89:370–375, 1982.

[CGH12] O. Cheong, X. Goaoc, and A. Holmsen. Lower bounds to Helly numbers of line transversals to disjoint congruent balls. *Israel J. Math.*, 190:213–228, 2012.

[CGH+08] O. Cheong, X. Goaoc, A. Holmsen, and S. Petitjean. Helly-type theorems for line transversals to disjoint unit balls. *Discrete Comput. Geom.*, 39:194–212, 2008.

[CGN05] O. Cheong, X. Goaoc, and H.-S. Na. Geometric permutations of disjoint unit spheres. *Comput. Geom.*, 30:253–270, 2005.

[CGP+94] S.E. Cappell, J.E. Goodman, J. Pach, R. Pollack, M. Sharir, and R. Wenger. Common tangents and common transversals. *Adv. Math.*, 106:198–215, 1994.

[CHZ15] S.R. Chestnut, R. Hildebrand, and R. Zenklusen. Sublinear bounds for a quantitative Doignon-Bell-Scarf theorem. Preprint, `arXiv:1512.07126`, 2015.

[CVGG14] É. Colin de Verdière, G. Ginot and X. Goaoc. Helly numbers of acyclic families. *Adv. Math.*, 253:163–193, 2014.

[Dan57] L. Danzer. Über ein Problem aus der kombinatorischen Geometrie. *Arch. Math.*, 8:347–351, 1957

[Dan86] L. Danzer. Zur Lösung des Gallaischen Problems über Kreisscheiben in der euklidischen Ebene. *Studia Sci. Math. Hungar.*, 21:111–134, 1986.

[Deb70] H. Debrunner. Helly type theorems derived from basic singular homology. *Amer. Math. Monthly*, 77:375–380, 1970.

[DGK63] L. Danzer, B. Grünbaum, and V. Klee. Helly's theorem and its relatives. In *Convexity*, vol. 7 of *Proc. Sympos. Pure Math.*, pages 101–180, AMS, Providence, 1963.

[DLL+15a] J.A. De Loera, R.N. La Haye, D. Rolnick, and P. Soberón. Quantitative Tverberg, Helly, & Carathéodory theorems. Preprint, `arXiv:1503.06116`, 2015.

[DLL+15b] J.A. De Loera, R.N. La Haye, D. Oliveros, and E. Roldán-Pensado. Helly numbers of algebraic subsets of $\mathbb{R}^d$. Preprint, `arxiv:1508.02380`, 2015. To appear in *Adv. Geom.*

[Doi73] J.P. Doignon. Convexity in cristallographical lattices. *J. Geometry*, 3:71–85, 1973).

[Eck03] J. Eckhoff. A survey on the Hadwiger–Debrunner (p, q)-problem. In B. Aronov et al., editors, *Discrete and Computational Geometry: The Goodman-Pollack Festschrift*, vol. 25 of *Algorithms and Combinatorics*, pages 347–377, Springer, Berlin, 2003.

[Eck73] J. Eckhoff. Transversalenprobleme in der Ebene. *Arch. Math.*, 24:195–202, 1973.

[Eck85] J. Eckhoff. An upper bound theorem for families of convex sets. *Geom. Dedicata*, 19:217–227, 1985.

[Eck93a] J. Eckhoff. A Gallai-type transversal problem in the plane. *Discrete Comput. Geom.*, 9:203–214, 1993.

[Eck93b] J. Eckhoff. Helly, Radon and Carathéodory type theorems. In P.M. Gruber and J.M. Wills, editors, *Handbook of Convex Geometry*, pages 389–448, North-Holland, Amsterdam, 1993.

[Ede85] H. Edelsbrunner. Finding transversals for sets of simple geometric figures. *Theoret. Comput. Sci.*, 35:55–69, 1985.

[EGS89] H. Edelsbrunner, L.J. Guibas, and M. Sharir. The upper envelope of piecewise linear functions: algorithms and applications. *Discrete Comput. Geom.*, 4:311–336, 1989.

[EN09] J. Eckhoff and K. Nischke. Morris's pigeonhole principle and the Helly theorem for unions of convex sets. *Bull. Lond. Math. Soc.*, 41:577–588, 2009.

[ES90] H. Edelsbrunner and M. Sharir. The maximum number of ways to stab n convex non-intersecting sets in the plane is $2n - 2$. *Discrete Comput. Geom.*, 5:35–42, 1990.

[EW89] P. Egyed and R. Wenger. Stabbing pairwise-disjoint translates in linear time. In *Proc. 5th Sympos. Comput. Geom.*, pages 364–369, ACM Press, 1989.

[Flø11] G. Fløystad. The colorful Helly theorem and colorful resolutions of ideals. *J. Pure Appl. Algebra*, 215:1255–1262, 2011.

[GM61] B. Grünbaum and T.S. Motzkin. On components of families of sets. *Proc. Amer. Math. Soc.*, 12:607-613, 1961.

[GP88] J.E. Goodman and R. Pollack. Hadwiger's transversal theorem in higher dimensions. *J. Amer. Math. Soc.* 1:301–309, 1988.

[GP+15] X. Goaoc, P. Paták, Z. Safernová, M. Tancer, and U. Wagner. Bounding Helly numbers via Betti numbers. In *Proc. 31st Sympos. Comput. Geom.*, vol. 34 of *LIPIcs*, pages 507–521, Schloss Dagstuhl, 2015.

[GPW93] J.E. Goodman, R. Pollack, and R. Wenger. Geometric transversal theory. In J. Pach, editor, *New Trends in Discrete and Computational Geometry*, vol. 10 of *Algorithms and Combinatorics*, pages 163–198, Springer, Berlin, 1993.

[GPW96] J.E. Goodman, R. Pollack, and R. Wenger. Bounding the number of geometric permutations induced by k-transversals. *J. Combin. Theory Ser. A*, 75:187–197, 1996.

[Grü58] B. Grünbaum. On common transversals. *Arch. Math.*, 9:465–469, 1958.

[Grü59] B. Grünbaum. On intersections of similar sets. *Portugal Math.*, 18:155–164, 1959.

[Grü62] B. Grünbaum. The dimension of intersections of convex sets. *Pacific J. Math.*, 12:197–202, 1962.

[Grü64] B. Grünbaum. Common secants for families of polyhedra. *Arch. Math.*, 15:76–80, 1964.

[Had57] H. Hadwiger. Über eibereiche mit gemeinsamer treffgeraden. *Portugal Math.*, 6:23–29, 1957.

[Hal09] N. Halman. Discrete and lexicographic Helly-type theorems. *Discrete Comput. Geom.*, 39:690–719, 2008.

[HD57] H. Hadwiger and H. Debrunner. Über eine Variante zum Helly'schen Satz. *Arch. Math.*, 8:309–313, 1957.

[Hel23] E. Helly. Über Mengen konvexer Körper mit gemeinschaftlichen Punkten. *Jahresber. Deutsch. Math.-Verein.*, 32:175–176, 1923.

[Hel30] E. Helly. Über Systeme abgeschlossener Mengen mit gemeinschaftlichen Punkten. *Monatsh. Math.*, 37:281–302, 1930.

[Hep05] A. Heppes. New upper bound on the transversal width of $T(3)$-families of discs. *Discrete Comput. Geom.*, 34:463–474, 2005.

[Hep07] A. Heppes. Proof of the Katchalski–Lewis transversal conjecture for $T(3)$-families of congruent discs. *Discrete Comput. Geom.*, 38:289–304, 2007.

[Her89] J. Hershberger. Finding the upper envelope of n line segments in $O(n \log n)$ time. *Inform. Process. Lett.*, 33:169–174, 1989.

[HII$^+$93] M.E. Houle, H. Imai, K. Imai, J.-M. Robert, and P. Yamamoto. Orthogonal weighted linear L_1 and L_∞ approximation and applications. *Discrete Appl. Math.*, 43:217–232, 1993.

[HM04] A. Holmsen and J. Matoušek. No Helly theorem for stabbing translates by lines in $\mathbb{R}^3$. *Discrete Comput. Geom.*, 31:405–410, 2004.

[Hof79] A.J. Hoffman. Binding constraints and Helly numbers. In *2nd International Conference on Combinatorial Mathematics*, vol. 319 of *Annals New York Acad. Sci*, pages 284–288, New York, 1979.

[Hol03] A. Holmsen. New bounds on the Katchalski-Lewis transversal problem. *Discrete Comput. Geom.*, 29:395–408, 2003.

[Hol07] A. Holmsen. The Katchalski-Lewis transversal problem in $\mathbb{R}^d$. *Discrete Comput. Geom.*, 37:341–349, 2007.

[Hol10] A.F. Holmsen. New results for $T(k)$-families in the plane. *Mathematika*, 56:86–92, 2010.

[Hol16] A.F. Holmsen. The intersection of a matroid and an oriented matroid. *Adv. Math.*, 290:1–14, 2016.

[HR16] A.F. Holmsen and E. Roldán-Pensado. The colored Hadwiger transversal theorem in $\mathbb{R}^d$. *Combinatorica*, 36:417–429, 2016.

[HXC01] Y. Huang, J. Xu, and D.Z. Chen. Geometric permutations of high dimensional spheres. In *Proc. 12th ACM-SIAM Sympos. Discrete Algorithms*, pages 244–245, 2001.

[Jer07] J. Jerónimo-Castro. Line transversals to translates of unit discs. *Discrete Comput. Geom.*, 37:409–417, 2007.

[JMS15] J. Jerónimo-Castro, A. Magazinov, and P. Soberón. On a problem by Dol'nikov. *Discrete Math.*, 338:1577–1585, 2015.

[Kal84] G. Kalai. Intersection patterns of convex sets. *Israel J. Math.*, 48:161–174, 1984.

[Kar00] R.N. Karasev. Transversals for families of translates of a two-dimensional convex compact set. *Discrete Comput. Geom.*, 24:345–353, 2000.

[Kar08] R.N. Karasev. Piercing families of convex sets with the d-intersection property in $\mathbb{R}^d$. *Discrete Comput. Geom.*, 39:766–777, 2008.

[Kat71] M. Katchalski. The dimension of intersections of convex sets. *Israel J. Math.*, 10:465–470, 1971.

[Kat80] M. Katchalski. Thin sets and common transversals. *J. Geom.*, 14:103–107, 1980.

[Kat86] M. Katchalski. A conjecture of Grünbaum on common transversals. *Math. Scand.*, 59:192–198, 1986.

[KGT99] D.J. Kleitman, A. Gyárfás, and G. Tóth. Convex sets in the plane with three of every four meeting. *Combinatorica*, 21:221–232, 2001.

[KL79] M. Katchalski and A. Liu. A problem of geometry in $\mathbb{R}^n$. *Proc. Amar. Math. Soc.*, 75:284–288, 1979.

[KL80a] M. Katchalski and A. Liu. Symmetric twins and common transversals. *Pacific J. Math*, 86:513–515, 1980.

[KL80b] M. Katchalski and T. Lewis. Cutting families of convex sets. *Proc. Amer. Math. Soc.*, 79:457–461, 1980.

[KL82] M. Katchalski and T. Lewis. Cutting rectangles in the plane. *Discrete Math.*, 42:67–71, 1982.

[Kle53] V. Klee. The critical set of a convex body. *Amer. J. Math.*, 75:178–188, 1953.

[KLL87] M. Katchalski, T. Lewis, and A. Liu. Geometric permutations of disjoint translates of convex sets. *Discrete Math.*, 65:249–259, 1987.

[KLL92] M. Katchalski, T. Lewis, and A. Liu. The different ways of stabbing disjoint convex sets. *Discrete Comput. Geom.*, 7:197–206, 1992.

[KM05] G. Kalai and R. Meshulam. A topological colorful Helly theorem. *Adv. Math.*, 191:305–311, 2005.

[KM08] G. Kalai and R. Meshulam. Leray numbers of projections and a topological Helly-type theorem. *J. Topol.*, 1:551–556, 2008.

[KN99] M. Katchalski and D. Nashtir. A Helly type conjecture. *Discrete Comput. Geom.*, 21:37–43, 1999.

[Kra74] D. Kramer. Transversalenprobleme vom Hellyschen und Gallaischen Typ. Dissertation, Universität Dortmund, 1974.

[KS03] V. Koltun and M. Sharir. The partition technique for overlays of envelopes. *SIAM J. Comput.*, 32:841–863, 2003.

[KSZ03] M. Katchalski, S. Suri, and Y. Zhou. A constant bound for geometric permutations of disjoint unit balls. *Discrete Comput. Geom.*, 29:161–173, 2003.

[KT08] J. Kynčl and M. Tancer. The maximum piercing number for some classes of convex sets with the $(4,3)$-property. *Electron. J. Combin.* 15:27, 2008

[KV01] M.J. Katz and K.R. Varadarajan. A tight bound on the number of geometric permutations of convex fat objects in $\Re^d$. *Discrete Comput. Geom.*, 26:543–548, 2001.

[LS09] M. Langberg and L.J. Schulman. Contraction and expansion of convex sets. *Discrete Comput. Geom.* 42:594–614, 2009.

[Mat97] J. Matoušek. A Helly-type theorem for unions of convex sets. *Discrete Comput. Geom.*, 18:1–12, 1997.

[Mat02] J. Matoušek. *Lectures on Discrete Geometry.* Vol. 212 of *Graduate Texts in Math.*, Springer, New York, 2002.

[Mat04] J. Matoušek. Bounded VC-dimension implies a fractional Helly theorem. *Discrete Comput. Geom.*, 31:251–255, 2004.

[Mon14] L. Montejano. A new topological Helly theorem and some transversal results. *Discrete Comput. Geom.*, 52:390–398, 2014.

[Mor73] H.C. Morris. *Two pigeon hole principles and unions of convexly disjoint sets.* Ph.D. thesis, Caltech, Pasadena, 1973.

[Mül13] T. Müller. A counterexample to a conjecture of Grünbaum on piercing convex sets in the plane. *Discrete Math.*, 313:2868–271, 2013.

[Nas16] M. Naszódi. Proof of a conjecture of Bárány, Katchalski and Pach. *Discrete Comput. Geom.* 55:243–248, 2016.

[NCG+16] J.-S. Haa, O. Cheong, X. Goaoc, and J. Yang. Geometric permutations of non-overlapping unit balls revisited. *Comput. Geom.*, 53:36–50, 2016.

[Pin15] R. Pinchasi. A note on smaller fractional Helly numbers. *Discrete Comput. Geom.*, 54:663–668, 2015.

[PS89] J. Pach and M. Sharir. The upper envelope of piecewise linear functions and the boundary of a region enclosed by convex plates: combinatorial analysis. *Discrete Comput. Geom.*, 4:291–309, 1989.

[PS92] M. Pellegrini and P.W. Shor. Finding stabbing lines in 3-space. *Discrete Comput. Geom.*, 8:191–208, 1992.

[PW90] R. Pollack and R. Wenger. Necessary and sufficient conditions for hyperplane transversals. *Combinatorica*, 10:307–311, 1990.

[RKS12] N. Rubin, H. Kaplan, and M. Sharir. Improved bounds for geometric permutations. *SIAM J. Comput*, 41:367–390, 2012.

[Rob97] J.-M. Robert. Geometric orderings of intersecting translates and their applications. *Comput. Geom.*, 7:59–72, 1997.

[RS17] D. Rolnick and P. Soberón. Quantitative (p, q) theorems in combinatorial geometry. *Discrete Math.*, 340:2516–2527, 2017.

[SA95] M. Sharir and P.K. Agarwal. *Davenport-Schinzel Sequences and Their Geometric Applications.* Cambridge University Press, 1995.

[San40] L.A. Santaló. Un teorema sóbre conjuntos de paralelepipedos de aristas paralelas. *Publ. Inst. Mat. Univ. Nac. Litoral*, 2:49–60, 1940.

[San57] R. De Santis. A generalization of Helly's theorem. *Proc. Amer. Math. Soc.*, 8:336–340, 1957.

[Sar92] K.S. Sarkaria. Tverberg's theorem via number fields. *Isreal. J. Math.*, 79:317–320, 1992.

[Sca77] H.E. Scarf. An observation on the structure of production sets with indivisibilities. *Proc. Nat. Acad. Sci. U.S.A.*, 74:3637–3641, 1977.

[SMS00] S. Smorodinsky, J.S.B. Mitchell, and M. Sharir. Sharp bounds on geometric permutations for pairwise disjoint balls in $\Re^d$. *Discrete Comput. Geom.*, 23:247–259, 2000.

[Sob16] P. Soberón. Helly-type theorems for the diameter. *Bull. London Math. Soc.* 48:577–588, 2016.

[Tan13] M. Tancer. Intersection patterns of convex sets via simplicial complexes, a survey. In J. Pach, editor, *Thirty Essays on Geometric Graph Theory*, pages 521–540, Springer, New York, 2013.

[Tve89] H. Tverberg. Proof of Grünbaum's conjecture on common transversals for translates. *Discrete Comput. Geom.*, 4:191–203, 1989.

[Tve91] H. Tverberg. On geometric permutations and the Katchalski-Lewis conjecture on partial transversals for translates. In J.E. Goodman, R. Pollack, and W. Steiger, editors, *Discrete and Computational Geometry*, vol. 6 of *DIMACS Ser. Discrete Math. Theor. Comp. Sci.*, pages 351–361, AMS, Providence, 1991.

[Vin39] P. Vincensini. Sur une extension d'un théorème de M. J. Radon sur les ensembles de corps convexes. *Bull. Soc. Math. France*, 67:115–119, 1939.

[Wen90a] R. Wenger. Upper bounds on geometric permutations for convex sets. *Discrete Comput. Geom.*, 5:27–33, 1990.

[Wen90b] R. Wenger. Geometric permutations and connected components. Technical Report TR-90-50, DIMACS, Piscataway, 1990.

[Wen90c] R. Wenger. A generalization of Hadwiger's transversal theorem to intersecting sets. *Discrete Comput. Geom.*, 5:383–388, 1990.