36 COMPUTATIONAL CONVEXITY

Peter Gritzmann and Victor Klee

INTRODUCTION

The subject of Computational Convexity draws its methods from discrete mathematics and convex geometry, and many of its problems from operations research, computer science, data analysis, physics, material science, and other applied areas. In essence, it is the study of the computational and algorithmic aspects of high-dimensional convex sets (especially polytopes), with a view to applying the knowledge gained to convex bodies that arise in other mathematical disciplines or in the mathematical modeling of problems from outside mathematics.

The name *Computational Convexity* is of more recent origin, having first appeared in print in 1989. However, results that retrospectively belong to this area go back a long way. In particular, many of the basic ideas of *Linear Programming* have an essentially geometric character and fit very well into the conception of Computational Convexity. The same is true of the subject of *Polyhedral Combinatorics* and of the *Algorithmic Theory of Polytopes and Convex Bodies*.

The emphasis in Computational Convexity is on problems whose underlying structure is the convex geometry of normed vector spaces of finite but generally *not* restricted dimension, rather than of fixed dimension. This leads to closer connections with the optimization problems that arise in a wide variety of disciplines. Further, in the study of Computational Convexity, the underlying model of computation is mainly the binary (Turing machine) model that is common in studies of computational complexity. This requirement is imposed by prospective applications, particularly in mathematical programming. For the study of algorithmic aspects of convex bodies that are not polytopes, the binary model is often augmented by additional devices called "oracles." Some cases of interest involve other models of computation, but the present discussion focuses on aspects of computational convexity for which binary models seem most natural. Many of the results stated in this chapter are qualitative, in the sense that they classify certain problems as being solvable in polynomial time, or show that certain problems are NP-hard or harder. Typically, the tasks remain to find optimal exact algorithms for the problems that are polynomially solvable, and to find useful approximation algorithms or heuristics for those that are NP-hard. In many cases, the known algorithms, even when they run in polynomial time, appear to be far from optimal from the viewpoint of practical application. Hence, the qualitative complexity results should in many cases be regarded as a guide to future efforts but not as final words on the problems with which they deal.

Some of the important areas of computational convexity, such as linear and convex programming, packing and covering, and geometric reconstructions, are covered in other chapters of this Handbook. Hence, after some remarks on presentations of polytopes in Section 36.1, the present discussion concentrates on the following areas that are not covered elsewhere in the Handbook: 36.2, Algorithmic Theory

Preliminary version (August 10, 2017). To appear in the Handbook of Discrete and Computational Geometry,
J.E. Goodman, J. O'Rourke, and C. D. Tóth (editors), 3rd edition, CRC Press, Boca Raton, FL, 2017.

of Convex Bodies; 36.3, Volume Computations; 36.4, Mixed Volumes; 36.5, Containment Problems; 36.6, Radii; 36.7, Constrained Clustering. There are various other classes of problems in computational convexity that will not be covered e.g., projections of polytopes [Fil90, BGK96], sections of polytopes [Fil92], Minkowski addition of polytopes [GS93], geometric tomography [Gar95, GG94, GG97, GGH17] or the Minkowski reconstruction of polytopes [GH99].

Because of the diversity of topics covered in this chapter, each section has a separate bibliography.

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RELATED CHAPTERS

Chapter 2: Packing and covering
Chapter 7: Lattice points and lattice polytopes
Chapter 15: Basic properties of convex polytopes
Chapter 34: Geometric reconstruction problems
Chapter 49: Linear programming

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36.1 PRESENTATIONS OF POLYTOPES

A convex polytope $P \subset \mathbb{R}^n$ can be represented in terms of its vertices or in terms of its facet inequalities. From a theoretical viewpoint, the two possibilities are equivalent. However, as the dimension increases, the number of vertices can grow exponentially in terms of the number of facets, and vice versa, so that different presentations may lead to different classifications concerning polynomial-time computability or NP-hardness. (See Sections 15.1 and 26.3 of this Handbook.)

For algorithmic purposes it is usually not the polytope P as a *geometric* object that is relevant, but rather its *algebraic presentation*. The discussion here is based mainly on the *binary* or *Turing machine* model of computation, in which the *size of the input* is defined as the length of the binary encoding needed to present the input data to a Turing machine and the *time-complexity* of an algorithm is also defined in terms of the operations of a Turing machine. Hence the algebraic presentation of the objects at hand must be finite.

Among important special classes of polytopes, the zonotopes are particularly interesting because they can be so compactly presented.

GLOSSARY

Convex body in $\mathbb{R}^n$: A compact convex subset of $\mathbb{R}^n$.

$\mathcal{K}^n$: The family of all convex bodies in $\mathbb{R}^n$.

Proper convex body in $\mathbb{R}^n$: A convex body in $\mathbb{R}^n$ with nonempty interior.

Polytope: A convex body that has only finitely many extreme points.

$\mathcal{P}^n$: The family of all convex polytopes in $\mathbb{R}^n$.

n-polytope: Polytope of dimension n.

Face of a polytope P: P itself, the empty set, or the intersection of P with some supporting hyperplane; $f_i(P)$ is the number of i-dimensional faces of P.

Facet of an n-polytope P: Face of dimension $n-1$.

Simple n-polytope: Each vertex is incident to precisely n edges or, equivalently, to precisely n facets.

Simplicial polytope: A polytope in which each facet is a simplex.

$\mathcal{V}$-presentation of a polytope P: A string $(n, m; v_1, \ldots, v_m)$, where $n, m \in \mathbb{N}$ and $v_1, \ldots, v_m \in \mathbb{R}^n$ such that $P = \text{conv}\{v_1, \ldots, v_m\}$.

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$\mathcal{H}$-presentation of a polytope P: A string $(n, m; A, b)$, where $n, m \in \mathbb{N}$, A is a real $m \times n$ matrix, and $b \in \mathbb{R}^m$ such that $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$.

irredundant $\mathcal{V}$- or $\mathcal{H}$-presentation of a polytope P: A $\mathcal{V}$- or $\mathcal{H}$-presentation $(n, m; v_1, \ldots, v_m)$ or $(n, m; A, b)$ of P with the property that none of the points $v_1, \ldots, v_m$ or none of the inequalities $Ax \leq b$ can be omitted without altering P, respectively.

$\mathcal{V}$-polytope P: A string $(n, m; v_1, \ldots, v_m)$, where $n, m \in \mathbb{N}$ and $v_1, \ldots, v_m \in \mathbb{Q}^n$. P is usually identified with the geometric object $\text{conv}\{v_1, \ldots, v_m\}$.

$\mathcal{H}$-polytope P: A string $(n, m; A, b)$, where $n, m \in \mathbb{N}$, A is a rational $m \times n$ matrix, $b \in \mathbb{Q}^m$, and the set $\{x \in \mathbb{R}^n \mid Ax \leq b\}$ is bounded. P is usually identified with this set.

Size of a $\mathcal{V}$- or an $\mathcal{H}$-polytope P: Number of binary digits needed to encode the string $(n, m; v_1, \ldots, v_m)$ or $(n, m; A, b)$, respectively.

Zonotope: The vector sum (Minkowski sum) of a finite number of line segments; equivalently, a polytope of which each face has a center of symmetry.

$\mathcal{S}$-presentation of a zonotope Z in $\mathbb{R}^n$: A string $(n, m; c; z_1, \ldots, z_m)$, where $n, m \in \mathbb{N}$ and $c, z_1, \ldots, z_m \in \mathbb{R}^n$, such that $Z = c + \sum_{i=1}^m [-1, 1] z_i$.

Parallelotope in $\mathbb{R}^n$: A zonotope $Z = c + \sum_{i=1}^m [-1, 1] z_i$, with $z_1, \ldots, z_m$ linearly independent.

$\mathcal{S}$-zonotope Z in $\mathbb{R}^n$: A string $(n, m; c; z_1, \ldots, z_m)$, where $n, m \in \mathbb{N}$ and $c, z_1, \ldots, z_m \in \mathbb{Q}^n$. Z is usually identified with the geometric object $c + \sum_{i=1}^m [-1, 1] z_i$.

36.1.1 CONVERSION OF ONE PRESENTATION INTO THE OTHER

Note, first, that from a given $\mathcal{V}$- or $\mathcal{H}$-representation of a polytope P, an irredundant $\mathcal{V}$- or $\mathcal{H}$-representation of P can be computed in polynomial time by means of linear programming, respectively; see also Section 26.2.

The following results indicate the difficulties that may be expected in converting the $\mathcal{H}$-presentation of a polytope into a $\mathcal{V}$-presentation or vice versa.

For $\mathcal{H}$-presented n-polytopes with m facets, the *maximum* possible number of vertices is

$$\mu(m, n) = \binom{m - \lfloor (n+1)/2 \rfloor}{m - n} + \binom{m - \lfloor (n+2)/2 \rfloor}{m - n},$$

and this is also the maximum possible number of facets for a $\mathcal{V}$-presented n-polytope with m vertices. The first maximum is attained within the family of simple n-polytopes, the second within the family of simplicial n-polytopes.

When n is fixed, the number of vertices is bounded by a polynomial in the number of facets, and vice versa, and it is possible to pass from either sort of presentation to the other in polynomial time. However, the degree of the polynomial goes to infinity with n. A consequence of this is that when the dimension n is permitted to vary in a problem concerning polytopes, the manner of presentation is often influential in determining whether the problem can be solved in polynomial time or is NP-hard. For the case of variable dimension, it is #P-hard even to determine the number of facets of a given $\mathcal{V}$-polytope, or to determine the number of vertices of a given $\mathcal{H}$-polytope, [Lin86].

For *simple* $\mathcal{H}$-presented n-polytopes with m facets, the *minimum* possible number of vertices is $(m - n)(n - 1) + 2$. The large gap between this number and the

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above sum of binomial coefficients makes it clear that, from a practical standpoint, the worst-case behavior of any conversion algorithm should be measured in terms of *both* input size and output size. In this respect, the following problem seems fundamental.

OPEN PROBLEM 36.1.1

What is the computational complexity of POLYTOPE VERIFICATION*: Given an $\mathcal{H}$-polytope P and a $\mathcal{V}$-polytope Q in $\mathbb{R}^n$, decide whether $P = Q$.*

The maximum number of j-dimensional faces of an n-dimensional zonotope formed as the sum of m segments is

$$2\binom{m}{j}\sum_{k=0}^{n-1-j}\binom{m-1-j}{k},$$

and hence, the number of vertices or of facets (or of faces of any dimension) of an $\mathcal{S}$-zonotope is not bounded by any polynomial in the size of the $\mathcal{S}$-presentation.

In combinatorial optimization one is particularly interested in "perfect formulations" of 0-1-polytopes in $\mathbb{R}^n$ associated with the underlying problems. A well-studied example is that of the traveling salesman polytopes, the convex hull of the incidence vectors of Hamiltonian cycles of the complete graph on the given number of cities. Since formulations in the "natural space" of the application often have an exponential number of inequalities, one tries to find small *extended formulations*, i.e., formulations with a polynomial number of inequalities, after allowing a polynomial number of extra variables; see e.g., [CCZ13]. In effect, one is asking for a polytope in higher (but not too high) dimension whose projection on $\mathbb{R}^n$ coincides with the original polytope. For some "oracular" results (in the spirit of the next section) see [BV08].

We end this section by mentioning two other ways of presenting polytopes.

A general result of Bröcker and Scheiderer (see [BCR98]) on semi-algebraic sets implies that for each n-polytope P in $\mathbb{R}^n$ (no matter how complicated its facial structure may be), there exists a system of $n(n+1)/2$ polynomial inequalities that has P as its solution-set, and that n polynomial inequalities suffice to describe the interior of P. More recently, Bröcker showed (see [AH11]) that for polytopes n polynomial inequalities actually always suffice. [AH11] give a fully constructive proof that any *simple* n-polytope can be described by n polynomial inequalities.

For a polytope P in $\mathbb{R}^n$ whose interior is known to contain the origin, [GKW95] shows that the entire face-lattice of P can be reconstructed with the aid of at most

$$f_0(P) + (n-1)f_{n-1}^2(P) + (5n-4)f_{n-1}(P)$$

queries to the *ray-oracle* of P. In each such query, one specifies a ray issuing from the origin and the oracle is required to tell where the ray hits the boundary of P. Related results were obtained in [DEY90].

For more on oracles, see Section 36.2 of this Handbook.

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RELATED CHAPTERS

Chapter 15: Basic properties of convex polytopes
Chapter 17: Face numbers of polytopes and complexes
Chapter 26: Convex hull computations

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36.2 ALGORITHMIC THEORY OF CONVEX BODIES

Polytopes may be $\mathcal{V}$-presented or $\mathcal{H}$-presented. However, a different approach is required to deal with convex bodies K that are not polytopes, since an enumeration of all the extreme points of K or of its polar is not possible. A convenient way to deal with the general situation is to assume that the convex body in question is given by an algorithm (called an *oracle*) that answers certain sorts of questions about the body. A small amount of a priori information about the body may be known, but aside from this, all information about the specific convex body must be obtained from the oracle, which functions as a "black box." In other words, while it is assumed that the oracle's answers are always correct, nothing is assumed about the manner in which it produces those answers. The algorithmic theory of convex bodies was developed in [GLS88] with a view to proper (i.e., n-dimensional) convex bodies in $\mathbb{R}^n$. For many purposes, provisions can be made to deal meaningfully with improper bodies as well, but that aspect is largely ignored in what follows.

GLOSSARY

Outer parallel body of a convex body K: $K(\epsilon) = K + \epsilon B^n$, where B^n is the Euclidean unit ball in $\mathbb{R}^n$.

Inner parallel body of a convex body K: $K(-\epsilon) = K \setminus \big((\mathbb{R}^n \setminus K) + \epsilon B^n\big)$.

Weak membership problem for a convex body K in $\mathbb{R}^n$: Given $y \in \mathbb{Q}^n$, and a rational number $\epsilon > 0$, conclude with one of the following: *report that* $y \in K(\epsilon)$; or *report that* $y \notin K(-\epsilon)$.

Weak separation problem for a convex body K in $\mathbb{R}^n$: Given a vector $y \in \mathbb{Q}^n$, and a rational number $\epsilon > 0$, conclude with one of the following: *report that* $y \in K(\epsilon)$; or *find a vector* $z \in \mathbb{Q}^n$ *such that* $\| z \|_{(\infty)} = 1$ *and* $z^T x < z^T y + \epsilon$ *for every* $x \in K(-\epsilon)$.

Weak (linear) optimization problem for a convex body K in $\mathbb{R}^n$: Given a vector $c \in \mathbb{Q}^n$ and a rational number $\epsilon > 0$, conclude with one of the following: *find a vector* $y \in \mathbb{Q}^n \cap K(\epsilon)$ *such that* $c^T x \le c^T y + \epsilon$ *for every* $x \in K(-\epsilon)$; or *report that* $K(-\epsilon) = \emptyset$.

Circumscribed convex body K: A positive rational number R is given explicitly such that $K \subset RB^n$.

Well-bounded convex body K: Positive rational numbers r, R are given explicitly such that $K \subset RB^n$ and K contains a ball of radius r.

Centered well-bounded convex body K: Positive rational numbers r, R and a vector $b \in \mathbb{Q}^n$ are given explicitly such that $b + rB^n \subset K$ and $K \subset RB^n$.

Weak membership oracle for a convex body K: Algorithm that solves the weak membership problem for K.

Weak separation oracle for K: Algorithm that solves the weak separation problem for K.

Weak (linear) optimization oracle for K: Algorithm that solves the weak (linear) optimization problem for K.

Preliminary version (August 10, 2017). To appear in the Handbook of Discrete and Computational Geometry,
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The three problems above are very closely related in the sense that when the classes of proper convex bodies are appropriately restricted to those that are circumscribed, well-bounded, or centered, and when input sizes are properly defined, an algorithm that solves any one of the problems in polynomial time can be used as a subroutine to solve the others in polynomial time also. The definition of input size involves the size of ϵ, the dimension of K, the given a priori information (size(r), size(R), and/or size(b)), and the input required by the oracle. The following theorem of [GLS88] contains a list of the precise relationships among the three basic oracles for proper convex bodies. The notation "$(\mathcal{A}; \mathit{prop}) \to_\pi \mathcal{B}$" indicates the existence of an (oracle-) polynomial-time algorithm that solves problem $\mathcal{B}$ for every proper convex body that is given by the oracle $\mathcal{A}$ and has all the properties specified in *prop*. (*prop*$= \emptyset$ means that the statement holds for general proper convex bodies.)

(Weak Membership; centered, well-bounded) $\to_\pi$ Weak Separation;
(Weak Membership; centered, well-bounded) $\to_\pi$ Weak Optimization;
(Weak Separation; $\emptyset$) $\to_\pi$ Weak Membership;
(Weak Separation; circumscribed) $\to_\pi$ Weak Optimization;
(Weak Optimization; $\emptyset$) $\to_\pi$ Weak Membership;
(Weak Optimization; $\emptyset$) $\to_\pi$ Weak Separation.

It should be emphasized that there are polynomial-time algorithms that, accepting as input a set P that is a proper $\mathcal{V}$-polytope, a proper $\mathcal{H}$-polytope, or a proper $\mathcal{S}$-zonotope, produce membership, separation, and optimization oracles for P, and also compute a lower bound on the inradius of P, an upper bound on its circumradius, and a "center" b_P for P. This implies that if an algorithm performs certain tasks for convex bodies given by some of the above (appropriately specified) oracles, then the same algorithm can also serve as a basis for procedures that perform these tasks for $\mathcal{V}$- or $\mathcal{H}$-polytopes and for $\mathcal{S}$-zonotopes. Hence the oracular framework, in addition to being applicable to convex bodies that are not polytopes, serves also to modularize the approach to algorithmic aspects of polytopes. On the other hand, there are lower bounds on the performance of approximate algorithms for the oracle model that do not carry over to the case of $\mathcal{V}$- or $\mathcal{H}$-polytopes or $\mathcal{S}$-zonotopes [BF87, BGK$^+$01]. However, if in polyhedral combinatorics certain tasks are known to be NP-hard then the above $\to_\pi$ implications can be used to show that certain other tasks are also hard. For instance, if the membership or the separation problem for the traveling salesman polytopes could be solved in polynomial time then optimization would also be tractable. Since the traveling salesman problem is known to be NP-hard, so are the membership or the separation problem for the traveling salesman polytopes. See [BV08] for approximations of convex bodies by sets for which a polynomial-time membership oracle is available.

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RELATED CHAPTERS

Chapter 7: Lattice points and lattice polytopes

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36.3 VOLUME COMPUTATIONS

It may be fair to say that the modern study of volume computations began with Kepler [Kep15] who derived the first *cubature formula* for measuring the capacities of wine barrels, and that it was the task of volume computation that motivated the general field of integration. The problem of computing or approximating volumes of convex bodies is certainly one of the basic problems in mathematics.

GLOSSARY

In the following, G is a subgroup of the group of all affine automorphisms of $\mathbb{R}^n$.

Dissection of an n-polytope P into n-polytopes $P_1, \ldots, P_k$: $P = P_1 \cup \cdots \cup P_k$, where the polytopes P_i have pairwise disjoint interiors.

Polytopes $P, Q \subset \mathbb{R}^n$ are ***G-equidissectable:*** For some k there exist dissections $P_1, \ldots, P_k$ of P and $Q_1, \ldots, Q_k$ of Q, and elements $g_1, \ldots, g_k$ of G, such that $P_i = g_i(Q_i)$ for all i.

Polytopes $P, Q \subset \mathbb{R}^n$ are ***G-equicomplementable:*** There are polytopes P_1, P_2 and Q_1, Q_2 such that P_2 is dissected into P and P_1, Q_2 is dissected into Q and Q_1, P_1 and Q_1 are G-equidissectable, and P_2 and Q_2 are G-equidissectable.

Decomposition of a set S: $S = S_1 \cup \cdots \cup S_k$, where the sets S_i are pairwise disjoint.

Sets S, T are ***G-equidecomposable:*** For some k there are decompositions $S_1, \ldots,$ S_k of S and $T_1, \ldots, T_k$ of T, and elements $g_1, \ldots, g_k$ of G, such that $S_i = g_i(T_i)$ for all i.

Valuation on a family $\mathcal{S}$ of subsets of $\mathbb{R}^n$: A functional $\varphi : \mathcal{S} \to \mathbb{R}$ with the property that $\varphi(S_1) + \varphi(S_2) = \varphi(S_1 \cup S_2) + \varphi(S_1 \cap S_2)$ whenever the sets $S_1, S_2, S_1 \cup S_2, S_1 \cap S_2 \in \mathcal{S}$.

G-invariant valuation φ: $\varphi(S) = \varphi(g(S))$ for all $S \in \mathcal{S}$ and $g \in G$.

Simple valuation φ: $\varphi(S) = 0$ whenever $S \in \mathcal{S}$ and S is contained in a hyperplane.

Monotone valuation φ: $\varphi(S_1) \leq \varphi(S_2)$ whenever $S_1, S_2 \in \mathcal{S}$ with $S_1 \subset S_2$.

Class $\mathcal{P}$ of $\mathcal{H}$-polytopes is ***near-simplicial:*** There is a nonnegative integer σ such that $\mathcal{P} = \bigcup_{n\in\mathbb{N}} \mathcal{P}_{\mathcal{H}}(n,\sigma)$, where $\mathcal{P}_{\mathcal{H}}(n,\sigma)$ is the family of all n-dimensional $\mathcal{H}$-polytopes P in $\mathbb{R}^n$ such that each facet of P has at most $n+1+\sigma$ vertices.

Class $\mathcal{P}$ of $\mathcal{V}$-polytopes is ***near-simple:*** There is a nonnegative integer τ such that $\mathcal{P} = \bigcup_{n\in\mathbb{N}} \mathcal{P}_{\mathcal{V}}(n,\tau)$, where $\mathcal{P}_{\mathcal{V}}(n,\tau)$ is the family of all n-dimensional $\mathcal{V}$-polytopes P in $\mathbb{R}^n$ such that each vertex of P is incident to at most $n+\tau$ edges.

Class $\mathcal{P}$ of $\mathcal{V}$-polytopes is ***near-parallelotopal:*** There is a nonnegative integer ζ such that $\mathcal{Z} = \bigcup_{n\in\mathbb{N}} \mathcal{Z}_{\mathcal{S}}(n,\zeta)$, where $\mathcal{Z}_{\mathcal{S}}(n,\zeta)$ is the family of all $\mathcal{S}$-zonotopes in $\mathbb{R}^n$ that are represented as the sum of at most $n+\zeta$ segments.

V: The functional that associates with a convex body K its volume.

$\mathcal{H}$-VOLUME: For a given $\mathcal{H}$-polytope P and a nonnegative rational ν, decide whether $V(P) \leq \nu$.

$\mathcal{V}$-VOLUME, $\mathcal{S}$-VOLUME: Similarly for $\mathcal{V}$-polytopes and $\mathcal{S}$-zonotopes.

$\boldsymbol{\lambda}$-APPROXIMATION for some functional ρ: Given a positive integer n and a well-bounded convex body K given by a weak separation oracle, determine a nonnegative rational μ such that

$$\rho(K) \leq (1+\lambda)\mu \quad \text{and} \quad \mu \leq (1+\lambda)\rho(K).$$

EXPECTED VOLUME COMPUTATION: Given a positive integer n, a centered well-bounded convex body K in $\mathbb{R}^n$ given by a weak membership oracle, and positive rationals β and ϵ, determine a positive rational random variable μ such that

$$\text{prob}\left\{\left|\frac{\mu}{V(K)} - 1\right| \leq \epsilon\right\} \geq 1-\beta.$$

36.3.1 CLASSICAL BACKGROUND, CHARACTERIZATIONS

The results in this subsection connect the subject matter of volume computation with related "classical" problems. In the following, G is a group of affine automorphisms of $\mathbb{R}^n$, as above, and D is the group of isometries.

(i) Two polytopes are G-equidissectable if and only if they are G-equicomplementable.

(ii) Two polytopes P and Q are G-equidissectable if and only if $\varphi(P) = \varphi(Q)$ for all G-invariant simple valuations on $\mathcal{P}^n$.

(iii) Two plane polygons are of equal area if and only if they are D-equidissectable.

Preliminary version (August 10, 2017). To appear in the Handbook of Discrete and Computational Geometry, J.E. Goodman, J. O'Rourke, and C. D. Tóth (editors), 3rd edition, CRC Press, Boca Raton, FL, 2017.

(iv) If one agrees that an a-by-b rectangle should have area ab, and also agrees that the area function should be a D-invariant simple valuation, it then follows from the preceding result that the area of any plane polygon P can be determined (at least in theory) by finding a rectangle R to which P is equidissectable. This provides a satisfyingly geometric theory of area that does not require any limiting considerations. The third problem of Hilbert [Hil00] asked, in effect, whether such a result extends to 3-polytopes. A negative answer was supplied by [Deh00], who showed that a regular tetrahedron and a cube of the same volume are not D-equidissectable.

(v) If P and Q are n-polytopes in $\mathbb{R}^n$, then for P and Q to be equidissectable under the group of all isometries of $\mathbb{R}^n$, it is necessary that $f^*(P) = f^*(Q)$ for each additive real function f such that $f(\pi) = 0$, where $f^*(P)$ is the so-called *Dehn invariant* of P associated with f. The condition is also sufficient for equidissectability when $n \leq 4$, but the matter of sufficiency is unsettled for $n \geq 5$.

(vi) Two plane polygons are of equal area if and only if they are D-equidecomposable.

(vii) In [Lac90], it was proved that any two plane polygons of equal area are equidecomposable under the group of translations. That paper also settled Tarski's old problem of "***squaring the circle***" by showing that a square and a circular disk of equal area are equidecomposable; there too, translations suffice. On the other hand, a disk and a square cannot be *scissors congruent*; i.e., there is no equidissection (with respect to rigid motions) into pieces that, roughly speaking, could be cut out with a pair of scissors.

(viii) If X and Y are bounded subsets of $\mathbb{R}^n$ (with $n \geq 3$), and each set has nonempty interior, then X and Y are D-equidecomposable. This is the famous ***Banach-Tarski paradox***.

(ix) Under the group of all volume-preserving affinities of $\mathbb{R}^n$, two n-polytopes are equidissectable if and only if they are of equal volume.

(x) If φ is a translation-invariant, nonnegative, simple valuation on $\mathcal{P}^n$ (resp. $\mathcal{K}^n$), then there exists a nonnegative real α such that $\varphi = \alpha V$.

(xi) A translation-invariant valuation on $\mathcal{P}^n$ that is homogeneous of degree n is a constant multiple of the volume.

(xii) A continuous, rigid-motion-invariant, simple valuation on $\mathcal{K}^n$ is a constant multiple of the volume.

(xiii) A nonnegative simple valuation on $\mathcal{P}^n$ (resp. $\mathcal{K}^n$) that is invariant under all volume-preserving linear maps of $\mathbb{R}^n$ is a constant multiple of the volume.

36.3.2 SOME VOLUME FORMULAS

Since simplex volumes can be computed so easily, the most natural approach to the problem of computing the volume of a polytope P is to produce a *triangulation* of P (see Chapter 16). Then compute the volumes of the individual simplices and add

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them up to find the volume of P. (This uses the fact that the volume is a simple valuation.) As a consequence, one sees that when the dimension n is fixed, the volume of $\mathcal{V}$-polytopes and of $\mathcal{H}$-polytopes can be computed in polynomial time.

Another equally natural method is to dissect P into pyramids with common apex over its facets. Since the volume of such a pyramid is just $1/n$ times the product of its height and the $(n-1)$-volume of its base, the volume can be computed recursively.

Another approach that has become a standard tool for many algorithmic questions in geometry is the *sweep-plane* technique. The general idea is to "sweep" a hyperplane through a polytope P, keeping track of the changes that occur when the hyperplane sweeps through a vertex. As applied to volume computation, this leads to the volume formula given below that does not explicitly involve triangulations, [BN83, Law91].

Suppose that $(n, m; A, b)$ is an irredundant $\mathcal{H}$-presentation of a simple polytope P. Let $b = (\beta_1, \ldots, \beta_m)^T$ and denote the row-vectors of A by $a_1^T, \ldots, a_m^T$. Let $M = \{1, \ldots, m\}$ and for each nonempty subset I of M, let A_I denote the submatrix of A formed by rows with indices in I and let b_I denote the corresponding right-hand side. Let $\mathcal{F}_0(P)$ denote the set of all vertices of the polytope $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$. For each $v \in \mathcal{F}_0(P)$, there is a set $I = I_v \subset M$ of cardinality n such that $A_I v = b_I$ and $A_{M \setminus I} v \leq b_{M \setminus I}$. Since P is assumed to be simple and its $\mathcal{H}$-presentation to be irredundant, the set I_v is unique.

Let $c \in \mathbb{R}^n$ be such that $\langle c, v_1 \rangle \neq \langle c, v_2 \rangle$ for any pair of vertices v_1, v_2 that form an edge of P. Then it turns out that

$$V(P) = \frac{1}{n!} \sum_{v \in \mathcal{F}_0(P)} \frac{\langle c, v \rangle^n}{\prod_{i=1}^n e_i^T A_{I_v}^{-1} c |\det(A_{I_v})|}.$$

The ingredients of this volume formula are those that are computed in the (dual) simplex algorithm. More precisely, $\langle c, v \rangle$ is just the value of the objective function at the current basic feasible solution v, $\det(A_{I_v})$ is the determinant of the current basis, and $A_{I_v}^{-1} c$ is the vector of reduced costs, i.e., the (generally infeasible) dual point that belongs to v.

For practical computations, this volume formula has to be combined with some vertex enumeration technique. Its closeness to the simplex algorithm suggests the use of a *reverse search* method [AF92], which is based on the simplex method with Bland's pivoting rule.

As it stands, the volume formula does not involve triangulation. However, when interpreted in a polar setting, it is seen to involve the faces of the simplicial polytope P° that is the polar of P. Accordingly, generalization to nonsimple polytopes involves polar triangulation. In fact, for general polytopes P, one may apply a "lexicographic rule" for moving from one basis to another, but this amounts to a particular triangulation of P°.

Another possibility for computing the volume of a polytope P is to study the *exponential integral* $\int_P e^{\langle c, x \rangle} dx$, where c is an arbitrary vector of $\mathbb{R}^n$; see [Bar93]. (Note that for $c = 0$, this integral just gives the volume of P.) Exponential integrals satisfy certain relations that make it possible to compute the integrals efficiently in some important cases. In particular, exponential sums can be used to obtain the tractability result for near-simple $\mathcal{V}$-polytopes stated in the next subsection.

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36.3.3 TRACTABILITY RESULTS

The volume of a polytope P can be computed in polynomial time in the following cases:

(i) when the dimension is fixed and P is a $\mathcal{V}$-polytope, an $\mathcal{H}$-polytope, or an $\mathcal{S}$-zonotope;

(ii) when the dimension is part of the input and P is a near-simple $\mathcal{V}$-polytope, a near-simplicial $\mathcal{H}$-polytope, or a near-parallelotopal $\mathcal{S}$-zonotope.

36.3.4 INTRACTABILITY RESULTS

(i) Since the output can have super polynomial size [Law91] there is no polynomial-space algorithm for exact computation of the volume of $\mathcal{H}$-polytopes.

(ii) $\mathcal{H}$-Volume is #P-hard even for the intersections of the unit cube with one rational halfspace.

(iii) $\mathcal{H}$-Volume is #P-hard in the strong sense. (This follows from the result of [BW92] that the problem of computing the number of linear extensions of a given partially ordered set $\mathcal{O} = (\{1, \ldots, n\}, \prec)$ is #P-complete, in conjunction with the fact that this number is equal to $n!V(P_{\mathcal{O}})$, where the set $P_{\mathcal{O}} = \{x = (\xi_1, \ldots, \xi_n)^T \in [0,1]^n \mid \xi_i \le \xi_j \iff i \prec j\}$ is the *order polytope* of $\mathcal{O}$ [Sta86].)

(iv) The problem of computing the volume of the convex hull of the regular $\mathcal{V}$-cross-polytope and an additional integer vector is #P-hard.

(v) $\mathcal{S}$-Volume is #P-hard.

36.3.5 DETERMINISTIC APPROXIMATION

(i) There exists an oracle-polynomial-time algorithm that, for any convex body K of $\mathbb{R}^n$ given by a weak optimization oracle, and for each $\epsilon > 0$, finds rationals μ_1 and μ_2 such that

$$\mu_1 \le V(K) \le \mu_2 \quad \text{and} \quad \mu_2 \le n!(1+\epsilon)^n \mu_1.$$

(ii) Suppose that

$$\lambda(n) < \left(\frac{n}{\log n}\right)^{n/2} - 1 \qquad \text{for all } n \in \mathbb{N}.$$

Then there exists no deterministic oracle-polynomial-time algorithm for λ-Approximation of the volume [BF87].

[DV13] give better approximations at higher computational cost. More precisely, it is shown that there is a deterministic algorithm, that accepts as input a well-bounded centrally symmetric convex body K given by a weak membership oracle and an $\epsilon \in (0,1]$, and computes a $(1+\epsilon)^n$-approximation of $V(K)$ in time $O(1/\epsilon)^{O(n)}$ and polynomial space. In view of the results of [BF87], this is optimal up to the constant in the exponent.

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36.3.6 RANDOMIZED ALGORITHMS

[DFK89] proved that there is a randomized algorithm for EXPECTED VOLUME COMPUTATION that runs in time that is oracle-polynomial in n, $1/\epsilon$, and $\log(1/\beta)$.

The first step is a rounding procedure, using an algorithmic version of John's theorem; see Section 36.5.4. For the second step, one may therefore assume that $B^n \subset K \subset (n+1)\sqrt{n}B^n$. Now, let

$$k = \left\lceil \frac{3}{2}(n+1)\log(n+1) \right\rceil, \qquad \text{and} \qquad K_i = K \cap \left(1+\frac{1}{n}\right)^i B^n \quad \text{for } i = 0,\ldots,k.$$

Then it suffices to estimate each ratio $V(K_i)/V(K_{i-1})$ up to a relative error of order $\epsilon/(n\log n)$ with error probability of order $\beta/(n\log n)$.

The main step of the algorithm of [DFK89] is based on a method for sampling nearly uniformly from within certain convex bodies K_i. It superimposes a chess-board grid of small cubes (say of edge length δ) on K_i, and performs a random walk over the set $\mathcal{C}_i$ of cubes in this grid that intersect a suitable parallel body $K_i + \alpha B^n$, where α is small. This walk is performed by moving through a facet with probability $1/f_{n-1}(C_n) = (2n)^{-1}$ if this move ends up in a cube of $\mathcal{C}_i$, and staying at the current cube if the move would lead outside of $\mathcal{C}_i$. The random walk gives a *Markov chain* that is irreducible (since the moves are connected), aperiodic, and hence ergodic. But this implies that there is a unique stationary distribution, the limit distribution of the chain, which is easily seen to be a *uniform distribution*. Thus after a sufficiently large (but polynomially bounded) number of steps, the current cube in the random walk can be used to sample nearly uniformly from $\mathcal{C}_i$. Having obtained such a uniformly sampled cube, one determines whether it belongs to $\mathcal{C}_{i-1}$ or to $\mathcal{C}_i \setminus \mathcal{C}_{i-1}$.

Now note that if ν_i is the number of cubes in $\mathcal{C}_i$, then the number $\mu_i = \nu_i/\nu_{i-1}$ is an estimate for the volume ratio $V(K_i)/V(K_{i-1})$. It is this number μ_i that can now be "randomly approximated" using the approximation constructed above of a uniform sampling over $\mathcal{C}_i$. In fact, a cube C that is reached after sufficiently many steps in the random walk will lie in $\mathcal{C}_{i-1}$ with probability approximately $1/\mu_i$; hence this probability can be approximated closely by repeated sampling.

This algorithm has been improved significantly by various authors. [LV06] achieved a bound where (except for logarithmic factors) n enters only to the fourth power—this is denoted by writing $O^*(n^4)$—which is currently the best running time in general. Recently, [CV16b] gave an $O^*(n^3)$ algorithm for convex bodies K containing B^n and being "mostly contained" in $O^*(\sqrt{n})B^n$, i.e., the expected value of $\|X\|^2$ for a uniform random point X of K is $O^*(n)$. Currently such a "well-rounding" can, however, only be achieved in time $O^*(n^4)$, [LV06].

The remarkable improvements over the original $O(n^{23})$ bound for the running time of [DFK89] rely on better initial rounding of the convex body and on improved sampling methods. In particular, the chain of bodies K_i (or equivalently their characteristic functions) were replaced by more general distributions f_i starting with one that is highly concentrated around a point close to an incenter of K (playing the role of K_0) and ending with a near uniform distribution. (In analogy to simulated annealing this process is called "cooling.") [CV16b] use Gaussian functions of the type

$$f_i(x) = \begin{cases} e^{-\frac{\|x\|^2}{2\sigma_i^2}} & \text{for } x \in K; \\ 0 & \text{otherwise,} \end{cases}$$

where the parameters σ_i are suitably adapted. The random walk then picks a random point q from a ball of suitable radius centered at the previous point x_j which is accepted as the next point x_{j+1} with probability $\min\{1, f_i(q)/f_i(x_j)\}$. (This is referred to as Gaussian sampling using the ball walk with Metropolis filter.) The ratio of the integrals of f_{i+1} and f_i is finally estimated by

$$\frac{1}{k} \sum_{j=1}^{k} \frac{f_{i+1}(x_j)}{f_i(x_j)}.$$

With this kind of running time, randomized volume computations are getting close to being practical; see [CV16a] for some corresponding study.

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RELATED CHAPTERS

Chapter 7: Lattice points and lattice polytopes
Chapter 15: Basic properties of convex polytopes
Chapter 16: Subdivisions and triangulations of polytopes
Chapter 44: Randomization and derandomization

Preliminary version (August 10, 2017). To appear in the Handbook of Discrete and Computational Geometry, J.E. Goodman, J. O'Rourke, and C. D. Tóth (editors), 3rd edition, CRC Press, Boca Raton, FL, 2017.

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36.4 MIXED VOLUMES

The study of mixed volumes, the *Brunn-Minkowski theory*, forms the backbone of classical convexity theory. It is also useful for applications in other areas, including combinatorics and algebraic geometry. A relationship to solving systems of polynomial equations is described at the end of this section.

Preliminary version (August 10, 2017). To appear in the Handbook of Discrete and Computational Geometry, J.E. Goodman, J. O'Rourke, and C. D. Tóth (editors), 3rd edition, CRC Press, Boca Raton, FL, 2017.

GLOSSARY

Mixed volume: Let $K_1, \ldots, K_s$ be convex bodies in $\mathbb{R}^n$, and let $\xi_1, \ldots, \xi_s$ be nonnegative reals. Then the function $V\left(\sum_{i=1}^{s} \xi_i K_i\right)$ is a homogeneous polynomial of degree n in the variables $\xi_1, \ldots, \xi_s$, and can be written in the form

$$V\left(\sum_{i=1}^{s} \xi_i K_i\right) = \sum_{i_1=1}^{s} \sum_{i_2=1}^{s} \cdots \sum_{i_n=1}^{s} \xi_{i_1} \xi_{i_2} \cdots \xi_{i_n} V(K_{i_1}, K_{i_2}, \ldots, K_{i_n}),$$

where the coefficients $V(K_{i_1}, K_{i_2}, \ldots, K_{i_n})$ are invariant under permutations of their argument. The coefficient $V(K_{i_1}, K_{i_2}, \ldots, K_{i_n})$ is called the mixed volume of the convex bodies $K_{i_1}, K_{i_2}, \ldots, K_{i_n}$.

36.4.1 MAIN RESULTS

Mixed volumes are nonnegative, monotone, multilinear, and continuous valuations. They generalize the ordinary volume in that $V(K) = V(\overbrace{K, \ldots, K}^{n})$. If A is an affine transformation, then $V(A(K_1), \ldots, A(K_n)) = |\det(A)| V(K_1, \ldots, K_n)$.

Among the most famous inequalities in convexity theory is the ***Aleksandrov-Fenchel inequality***,

$$V(K_1, K_2, K_3, \ldots, K_n)^2 \geq V(K_1, K_1, K_3, \ldots, K_n)\ V(K_2, K_2, K_3, \ldots, K_n),$$

and its consequence, the ***Brunn-Minkowski theorem***, which asserts that for each $\lambda \in [0,1]$,

$$V^{\frac{1}{n}}((1-\lambda)K_0 + \lambda K_1) \geq (1-\lambda)V^{\frac{1}{n}}(K_0) + \lambda V^{\frac{1}{n}}(K_1).$$

OPEN PROBLEM 36.4.1

Provide a useful geometric characterization of the sequences $(K_1, \ldots, K_n)$ for which equality holds in the Aleksandrov-Fenchel inequality.

36.4.2 TRACTABILITY RESULTS

When n is fixed, there is a polynomial-time algorithm whereby, given s ($\mathcal{V}$- or $\mathcal{H}$-) polytopes $P_1, \ldots, P_s$ in $\mathbb{R}^n$, all the mixed volumes $V(P_{i_1}, \ldots, P_{i_n})$ can be computed.

When the dimension is part of the input, it follows at least that mixed volume computation is not harder than volume computation. In fact, computation (for $\mathcal{V}$-polytopes or $\mathcal{S}$-zonotopes) or approximation (for $\mathcal{H}$-polytopes) of any single mixed volume is #P-easy.

There is a polynomial-time algorithm for approximating the mixed volume of n convex bodies up to a simply exponential error, [Gur09].

36.4.3 INTRACTABILITY RESULTS

Since mixed volumes generalize the ordinary volume, it is clear that mixed volume computation cannot be easier, in general, than volume computation. In addition,

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there are hardness results for mixed volumes that do not trivially depend on the hardness of volume computations. One such result is described next.

As the term is used here, a *box* is a rectangular parallelotope with axis-aligned edges. Since the vector sum of boxes $V(Z_1, \ldots, Z_n)$ is again a box, the volume of the sum is easy to compute. Nevertheless, computation of the mixed volume $V(Z_1, \ldots, Z_n)$ is hard; see [DGH98]. This is in interesting contrast to the fact that the volume of a sum of segments (a zonotope) is hard to compute even though each of the mixed volumes can be computed in polynomial time.

36.4.4 RANDOMIZED ALGORITHMS

Since the mixed volumes of convex bodies $K_1, \ldots, K_s$ are coefficients of the polynomial $\varphi(\xi_1, \ldots, \xi_s) = V(\sum_{i=1}^{s} \xi_i K_i)$, it seems natural to estimate these coefficients by combining an interpolation method with a randomized volume algorithm. However, there are significant obstacles to this approach, even for the case of two bodies. First, for a general polynomial φ there is *no* way of obtaining *relative* estimates of its coefficients from *relative* estimates of the values of φ. This can be overcome in the case of two bodies by using the special structure of the polynomial $p(x) = V(K_1 + xK_2)$. However, even then the absolute values of the entries of the "inversion" that is used to express the coefficients of the polynomial in terms of its approximate values are not bounded by a polynomial, while the randomized volume approximation algorithm is polynomial only in $\frac{1}{\tau}$ but not in size(τ).

Suppose that $\psi : \mathbb{N} \to \mathbb{N}$ is nondecreasing with

$$\psi(n) \le n \qquad \text{and} \qquad \psi(n) \log \psi(n) = o(\log n).$$

Then there is a polynomial-time algorithm for the problem whose instance consists of $n, s \in \mathbb{N}$, $m_1, \ldots, m_s \in \mathbb{N}$ with $m_1 + m_2 + \cdots + m_s = n$ and $m_1 \ge n - \psi(n)$, of well-bounded convex bodies $K_1, \ldots, K_s$ of $\mathbb{R}^n$ given by a weak membership oracle, and of positive rational numbers ϵ and β, and whose output is a random variable $\hat{V}_{m_1,\ldots,m_s} \in \mathbb{Q}$ such that

$$\text{prob}\left\{ \frac{|\hat{V}_{m_1,\ldots,m_s} - V_{m_1,\ldots,m_s}|}{V_{m_1,\ldots,m_s}} \ge \epsilon \right\} \le \beta,$$

where

$$V_{m_1,\ldots,m_s} = V(\overbrace{K_1, \ldots, K_1}^{m_1}, \ldots, \overbrace{K_s, \ldots, K_s}^{m_s}).$$

Note that the hypotheses above require that m_1 is close to n, and hence that the remaining m_i's are relatively small. A special feature of an interpolation method as used for the proof of this result is that in order to compute *a specific* coefficient of the polynomial under consideration, it computes essentially *all previous* coefficients. Since there can be a polynomial-time algorithm for computing *all such* mixed volumes only if $\psi(n) \le \log n$, the above result is essentially best-possible for any interpolation method.

OPEN PROBLEM 36.4.2 [DGH98]

Is there a polynomial-time randomized algorithm that, for any $n, s \in \mathbb{N}$, $m_1, \ldots, m_s \in \mathbb{N}$ with $m_1 + m_2 + \cdots + m_s = n$, well-bounded convex bodies $K_1, \ldots, K_s$ in $\mathbb{R}^n$ given *by a weak membership oracle, and positive rationals ϵ and β, computes a random variable $\hat{V}_{m_1,\ldots,m_s} \in \mathbb{Q}$ such that* $\text{prob}\{|\hat{V}_{m_1,\ldots,m_s} - V_{m_1,\ldots,m_s}|/V_{m_1,\ldots,m_s} \geq \epsilon\} \leq \beta$*?*

Even the case $s = n$, $m_1 = \cdots = m_s = 1$ is open in general. See, however, [Bar97] for some partial results and [Mal16] for performance bounds in terms of geometric invariants.

AN APPLICATION

Let $S_1, S_2, \ldots, S_n$ be subsets of $\mathbb{Z}^n$, and consider a system $F = (f_1, \ldots, f_n)$ of Laurent polynomials in n variables, such that the exponents of the monomials in f_i are in S_i for all $i = 1, \ldots, n$. For $i = 1, \ldots, n$, let

$$f_i(x) = \sum_{q \in S_i} c_q^{(i)} x^q,$$

where $f_i \in \mathbb{C}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$, and x^q is an abbreviation for the monomial $x_1^{q_1} \cdots x_n^{q_n}$; $x = (x_1, \ldots, x_n)$ is the vector of indeterminates and $q = (q_1, \ldots, q_n)$ the vector of exponents. Further, let $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

Now, if the coefficients $c_q^{(i)}$ $(q \in S_i)$ are chosen "generically," then the number $L(F)$ of distinct common roots of the system F in $(\mathbb{C}^*)^n$ depends only on the ***Newton polytopes*** $P_i = \text{conv}(S_i)$ of the polynomials. More precisely,

$$L(F) = n! \cdot V(P_1, P_2, \ldots, P_n).$$

In general, $L(F) \leq n! \cdot V(P_1, P_2, \ldots, P_n)$. These connections can be utilized to develop a numerical continuation method for computing the isolated solutions of sparse polynomial systems; see [CLO98, DE05, Stu02].

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RELATED CHAPTERS

Chapter 15: Basic properties of convex polytopes
Chapter 44: Randomization and derandomization

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36.5 CONTAINMENT PROBLEMS

Typically, containment problems involve two fixed sequences, Γ and Ω, that are given as follows: for each $n \in \mathbb{N}$, let $\mathcal{C}_n$ denote a family of closed convex subsets of $\mathbb{R}^n$, and let $\omega_n : \mathcal{C}_n \longrightarrow \mathbb{R}$ be a functional that is nonnegative and is monotone with respect to inclusion. Then $\Gamma = (\mathcal{C}_n)_{n\in\mathbb{N}}$ and $\Omega = (\omega_n)_{n\in\mathbb{N}}$.

GLOSSARY

(Γ, Ω)-Inbody: Accepts as input a positive integer n, a body K in $\mathbb{R}^n$ that is given by an oracle or is an $\mathcal{H}$-polytope, a $\mathcal{V}$-polytope, or an $\mathcal{S}$-zonotope, and a positive rational λ. It answers the question of whether there is a $C \in \mathcal{C}_n$ such that $C \subset K$ and $\omega_n(C) \geq \lambda$.

(Γ, Ω)-Circumbody is defined similarly for $C \supset K$.

j-simplex S ***bound to a polytope*** P: Each vertex of S is a vertex of P.

Largest j-simplex in a given polytope: One of maximum j-measure.

36.5.1 THE GENERAL CONTAINMENT PROBLEM

The general containment problem deals with the question of computing, approximating, or measuring extremal bodies of a given class that are contained in or contain a given convex body. The broad survey [GK94a] can (yet being somewhat older) still be used as a starting point for getting acquainted with the subject and its many applications. Here we want to minimize the overlap with other chapters, and restrict the exposition to some selected examples. In particular we do not focus on *coresets* (see AHV07), as they are covered in Chapter 48.

Preliminary version (August 10, 2017). To appear in the Handbook of Discrete and Computational Geometry, J.E. Goodman, J. O'Rourke, and C. D. Tóth (editors), 3rd edition, CRC Press, Boca Raton, FL, 2017.

The emphasis here will be on containment under homothety and affinity. For some results on containment under similarity see [GK94a, Sec. 7]; see also [Fir15] for numerical computations in the case of $\mathcal{V}$-polytopes in $\mathcal{H}$-polytopes.

36.5.2 OPTIMAL CONTAINMENT UNDER HOMOTHETY

The results on (Γ, Ω)-INBODY and (Γ, Ω)-CIRCUMBODY are summarized below for the case in which each C_n is a fixed polytope,

$$\mathcal{C}_n = \{g(C_n) \mid g \text{ is a homothety}\},$$

and

$$\omega_n(g(C_n)) = \rho, \quad \text{when } g(C_n) = a + \rho C_n \text{ for some } a \in \mathbb{R}^n \text{ and } \rho \geq 0.$$

As an abbreviation, these specific problems are denoted by $\mathcal{E}^{\text{Hom}}$-INBODY and $\mathcal{E}^{\text{Hom}}$-CIRCUMBODY, respectively, where $\mathcal{E} = (C_n)_{n\in\mathbb{N}}$ and a subscript ($\mathcal{V}$ or $\mathcal{H}$) is used to indicate the manner in which each C_n is presented.

There are polynomial-time algorithms for the following problems:

$\mathcal{E}_{\mathcal{V}}^{\text{Hom}}$-INBODY for $\mathcal{V}$-polytopes P;	$\mathcal{E}_{\mathcal{V}}^{\text{Hom}}$-CIRCUMBODY for $\mathcal{V}$-polytopes P;
$\mathcal{E}_{\mathcal{V}}^{\text{Hom}}$-INBODY for $\mathcal{H}$-polytopes P;	$\mathcal{E}_{\mathcal{H}}^{\text{Hom}}$-CIRCUMBODY for $\mathcal{V}$-polytopes P;
$\mathcal{E}_{\mathcal{H}}^{\text{Hom}}$-INBODY for $\mathcal{H}$-polytopes P;	$\mathcal{E}_{\mathcal{H}}^{\text{Hom}}$-CIRCUMBODY for $\mathcal{H}$-polytopes P.

These positive results are best possible in the sense that the cases not listed above contain instances of NP-hard problems. In fact, the problem $\mathcal{E}_{\mathcal{H}}^{\text{Hom}}$-INBODY is coNP-complete even when C_n is the standard unit $\mathcal{H}$-cube while P is restricted to the class of all affinely regular $\mathcal{V}$-cross-polytopes centered at the origin. The problem $\mathcal{E}_{\mathcal{V}}^{\text{Hom}}$-CIRCUMBODY is coNP-complete even when C_n is the standard $\mathcal{V}$-cross-polytope while P is restricted to the class of all $\mathcal{H}$-parallelotopes centered at the origin.

There are some results for bodies that are more general than polytopes. Suppose that for each $n \in \mathbb{N}$, C_n is a centrally symmetric body in $\mathbb{R}^n$, and that there exists a number μ_n whose size is bounded by a polynomial in n and an n-dimensional $\mathcal{S}$-parallelotope Z that is strictly inscribed in $\mu_n C_n$ (i.e., the intersection of Z with the boundary of $\mu_n C_n$ consists of the vertex set of Z), the size of the presentation being bounded by a polynomial in n. Then with $\mathcal{E} = (C_n)_{n\in\mathbb{N}}$, (an appropriate variant of) the problem $\mathcal{E}^{\text{Hom}}$-CIRCUMBODY is NP-hard for the classes of all centrally symmetric $(n-1)$-dimensional $\mathcal{H}$-polytopes in $\mathbb{R}^n$. With the aid of polarity, similar results for $\mathcal{E}^{\text{Hom}}$-INBODY can be obtained. A particularly important special case is that of *norm maximization*, i.e., maximizing the Euclidean (or some other norm) over a polytope.

Besides the obvious examples of unit balls of norms and polytopes, containment problems have also been studied for *spectrahedra* which arise in convex algebraic geometry [HN12] and generalize the class of polyhedra. [KTT13] extend known complexity results to spectrahedra. For instance, they show that deciding whether a given $\mathcal{V}$-polytope is contained in a given spectrahedron can be decided in polynomial time, while deciding whether a spectrahedron is contained in a $\mathcal{V}$-polytope is co-NP-hard. As spectrahedra arise as feasible regions of semidefinite programs they also give semidefinite conditions to certify containment.

36.5.3 OPTIMAL CONTAINMENT UNDER AFFINITY: SIMPLICES

This section focuses on the problem of finding a largest j-dimensional simplex in a given n-dimensional polytope, where *largest* means of maximum j-measure.

When an n-polytope P has m vertices, it contains at most $\binom{m}{j+1}$ bound j-simplices. There is always a largest j-simplex that is bound, and hence there is a finite algorithm for finding a largest j-simplex contained in P.

Each largest j-simplex in P contains at least two vertices of P. However, there are polytopes P of arbitrarily large dimension, with an arbitrarily large number of vertices, such that some of the largest n-simplices in P have only two vertices in the vertex-set of P. Hence for $j \geq 2$ it is not clear whether there is a finite algorithm for producing a useful presentation of *all* the largest j-simplices in a given n-polytope.

The problem of finding a largest j-simplex in a $\mathcal{V}$- or $\mathcal{H}$-polytope can be solved in polynomial time when the dimension n of the polytope is fixed. Further, for fixed j, the volumes of all bound j-simplices in a given $\mathcal{V}$-polytope can be computed in polynomial time (even for varying n).

Suppose that the functions $\psi : \mathbb{N} \to \mathbb{N}$ and $\gamma : \mathbb{N} \to \mathbb{N}$ are both of order $\Omega(n^{1/k})$ for some $k \in \mathbb{N}$, and that $1 \leq \gamma(n) \leq n$ for each $n \in \mathbb{N}$. Then the following problem is NP-complete: Given $n, \lambda \in \mathbb{N}$, and the vertex set V of an n-dimensional $\mathcal{V}$-polytope $P \subset \mathbb{R}^n$ with $|V| \leq n + \psi(n)$, and given $j = \gamma(n)$, decide whether P contains a j-simplex S such that $(j!)^2 \text{vol}(S)^2 \geq \lambda$. Note that the conditions for γ are satisfied when $\gamma(n) = \max\{1, n - \mu\}$ for a nonnegative integer constant μ, and also when $\gamma(n) = \max\{1, \lfloor \mu n \rfloor\}$ for a fixed rational μ with $0 < \mu \leq 1$.

A similar hardness result holds for $\mathcal{H}$-polytopes. There the question is the same, but the growth condition on the function γ is that $1 \leq \gamma(n) \leq n$ and that there exists a function $f : \mathbb{N} \to \mathbb{N}$, bounded by a polynomial in n, such that for each $n \in \mathbb{N}$, $f(n) - \gamma(f(n)) = n$. Note that such an f exists when the function γ is constant, and also when $\gamma(n) = \lfloor \mu n \rfloor$ for fixed rational μ with $0 < \mu < 1$.

Under the assumption that the function $\gamma : \mathbb{N} \to \mathbb{N}$ is such that $\gamma(n) = \Omega(n^{1/k})$ for some fixed $k > 0$, [Pac02] gives a unifying approach for proving the NP-hardness of the problems, for which an instance consists of $n \in \mathbb{N}$, an $\mathcal{H}$-polytope or $\mathcal{V}$-polytope P in $\mathbb{R}^n$, and a rational $\lambda > 0$, and the question is whether there exists an $\gamma(n)$-simplex $S \subset P$ with $\text{V}^2(S) \geq \lambda$.

The following conjecture is, however, still open.

CONJECTURE 36.5.1 [GKL95]

For each function $\gamma : \mathbb{N} \to \mathbb{N}$ with $1 \leq \gamma(n) \leq n$, the problem of finding a largest j-simplex in a given n-dimensional $\mathcal{H}$-parallelotope P is NP*-hard.*

The "dual" problem of finding smallest simplices containing a given polytope P seems even harder, since the relationship between a smallest such simplex and the faces of P is much weaker. However, [Pac02] gives the following hardness results for *j-simplicial cylinders* C which are cylinders of the form $C = S + L$, where S is a j-simplex with $0 \in \text{aff}(S)$ and $L = \text{aff}(S)^\perp$. Let again the function $\gamma : \mathbb{N} \to \mathbb{N}$ be such that $\gamma(n) = \Omega(n^{1/k})$ for some fixed $k > 0$. Then it is NP-hard to decide whether for given $n \in \mathbb{N}$, positive rational λ and an $\mathcal{H}$-polytope or $\mathcal{V}$-polytope P in $\mathbb{R}^n$ there exists an $\gamma(n)$-simplicial cylinder C with $P \subset C$ and $\text{V}^2(S) \leq \lambda$. Note that the condition on γ particularly includes the case $j = n$, i.e., that of an ordinary n-dimensional simplex.

Preliminary version (August 10, 2017). To appear in the Handbook of Discrete and Computational Geometry, J.E. Goodman, J. O'Rourke, and C. D. Tóth (editors), 3rd edition, CRC Press, Boca Raton, FL, 2017.

These results have been complemented in various ways. [Kou06] shows that the decision problem related to finding a largest j-simplex in a given $\mathcal{V}$-polytope is $W[1]$-complete with respect to the parameter j. See e.g., [FG06] for background information on parametrized complexity. Also deterministic approximation and nonapproximability results have been given [BGK00a, Pac04, Kou06, DEFM15, Nik15] showing in particular that the problem of finding a largest j-simplex in a given $\mathcal{V}$-polytope can on the one hand be approximated in polynomial time up to a factor of $e^{j/2+o(j)}$; yet, on the other hand, there is a constant $\mu > 1$ such that it is NP-hard to approximate within a factor of μ^j.

APPLICATIONS

Applications of this problem and its relatives include that of finding submatrices of maximum determinant, and, in particular, the Hadamard determinant problem, of finding optimal weighing designs, and bounding the growth of pivots in Gaussian elimination with complete pivoting; see [GK94a], [Nik15].

36.5.4 OPTIMAL CONTAINMENT UNDER AFFINITY: ELLIPSOIDS

For an arbitrary proper body K in $\mathbb{R}^n$, there is a unique ellipsoid E_0 of maximum volume contained in K, and it is concentric with the unique ellipsoid E of minimum volume containing K. If a is the common center, then $K \subset a + n(E_0 - a)$, where the factor n can be replaced by $\sqrt{n}$ when K is centrally symmetric. E is called the ***Löwner-John ellipsoid*** of K, and it plays an important role in the algorithmic theory of convex bodies.

Algorithmic approximations of the Löwner-John ellipsoid can be obtained by use of the ellipsoid method [GLS88]: There exists an oracle-polynomial-time algorithm that, for any well-bounded body K of $\mathbb{R}^n$ given by a weak separation oracle, finds a point a and a linear transformation A such that

$$a + A(B^n) \subset K \subset a + (n+1)\sqrt{n}A(B^n).$$

Further, the dilatation factor $(n+1)\sqrt{n}$ can be replaced by $\sqrt{n(n+1)}$ when K is symmetric, by $(n+1)$ when K is an $\mathcal{H}$-polytope, and by $\sqrt{n+1}$ when K is a symmetric $\mathcal{H}$-polytope.

[TKE88] and [KT93] give polynomial-time algorithms for approximating the ellipsoid of maximum volume E_0 that is contained in a given $\mathcal{H}$-polytope. For each rational $\gamma < 1$, there exists a polynomial-time algorithm that, given $n, m \in \mathbb{N}$ and $a_1, \dots, a_m \in \mathbb{Q}^n$, computes an ellipsoid $E = a + A(B^n)$ such that

$$E \subset P = \{x \in \mathbb{R}^n \mid \langle a_i x \rangle \le 1, \text{ for } i = 1, \dots, m\} \quad \text{and} \quad \frac{V(E)}{V(E_0)} \ge \gamma.$$

The running time of the algorithm is

$$O\left(m^{3.5} \log\bigl(mR/(r\log(1/\gamma))\bigr) \log\bigl(nR/(r\log(1/\gamma))\bigr)\right),$$

where the numbers r and R are, respectively, a lower bound on the inradius of P and an upper bound on its circumradius.

It is not known whether a similar result holds for $\mathcal{V}$-polytopes.

Preliminary version (August 10, 2017). To appear in the Handbook of Discrete and Computational Geometry, J.E. Goodman, J. O'Rourke, and C. D. Tóth (editors), 3rd edition, CRC Press, Boca Raton, FL, 2017.

As shown in [TKE88], an approximation of E_0 of the kind given above leads to the following inclusion:

$$a + A(B^n) \subset K \subset a + \frac{n(1+3\sqrt{1-\gamma})}{\gamma} A(B^n).$$

Other important ellipsoids related to convex bodies K are the *M-ellipsoids*; see e.g., [Pis89]. Intuitively an M-ellipsoid E is an ellipsoid with small covering number with respect to K. More precisely, for two sets A, B let $N(A;B)$ denote the number of translates of B needed to cover A. Then every convex body K in $\mathbb{R}^n$ admits an ellipsoid E for which $N(K;E)N(E;K)$ is bounded by $2^{O(n)}$, [Mil86]; this is best possible up to the constant in the exponent. [DV13] give a deterministic algorithm for computing an M-ellipsoid for a well-bounded convex body K in $\mathbb{R}^n$ given by a weak membership oracle in time $2^{O(n)}$. This is best possible up to the constant in the exponent.

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RELATED CHAPTERS

Chapter 48: Coresets and sketches

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36.6 RADII

The diameter, width, circumradius, and inradius of a convex body are classical functionals that play an important role in convexity theory and in many applications. For other applications, generalizations have been introduced. Here we focus on the case that the underlying space is a ***Minkowski space*** (i.e., a finite-dimensional normed space) $\mathbb{M} = (\mathbb{R}^n, \| \quad \|)$. Let B denote its unit ball, j a positive integer, and K a convex body. For some generalizations to the case of non symmetric "unit balls" see [BK13, BK14].

GLOSSARY

Outer j-radius $R_j(K)$ of K: Infimum of the positive numbers ρ such that the space contains an $(n-j)$-flat F for which $K \subset F + \rho B$.

j-ball of radius ρ: Set of the form $(q + \rho B) \cap F = \{x \in F \mid \|x - q\| \leq \rho\}$ for some j-flat F in $\mathbb{R}^n$ and point $q \in F$.

Inner j-radius $r_j(K)$ of K: Maximum of the radii of the j-balls contained in K.

Diameter of K: $2r_1(K)$.

Width of K: $2R_1(K)$.

Inradius of K: $r_n(K)$.

Circumradius of K: $R_n(K)$.

Note that for a convex body K that is symmetric about the origin $r_1(K)$ coincides with the norm-maximum $\max_{x \in K} \|x\|$ over K.

For the case of variable dimension (i.e., the dimension is part of the input), Tables 36.6.1, 36.6.2, and 36.6.3 provide a rapid indication of the main complexity results for the most important radii: r_1, R_1, r_n, and R_n; and for the three most important ℓ_p spaces: $\mathbb{R}^n_2$, $\mathbb{R}^n_1$, and $\mathbb{R}^n_\infty$. The designations P, NPC, and NPH indicate respectively polynomial-time computability, NP-completeness, and NP-hardness. The tables provide only a rough indication of results. They are imprecise in the following respects: (i) the diameter and width are actually equal to $2r_1$ and $2R_1$ respectively; (ii) the results for $\mathbb{R}^n_2$ involve the square of the radius rather than the radius itself; (iii) some of the P entries are based on polynomial-time approximability rather than polynomial-time computability; (iv) the designations NPC and NPH do not refer to computability per se, but to the appropriately related decision problems involving the establishment of lower or upper bounds for the radii in question.

TABLE 36.6.1 Complexity of radii in $\mathbb{R}^n_2$.

Polytope functional		$\mathcal{H}$-polytopes general	$\mathcal{H}$-polytopes symmetric	$\mathcal{V}$-polytopes general	$\mathcal{V}$-polytopes symmetric
Diameter	r_1^2	NPC	NPC	P	P
Inradius	r_n^2	P	P	NPH	NPC
Width	R_1^2	NPC	P	NPC	NPC
Circumradius	R_n^2	NPC	NPC	P	P

TABLE 36.6.2 Complexity of radii in $\mathbb{R}^n_1$.

Polytope functional		$\mathcal{H}$-polytopes general	$\mathcal{H}$-polytopes symmetric	$\mathcal{V}$-polytopes general	$\mathcal{V}$-polytopes symmetric
Diameter	r_1	NPC	NPC	P	P
Inradius	r_n	P	P	P	P
Width	R_1	P	P	P	P
Circumradius	R_n	NPC	NPC	P	P

TABLE 36.6.3 Complexity of radii in $\mathbb{R}^n_\infty$.

Polytope functional		$\mathcal{H}$-polytopes general	$\mathcal{H}$-polytopes symmetric	$\mathcal{V}$-polytopes general	$\mathcal{V}$-polytopes symmetric
Diameter	r_1	P	P	P	P
Inradius	r_n	P	P	NPC	NPC
Width	R_1	NPC	P	NPC	NPC
Circumradius	R_n	P	P	P	P

Preliminary version (August 10, 2017). To appear in the Handbook of Discrete and Computational Geometry, J.E. Goodman, J. O'Rourke, and C. D. Tóth (editors), 3rd edition, CRC Press, Boca Raton, FL, 2017.

For inapproximability results in the Turing machine model see [BGK00b] and [Bri02]; for sharp bounds on the approximation error of polynomial-time algorithms in the oracle model see [BGK$^+$01]. In view of the results in Section 36.3 on volume computation where there is a sharp contrast between the performance of deterministic and randomized algorithms it may be worth noting that, generally, for radii *randomization does not help!* This means that the same limitations on the error of polynomial-time approximations that apply for deterministic algorithms also apply for randomized algorithms, [BGK$^+$01].

Parametrized complexity (see e.g., [FG06]) has been used in [KKW15] to analyze more sharply how the hardness of norm maximization and radius computation depends on the dimension n. In particular, it is show that for $p = 1$ the problem of maximizing the p-th power of the ℓ_p-norm over $\mathcal{H}$-polytopes is fixed parameter tractable but that for each $p \in \mathbb{N} \setminus \{1\}$ norm maximization is $W[1]$-hard.

APPLICATIONS

Applications of radii include conditioning in global optimization, sensitivity analysis of linear programs, orthogonal minimax regression, computer graphics and computer vision, chromosome classification, set separation, and design of membranes and sieves; see [GK93b].

FURTHER READING

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36.7 CONSTRAINED CLUSTERING

Clustering has long been known as an instrumental part of data analytics. In increasingly many applications additional constraints are imposed on the clusters, for instance, bounding the cluster sizes. As has been observed in numerous fields, good clusterings are closely related to geometric diagrams, i.e., various generalizations of Voronoi diagrams. Applications of constrained clustering include the representation of polycrystals (grain maps) in material science, [ABG+15], farmland consolidation, [BBG14], facility and robot network design [Cor10], and electoral district design, [BGK17].

As Voronoi diagrams are covered in great detail in Chapter 27, we will concentrate in the following on some geometric aspects of the relation between diagrams and constrained clusterings.

GLOSSARY

Instance of a constrained clustering problem: $(k, m, n, X, \omega, \kappa^-, \kappa^+)$ (***weakly balanced***) or $(k, m, n, X, \omega, \kappa)$ (***strongly balanced***), where $k, m, n \in \mathbb{N}$, $X = \{x_1, \ldots, x_m\} \subset \mathbb{R}^n$, $\omega : X \to (0, \infty)$, $\kappa, \kappa^-, \kappa^+ : \{1, \ldots, k\} \to (0, \infty)$, such that $\kappa^- \le \kappa^+$ and $\sum_{i=1}^k \kappa^-(i) \le \sum_{j=1}^m \omega(x_j) \le \sum_{i=1}^k \kappa^+(i)$ in the weakly balanced case and $\sum_{j=1}^m \omega(x_j) = \sum_{i=1}^k \kappa(i)$ in the strongly balanced case.

Of course, n is again the dimension of space, m is the number of points of the given set X in $\mathbb{R}^n$, and $\omega(x)$ specifies the weight of each point $x \in X$. The set X has to be split (in a fractional or integer fashion that will be specified explicitly next) into clusters $C_1, \ldots, C_k$ whose total weights lie in the given intervals $[\kappa^-(i), \kappa^+(i)]$ in the weakly balanced case or actually coincide with the prescibed weight $\kappa(i)$ in the strongly balanced case, respectively.

Balanced clustering $\mathcal{C}$ for an instance $(k, m, n, X, \omega, \kappa^-, \kappa^+)$, or $(k, m, n, X, \omega, \kappa)$: $\mathcal{C} = \{C_1, \ldots, C_k\}$ with $C_i = (\xi_{i,1}, \ldots, \xi_{i,m}) \in [0,1]^m$ for $i \in \{1, \ldots, k\}$, such that $\sum_{i=1}^k \xi_{i,j} = 1$ for $j \in \{1, \ldots, m\}$ and $\kappa^-(i) \le \sum_{j=1}^m \omega(x_j)\xi_{i,j} \le \kappa^+(i)$ for $i \in \{1, \ldots, k\}$ in the weakly balanced case and $\sum_{j=1}^m \omega(x_j)\xi_{i,j} = \kappa(i)$ in the strongly balanced case. C_i is the ith ***cluster***. Note that $\xi_{i,j}$ is the fraction of x_j assigned to the cluster C_i.

Integer clustering $\mathcal{C}$***:*** $C_i \in \{0,1\}^m$ for $i \in \{1, \ldots, k\}$.

Constrained clustering problem: Given an instance of a constrained clustering problem, find a balanced (integer) clustering $\mathcal{C}$ which optimizes some given objective function (examples of which will be given later).

Gravity vector of a given clustering $\mathcal{C}$: $\mathbf{g}(\mathcal{C}) = (g(C_1)^T, \ldots, g(C_k)^T)^T$, where $g(C_i) = \left(\sum_{j=1}^m \xi_{i,j}\omega(x_j)x_j\right) / \left(\sum_{j=1}^m \xi_{i,j}\omega(x_j)\right)$ is the ***center of gravity*** of C_i for $i \in \{1, \ldots, k\}$.

Gravity body for a given instance $(k, m, n, X, \omega, \kappa^-, \kappa^+)$, or $(k, m, n, X, \omega, \kappa)$: $Q = \text{conv}\{\mathbf{g}(\mathcal{C}) \mid \mathcal{C} \text{ is a balanced clustering}\}$.

Support $\mathrm{supp}(C_i)$ ***of a cluster*** C_i***:*** $\mathrm{supp}(C_i) = \{x_j \in X \mid \xi_{i,j} > 0\}$.

Support multi-graph $G(\mathcal{C})$ of a clustering $\mathcal{C}$: vertices: $C_1, \ldots, C_k$; edges: $\{C_i, C_l\}$ for every j for which $x_j \in \mathrm{supp}(C_i) \cap \mathrm{supp}(C_l)$; label of an edge $\{C_i, C_l\}$: x_j. A cycle in $G(\mathcal{C})$ is ***colored*** if not all of its labels coincide. $G(\mathcal{C})$ is ***c-cycle-free*** if it does not contain any colored cycle.

Given $\mathcal{F} = \{\varphi_1, \ldots, \varphi_k\}$ with functions $\varphi_i : \mathbb{R}^n \to \mathbb{R}$ for $i \in \{1, \ldots, k\}$; $\mathcal{F}$***-diagram*** $\mathcal{P}$***:*** $\mathcal{P} = \{P_1, \ldots, P_k\}$ with $P_i = \{x \in \mathbb{R}^n \mid \varphi_i(x) \le \varphi_l(x)\ \forall l \in \{1, \ldots, k\}\}$ for all i.

An $\mathcal{F}$-diagram $\mathcal{P}$ is ***feasible*** for a clustering $\mathcal{C}$: $\mathrm{supp}(C_i) \subset \mathrm{P}_i$ for all i.

An $\mathcal{F}$-diagram $\mathcal{P}$ ***supports*** a clustering $\mathcal{C}$: $\mathrm{supp}(C_i) = \mathrm{P}_i \cap X$ for all i.

An $\mathcal{F}$-diagram $\mathcal{P}$ is ***strongly feasible*** for a clustering $\mathcal{C}$: $\mathcal{P}$ supports $\mathcal{C}$ and $G(\mathcal{C})$ is c-cycle-free.

$(\mathcal{D}, h, \mathcal{S}, \mathcal{M})$***-diagram***: $\mathcal{F}$-diagram for the functions $\varphi_i(x) := h(d_i(s_i, x)) - \mu_i$, where $\mathcal{D} = (d_1, \ldots, d_k)$ is a k-tuple of metrics (or more general distance measures) in $\mathbb{R}^n$, $h : [0, \infty) \to [0, \infty)$ is monotonically increasing, $\mathcal{S} = \{s_1, \ldots, s_k\} \subset \mathbb{R}^n$, and $\mathcal{M} = (\mu_1, \ldots, \mu_k)^T \in \mathbb{R}^k$. The vectors s_i are called ***sites***. If the metrics d_i are all identical, the resulting diagram is ***isotropic***, otherwise it is ***anisotropic***.

Centroidal: A $(\mathcal{D}, h, \mathcal{S}, \mathcal{M})$-diagram that supports a balanced clustering $\mathcal{C}$ is ***centroidal*** if the sites s_i coincide with the centers of gravity $g(C_i)$ of the clusters.

IMPORTANT SPECIAL CASES

Additively weighted Voronoi diagram: $d_1, \ldots, d_k = \|\,.\,\|_{(2)}$, $h = \mathrm{id}$, i.e., $\varphi_i(x) = \|x - s_i\|_{(2)} - \mu_i$ for all i.

Power diagram: $d_1, \ldots, d_k = \|\,.\,\|_{(2)}$, $h = (.)^2$, i.e., $\varphi_i(x) = \|x - s_i\|^2_{(2)} - \mu_i$ for all i.

Anisotropic additively weighted Voronoi diagram (with ellipsoidal norms): For $i = 1, \ldots, k$ each d_i is induced by an ellipsoidal norm $\|\,.\,\|_{M_i}$, i.e., $\|x\|_{M_i} = \sqrt{x^T M_i x}$ for a symmetric positive definite matrix M_i, $h = \mathrm{id}$, i.e., $\varphi_i(x) = \|x - s_i\|_{M_i} - \mu_i$.

Anisotropic power diagram (with ellipsoidal norms): For $i = 1, \ldots, k$ each d_i is induced by an ellipsoidal norm $\|\,.\,\|_{M_i}$, $h = (.)^2$, i.e., $\varphi_i(x) = \|x - s_i\|^2_{M_i} - \mu_i$.

Anisotropic additively weighted Voronoi diagrams have been used by [LS03] and [CG11] for mesh generation. Anisotropic power diagrams were used in [ABG+15] for the reconstruction of polycrystals from information about grain volumes, centers and moments. In [CCD14] districts are designed so as to balance the workload of service vehicles. The effect of different diagrams for electoral district design is studied in [BGK17].

BASIC FACTS

In general, an $\mathcal{F}$-diagram $\mathcal{P}$ does not constitute a dissection of $\mathbb{R}^n$. If, however, $\mathcal{D}$ is a family of metrics induced by strictly convex norms, $h : \mathbb{R} \to \mathbb{R}$ is injective, and

$\mathcal{S} = \{s_1, \ldots, s_k\} \subset \mathbb{R}^n$ is such that $s_i \neq s_l$ for $i \neq l$, then the $(\mathcal{D}, h, \mathcal{S}, \mathcal{M})$-diagram $\mathcal{P} = \{P_1, \ldots, P_k\}$ has the property that $\mathrm{int}(P_i) \cap \mathrm{int}(P_l) = \emptyset$ whenever $i \neq l$.

Given an instance of a constrained clustering problem and any choice of metrics $\mathcal{D}$, functions h and sites $\mathcal{S}$, and let $\mathcal{C}^*$ be a minimizer of

$$\sum_{i=1}^{k} \sum_{j=1}^{m} \xi_{i,j} \cdot \omega(x_j) \cdot \varphi_i(x_j).$$

Then there exists a choice of the additive parameter tuple $\mathcal{M}$, such that the corresponding $(\mathcal{D}, h, \mathcal{S}, \mathcal{M})$-diagram supports $\mathcal{C}^*$; see [BGK17].

Thus, $\mathcal{D}, h, \mathcal{S}$ can be regarded as *structural parameters* while $\mathcal{M}$ is the *feasibility parameter*. Typically, $\mathcal{D}$ and h are defined by requirements on the clusters for a given specific application. Optimization over $\mathcal{S}$ can be done with respect to different criteria. Natural choices involve the total variances or the intercluster distance. For any choice of structural parameters, the feasibility parameter $\mathcal{M}$ is then provided by the dual variables of a certain linear program; see [BG12], [CCD16], [BGK17].

Unless the weights are all the same, this approach does not automatically yield integral assignments in general but may require subsequent rounding. However, the number of fractionally assigned points can be controlled to be at most $k-1$.

POWER DIAGRAMS AND GRAVITY BODIES

Power diagrams constitute cell-decompositions of space into polyhedra. As it turns out all relevant power diagrams in $\mathbb{R}^n$ can be encoded in a certain convex body in $\mathbb{R}^{nk}$. More precisely, we are dealing with the power cells

$$P_i = \{x \in \mathbb{R}^n \mid \|x - s_i\|_{(2)}^2 - \mu_i \leq \|x - s_l\|_{(2)}^2 - \mu_l \ \forall l \in \{1, \ldots, k\}\}.$$

For given sites s_i, the inequalities for x are in fact linear. Hence P_i is a polyhedron and thus, particularly, convex. The relation of power diagrams to *least-square clustering*, i.e., clusterings minimizing the objective function

$$\sum_{i=1}^{k} \sum_{j=1}^{m} \xi_{i,j} \cdot \omega(x_j) \cdot \|x_j - s_i\|_{(2)}^2,$$

has been studied in [BHR92, AHA98, BG12]; see also [BGK17].

In the strongly balanced case, the gravity body Q is a polytope. Its vertices are precisely the gravity vectors of all strongly balanced clusterings that admit a strongly feasible power diagram. In the weakly balanced case, Q does in general have more than finitely many extreme points. However, each extreme point is still the gravity vector of a clustering that admits a strongly feasible power diagram, [BG12].

CENTROIDAL POWER DIAGRAMS AND CLUSTERING BODIES

A quite natural (and in spite of the NP-hardness of the problem practically efficient) approach was introduced in [BG04] for the problem of consolidation of farmland. It models optimal balanced clustering as a convex maximization problem that involves two norms, a norm $\|\,.\,\|$ on $\mathbb{R}^n$ and a second norm $\|\,.\,\|_\diamond$ on $\mathbb{R}^{k(k-1)/2}$. $\|\,.\,\|_\diamond$

is required to be *monotone*, i.e., $\|x\|_\diamond \leq \|y\|_\diamond$ whenever $x, y \in \mathbb{R}^{k(k-1)/2}$ with $0 \leq x \leq y$. A balanced clustering is desired that maximizes

$$\left\| \left(\|c_1 - c_2\|, \|c_1 - c_3\|, \ldots, \|c_{k-1} - c_k\| \right)^T \right\|_\diamond$$

where c_i is a suitable approximation of the center $g(C_i)$ which actually coincides with $g(C_i)$ in the strongly balanced case. In this model, intuitively, a feasible clustering is optimal, if the corresponding "inexact" centers of gravity c_i are pushed apart as far as possible.

This model leads to the study of *clustering bodies*

$$C = \left\{ \mathbf{z} = (z_1^T, \ldots, z_k^T)^T \in \mathbb{R}^{kn} \mid \left\| \left(\|z_1 - z_2\|, \ldots, \|z_{k-1} - z_k\| \right)^T \right\|_\diamond \leq 1 \right\}$$

in $\mathbb{R}^{kn}$. Note that these sets live in $\mathbb{R}^{kn}$ rather than in the typically much higher dimensional space $\mathbb{R}^{km}$ of the optimization problem. Further, note that C has a non-trivial lineality space $\mathrm{ls}(C)$ since a translation applied to all component vectors leaves it invariant. Of course, C and $C \cap (\mathrm{ls}(C))^\perp$ can be regarded as the unit ball of a seminorm or norm, respectively. Hence we are, in effect dealing with the problem of (semi-) norm maximization over polytopes (as in Sections 36.5 and 36.6). In particular, in the strongly balanced case, centroidal power diagrams correspond to the local maxima of the ellipsoidal function $\psi : \mathbb{R}^{kn} \to [0, \infty)$ defined by $\psi(\mathbf{z}) = \sum_{i=1}^{k} \kappa_i \|z_i\|_{(2)}^2$ for $\mathbf{z} = (z_1^T, \ldots, z_k^T)^T \in \mathbb{R}^{kn}$; see [BG12] for additional results.

As it turns out, clustering bodies provide a rich class of sets which include polytopal and smooth bodies but also "mixtures." For some choices of norms one can find permutahedral substructures. [BG10] gives tight bounds for the approximability of such clustering bodies by polyhedra with only polynomially many facets. The proposed algorithm then solves a linear program in $\mathbb{R}^{km}$ for each facet of such an approximating polyhedron. In spite of the NP-hardness of the general balanced clustering problem one obtains good approximate solutions very efficiently, [BG04]; see also [BBG14].

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RELATED CHAPTERS

Chapter 27: Voronoi diagrams and Delaunay triangulations

Preliminary version (August 10, 2017). To appear in the Handbook of Discrete and Computational Geometry,
J.E. Goodman, J. O'Rourke, and C. D. Tóth (editors), 3rd edition, CRC Press, Boca Raton, FL, 2017.

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