12 DISCRETE ASPECTS OF STOCHASTIC GEOMETRY

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INTRODUCTION

Stochastic geometry studies randomly generated geometric objects. The present chapter deals with some discrete aspects of stochastic geometry. We describe work that has been done on finite point sets, their convex hulls, infinite discrete point sets, arrangements of flats, and tessellations of space, under various assumptions of randomness. Typical results concern expectations of geometrically defined random variables, or probabilities of events defined by random geometric configurations. The selection of topics must necessarily be restrictive. We leave out the large number of special elementary geometric probability problems that can be solved explicitly by direct, though possibly intricate, analytic calculations. We pay special attention to either asymptotic results, where the number of points considered tends to infinity, or to inequalities, or to identities where the proofs involve more delicate geometric or combinatorial arguments. The close ties of discrete geometry with convexity are reflected: we consider convex hulls of random points, intersections of random halfspaces, and tessellations of space into convex sets.

There are many topics that one might classify under 'discrete aspects of stochastic geometry', such as optimization problems with random data, the average-case analysis of geometric algorithms, random geometric graphs, random coverings, percolation, shape theory, and several others. All of these have to be excluded here.

12.1 RANDOM POINTS

The setup is a finite number of random points or, alternatively, a point process, in a topological space Σ. Mostly, the space Σ is $\mathbb{R}^d$ ($d \geq 2$), the d-dimensional Euclidean space, with scalar product $\langle \cdot\,, \cdot \rangle$ and norm $\|\cdot\|$. Later, Σ can also be a space of r-flats in $\mathbb{R}^d$, or a space of convex polytopes. By $B^d := \{x \in \mathbb{R}^d : \|x\| \leq 1\}$ we denote the unit ball of $\mathbb{R}^d$, and by $S^{d-1} := \{x \in \mathbb{R}^d : \|x\| = 1\}$ the unit sphere. The volume of B^d is denoted by κ_d.

GLOSSARY

Random point in Σ: A Borel-measurable mapping from some probability space into Σ.

Distribution of a random point X in Σ: The probability measure μ on Σ such that $\mu(B)$, for a Borel set $B \subset \Sigma$, is the probability that $X \in B$.

i.i.d. random points: Stochastically independent random points (on the same probability space) with the same distribution.

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12.1.1 NATURAL DISTRIBUTIONS OF RANDOM POINTS

In geometric problems about i.i.d. random points, a few distributions have been considered as particularly natural.

TABLE 12.1.1 Natural distributions of a random point in $\mathbb{R}^d$.

NAME OF DISTRIBUTION	PROBABILITY DENSITY AT $x \in \mathbb{R}^d$
Uniform in K, μ_K	$\propto$ indicator function of K at x
Standard normal, γ_d	$\propto \exp\left(-\frac{1}{2}\|x\|^2\right)$
Beta type 1	$\propto (1-\|x\|^2)^q \times$ indicator function of B^d at x, $q > -1$
Beta type 2	$\propto \|x\|^{\alpha-1}(1+\|x\|)^{-(\alpha+\beta)}, \alpha, \beta > 0$
Spherically symmetric	function of $\|x\|$

Here $K \subset \mathbb{R}^d$ is a given closed set of positive, finite volume, often a ***convex body*** (a compact, convex set with interior points). In that case, μ_K denotes Lebesgue measure, restricted to K and normalized. Usually, the name of a distribution of a random point is also associated with the random point itself. A ***uniform point*** in K is a random point with distribution μ_K. If F is a smooth compact hypersurface in $\mathbb{R}^d$, a random point is uniform on F if its distribution is proportional to the volume measure on F.

By $\mathbb{P}$ we denote probability and by $\mathbb{E}$ we denote mathematical expectation.

12.1.2 POINT PROCESSES

For randomly generated infinite discrete point sets, suitable models are provided by stochastic point processes. They are considered in spaces more general than the Euclidean, so that the 'points' may be other geometric objects, e.g., hyperplanes.

GLOSSARY

Locally finite: $M \subset \Sigma$ (Σ a locally compact Hausdorff space with countable base) is locally finite if $\mathrm{card}\,(M \cap B) < \infty$ for every compact set $B \subset \Sigma$.

$\mathcal{M}$: The set of all locally finite subsets of Σ.

M: The smallest σ-algebra on $\mathcal{M}$ for which every function $M \mapsto \mathrm{card}\,(M \cap B)$, with $B \subset \Sigma$ a Borel set, is measurable.

(Simple) point process X on Σ: A measurable map X from some probability space (Ω, A, P) into the measurable space $(\mathcal{M}, \mathsf{M})$. When convenient, X is interpreted as a measure, identifying $\mathrm{card}(X \cap A)$ with $X(A)$, for Borel sets A.

Distribution of X: The image measure P_X of P under X.

Intensity measure Λ of X: $\Lambda(B) = \mathbb{E}\,\mathrm{card}\,(X \cap B)$, for Borel sets $B \subset \Sigma$.

Stationary (or ***homogeneous***): If a translation group operates on Σ (e.g., if Σ is $\mathbb{R}^d$, or the space of r-flats, or the space of convex bodies in $\mathbb{R}^d$), then X is a stationary point process if the distribution P_X is invariant under translations.

The point process X on Σ, with intensity measure Λ (assumed to be finite on compact sets), is a ***Poisson process*** if, for any finitely many pairwise disjoint

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Borel sets $B_1, \ldots, B_m$, the random variables card $(X \cap B_1), \ldots,$ card $(X \cap B_m)$ are independent and Poisson distributed, namely

$$\mathbb{P}(card\,(X \cap B) = k) = e^{-\Lambda(B)} \frac{\Lambda(B)^k}{k!}$$

for $k \in \mathbb{N}_0$ and every Borel set B. If $\Sigma = \mathbb{R}^d$ and the process is stationary, then the intensity measure Λ is γ times the Lebesgue measure, and the number γ is called the ***intensity*** of X. Let X be a stationary Poisson process on $\mathbb{R}^d$ and $C \subset \mathbb{R}^d$ a compact set, and let $k \in \mathbb{N}_0$. Under the condition that exactly k points of the process fall into C, these points are stochastically equivalent to k i.i.d. uniform points in C. For an introduction, see also [ScW08, Section 3.2].

A detailed study of geometric properties of stationary Poisson processes in the plane was made by Miles [Mil70].

12.2 CONVEX HULLS OF RANDOM POINTS

We consider convex hulls of finitely many i.i.d. random points in $\mathbb{R}^d$. A random polytope in a convex body K according to the ***uniform model*** (also called *binomial model*) is defined by $K_n := \mathrm{conv}\{X_1, \ldots, X_n\}$, where $X_1, \ldots, X_n$ are i.i.d. uniform random points in K. A random polytope in K according to the ***Poisson model*** is the convex hull, $\widetilde{K}_\lambda$, of the points of a Poisson process with intensity measure $\lambda\mu_K$, for $\lambda > 0$.

A convex body K is of class C^k_+ (where $k \geq 2$) if its boundary ∂K is a k-times continuously differentiable, regular hypersurface with positive Gauss–Kronecker curvature; the latter is denoted by κ.

By $\Omega(K)$ we denote the affine surface area of K (see, e.g., [Sch14, Sec. 10.5]).

NOTATION

$X_1, \ldots, X_n$	i.i.d. random points in $\mathbb{R}^d$
μ	the common probability distribution of X_i
φ	a measurable real function defined on polytopes in $\mathbb{R}^d$
$\varphi(\mu, n)$	the random variable $\varphi(\mathrm{conv}\{X_1, \ldots, X_n\})$
$\varphi(K, n)$	$= \varphi(K_n) = \varphi(\mu_K, n)$
f_k	number of k-faces
ψ_j	1 on polytopes with j vertices, 0 otherwise
V_j	jth intrinsic volume (see Chapter 15); in particular:
V_d	d-dimensional volume
S	$= 2V_{d-1}$, surface area; dS element of surface area
$D_j(K, n)$	$= V_j(K) - V_j(K, n)$

12.2.1 SOME GENERAL IDENTITIES

There are a few general identities and recursion formulas for random variables of type $\varphi(K, n)$. Some of them hold for quite general distributions of the random points.

A classical result due to Wendel (1962; reproduced in [ScW08, 8.2.1]) concerns the probability, say $p_{d,n}$, that $0 \notin \mathrm{conv}\{X_1, \ldots, X_n\}$. If the distribution of the i.i.d. random points $X_1, \dots, X_n \in \mathbb{R}^d$ is symmetric with respect to 0 and assigns measure zero to every hyperplane through 0, then

$$p_{d,n} = \frac{1}{2^{n-1}} \sum_{k=0}^{d-1} \binom{n-1}{k}. \tag{12.2.1}$$

This follows from a combinatorial result of Schläfli, on the number of d-dimensional cells in a partition of $\mathbb{R}^d$ by n hyperplanes through 0 in general position. It was proved surprisingly late that the symmetric distributions are extremal: Wagner and Welzl [WaW01] showed that if the distribution of the points is absolutely continuous with respect to Lebesgue measure, then $p_{d,n}$ is at least the right-hand side of (12.2.1).

Some of the expectations of $\varphi(K, n)$ for different functions φ are connected by identities. Two classical results of Efron [Efr65],

$$\mathbb{E}\psi_{d+1}(K, n) = \binom{n}{d+1} \frac{\mathbb{E}V_d(K, d+1)^{n-d-1}}{V_d(K)^{n-d-1}} \tag{12.2.2}$$

and

$$\mathbb{E}f_0(K, n+1) = \frac{n+1}{V_d(K)} \mathbb{E}D_d(K, n), \tag{12.2.3}$$

have found far-reaching generalizations in work of Buchta [Buc05]. He extended (12.2.3) to higher moments of the volume, showing that

$$\frac{\mathbb{E}V_d(K, n)^k}{V_d(K)^k} = \mathbb{E} \prod_{i=1}^{k} \left(1 - \frac{f_0(K, n+k)}{n+i}\right) \tag{12.2.4}$$

for $k \in \mathbb{N}$ (this is now known as the Efron–Buchta identity). As a consequence, the kth moment of $V_d(K, n)/V_d(K)$ can be expressed linearly by the first k moments of $f_0(K, n+k)$. Conversely, the distribution of $f_0(K, n)$ is determined by the moments $\mathbb{E}[V_d(K, j)/V_d(K)]^{n-j}$, $j = d+1, \dots, n$. Further consequences are variance estimates for $D_d(K, n)$ and $f_0(K, n)$ for sufficiently smooth convex bodies K. Relation (12.2.4) holds for more general distributions, if the volume is replaced by the probability content.

Of combinatorial interest is the expectation $\mathbb{E}\psi_i(K, n)$, which is the probability that K_n has exactly i vertices. For this, Buchta [Buc05] proved that

$$\mathbb{E}\psi_i(K, n) = (-1)^i \binom{n}{i} \sum_{j=d+1}^{i} (-1)^j \binom{i}{j} \frac{\mathbb{E}V_d(K, j)^{n-j}}{V_d(K)^{n-j}},$$

which for $i = d+1$ reduces to (12.2.2).

The expected values $\mathbb{E}V_d(\mu, n)$ for different numbers n are connected by a sequence of identities. For an arbitrary probability distribution μ on $\mathbb{R}^d$, Buchta [Buc90] proved the recurrence relations

$$\mathbb{E}V_d(\mu, d+2m) = \frac{1}{2} \sum_{k=1}^{2m-1} (-1)^{k+1} \binom{d+2m}{k} \mathbb{E}V_d(\mu, d+2m-k)$$

and, consequently,

$$\mathbb{E}V_d(\mu, d+2m) = \sum_{k=1}^{m} \left(2^{2k} - 1\right) \frac{B_{2k}}{k} \binom{d+2m}{2k-1} \mathbb{E}V_d(\mu, d+2m-2k+1)$$

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for $m \in \mathbb{N}$, where the constants B_{2k} are the Bernoulli numbers. Cowan [Cow07] deduced these relations by integrating a new pointwise identity. Of this, he made other applications. For a distribution μ that assigns measure zero to each hyperplane, Cowan [Cow10] proved that

$$\mathbb{E}f_0(\mu,n) = \frac{n}{2} + \frac{1}{2}\sum_{j=1}^{n-1}(-1)^{j-1}\binom{n}{j}\mathbb{E}f_0(\mu,n-j)$$

if $n \geq d+3$ and n is odd, and similar formulas for $\mathbb{E}f_{d-1}(\mu,n)$ and $\mathbb{E}f_{d-2}(\mu,n)$.

For a slightly different model of random polytopes, Beermann and Reitzner [BeR15] were able to go beyond the Efron–Buchta identity, by linking generating functions. Let μ be a probability measure on $\mathbb{R}^d$ which is absolutely continuous with respect to Lebesgue measure. Let η be a Poisson point process on $\mathbb{R}^d$ with intensity measure $t\mu$, for $t > 0$. The convex hull, Π_t, of η is a random polytope. Let $g_{I(\Pi_t)}$ denote the probability-generating function of the random variable $I(\Pi_t) := \eta(\mathbb{R}^d) - f_0(\Pi_t)$ (the number of inner points of Π_t), and let $h_{\mu(\Pi_t)}$ denote the moment-generating function of $\mu(\Pi_t)$ (the μ-content of Π_t). It is proved in [BeR15] that

$$g_{I(\Pi_t)}(z+1) = h_{\mu(\Pi_t)}(tz) \quad \text{for } z \in \mathbb{C}.$$

12.2.2 RANDOM POINTS IN CONVEX BODIES—EXPECTATIONS

We consider the random variables $\varphi(K,n)$, for convex bodies K and basic geometric functionals φ, such as volume and vertex number or, more generally, intrinsic volumes and numbers of k-faces.

A few cases where information on the whole distribution is available, are quoted in [ScW08, Note 2 on p. 312]. This section deals mainly with expectations. The table below lists the rare cases where $\mathbb{E}\,\varphi(K,n)$ is known explicitly.

TABLE 12.2.1 Expected value of $\varphi(K,n)$.

DIMENSION d	CONVEX BODY K	FUNCTIONAL φ	SOURCES
2	polygon	V_2	Buchta [Buc84a]
2	polygon	f_0	Buchta and Reitzner [BuR97a]
2	ellipse	V_2	Buchta [Buc84b]
3	ellipsoid	V_3	Buchta [Buc84b]
≥ 2	ball	S, mean width, f_{d-1}	Buchta and Müller [BuM84]
≥ 2	ball	V_d	Affentranger [Aff88]
3	tetrahedron	V_3	Buchta and Reitzner [BuR01]

The value of $\mathbb{E}V_3(C^3,4)$ for the three-dimensional cube C^3 was determined by Zinani [Zin03].

For information on inequalities for expectations $\mathbb{E}\,\varphi(K,n)$ (some of them classical), we refer to [ScW08, Note 1 on p. 311 and Section 8.6].

We turn to inequalities that exhibit the behavior of $\mathbb{E}\,\varphi(K,n)$ for large n. For a convex body $K \subset \mathbb{R}^d$, Schneider [Sch87] showed the existence of positive constants $a_1(K), a_2(K)$ such that

$$a_1(K)n^{-\frac{2}{d+1}} < \mathbb{E}D_1(K,n) < a_2(K)n^{-\frac{1}{d}} \tag{12.2.5}$$

for $n \in \mathbb{N}$, $n \geq d+1$. Smooth bodies (left) and polytopes (right) show that the orders are best possible.

For general convex bodies K, a powerful method for investigating the random polytopes K_n was invented by Bárány and Larman [BáL88] (for introductions and surveys, see [Bár00], [Bár07], [Bár08]). For K of volume one and for sufficiently small $t > 0$, they introduced the floating body

$$K[t] := \{x \in K \mid V_d(K \cap H) \geq t \text{ for every halfspace } H \text{ with } x \in H\}.$$

Their main result says that K_n and $K[1/n]$ approximate K of the same order and that $K \setminus K_n$ is close to $K \setminus K[1/n]$ in a precise sense. From this, several results on the expectations $\mathbb{E}\,\varphi(K,n)$ for various φ were obtained by Bárány and Larman [BáL88], by Bárány [Bár89], for example

$$c_1(d)(\log n)^{d-1} < \mathbb{E} f_j(K,n) < c_2(d) n^{\frac{d-1}{d+1}} \tag{12.2.6}$$

for $j \in \{0,\dots,d\}$ with positive constants $c_i(d)$ (the orders are best possible), and by Bárány and Vitale [BáV93].

The inequalities (12.2.5) show that for general K the approximation, measured in terms of $D_1(K,\cdot)$, is not worse than for polytopes and not better than for smooth bodies. For approximation measured by $D_d(K,\cdot)$, the class of polytopes and the class of smooth bodies interchange their roles, since

$$b_1(K) n^{-1} (\log n)^{d-1} < \mathbb{E} D_d(K,n) < b_2(K) n^{-\frac{2}{d+1}},$$

as follows from [BáL88] (or from (12.2.6) for $j=0$ and (12.2.3)).

Precise asymptotic behavior for some cases of $\mathbb{E}\,\varphi(K,n)$ is known if K is either a polytope or sufficiently smooth. For early results and for work in the plane, we refer to the survey [Sch88, Section 5] and to [BuR97a]. For d-dimensional polytopes P, Bárány and Buchta [BáB93] showed that

$$\mathbb{E} f_0(P,n) = \frac{T(P)}{(d+1)^{d-1}(d-1)!} \log^{d-1} n + O(\log^{d-2} n \log\log n), \tag{12.2.7}$$

where $T(P)$ denotes the number of chains $F_0 \subset F_1 \subset \cdots \subset F_{d-1}$ where F_i is an i-dimensional face of P. They established a corresponding relation for the volume, from which (12.2.7) follows by (12.2.3). This work was the culmination of a series of papers by other authors, among them Affentranger and Wieacker [AfW91], who settled the case of simple polytopes, which is applied in [BáB93]. An extension of (12.2.7) was proved by Reitzner [Rei05a], in the form

$$\mathbb{E} f_k(P,n) = c(d,k,P) \log^{d-1} n + o(\log^{d-1} n),$$

for $k = 0,\dots,d-1$.

For convex bodies $K \subset \mathbb{R}^d$ of class C^2_+, Reitzner [Rei05a] succeeded with showing that

$$\mathbb{E} f_k(K,n) = c_{d,k} \Omega(K)\, n^{\frac{d-1}{d+1}} + o\left(n^{\frac{d-1}{d+1}}\right)$$

with a constant $c_{d,k}$.

A relation of the form

$$\mathbb{E} D_j(K,n) = c^{(j,d)}(K)\, n^{-\frac{2}{d+1}} + o\left(n^{-\frac{2}{d+1}}\right) \tag{12.2.8}$$

for $j = 1, \dots, d$ was first proved for convex bodies K of class C^3_+ by Bárány [Bár92] (for $j = 1$, such a result, with explicit $c^{(1,d)}$, was obtained earlier by Schneider and Wieacker [ScWi80]; this has been extended in [BöFRV09] to convex bodies in which a ball rolls freely). Reitzner [Rei04] proved (12.2.8) for bodies of class C^2_+ and showed that the coefficients are given by

$$c^{(j,d)}(K) = c^{(j,d)} \int_{\partial K} \kappa^{\frac{1}{d+1}} H_{d-j}\, dS, \qquad (12.2.9)$$

where H_{d-j} denotes the $(d-j)$th normalized elementary symmetric function of the principal curvatures of ∂K and $c^{(j,d)}$ depends only on j and d. Under stronger differentiability assumptions, Reitzner also obtained more precise asymptotic expansions. (Earlier, Gruber [Gru96] had obtained analogous expansions for $\eta(C) - \mathbb{E}\eta(C, n)$, where $\eta(C)$ is the value of the support function of the convex body C at a given vector $u \in S^{d-1}$.) Relation (12.2.8) was extended by Böröczky, Hoffmann and Hug [BöHH08] to convex bodies K satisfying only the assumption that a ball rolls freely in K.

Let $V_d(K) = 1$. The coefficient $c^{(d,d)}(K)$ in (12.2.8) is a (dimension-dependent) constant multiple of the affine surface area $\Omega(K)$. The limit relation

$$\lim_{n\to\infty} n^{\frac{2}{d+1}} \mathbb{E} D_d(K, n) = c^{(d,d)} \int_{\partial K} \kappa^{\frac{1}{d+1}}\, dS$$

was extended by Schütt [Schü94] to arbitrary convex bodies (of volume one), with the Gauss–Kronecker curvature generalized accordingly. A generalization, with a modified proof, is found in Böröczky, Fodor and Hug [BöFH10, Thm. 3.1].

The following concerns the Hausdorff distance δ. For $K \subset \mathbb{R}^d$ of class C^2_+, Bárány [Bár89] showed that there are positive constants $c_1(K), c_2(K)$ such that

$$c_1(K)(n^{-1}\log n)^{\frac{2}{d+1}} < \mathbb{E}\,\delta(K, K_n) < c_2(K)(n^{-1}\log n)^{\frac{2}{d+1}}.$$

OPEN PROBLEMS

PROBLEM 12.2.1 (often posed)

For a convex body $K \subset \mathbb{R}^d$ of given positive volume, is $\mathbb{E}V_d(K, d+1)$ maximal if K is a simplex? This problem is related to some other major open problems in convex geometry, such as the slicing problem. (This problem asks whether there is a constant $c > 0$, independent of the dimension, such that every d-dimensional convex body of volume 1 has a hyperplane section of $(d-1)$-dimensional volume at least c.) See, e.g., Milman and Pajor [MiP89].

PROBLEM 12.2.2 (Vu [Vu06])

For a convex body K in $\mathbb{R}^d$, is $\mathbb{E}f_0(K, n)$ an increasing function of n? For $d = 2$, this was proved in [DeGG$^+$13].

PROBLEM 12.2.3 (M. Meckes)

Is there a universal constant $c > 0$ such that, for convex bodies $K, L \subset \mathbb{R}^d$, the inclusion $K \subset L$ implies $\mathbb{E}V_d(K, d+1) \le c^d\, \mathbb{E}V_d(L, d+1)$? This question is equivalent to the slicing problem; see Rademacher [Rad12]. A main result of [Rad12] is that $K \subset L$ implies $\mathbb{E}V_d(K, d+1) \le \mathbb{E}V_d(L, d+1)$ for $d \le 2$, but not for $d \ge 4$.

12.2.3 VARIANCES, HIGHER MOMENTS, LIMIT THEOREMS

The step from expectations, as considered in the previous section, to higher moments, concentration inequalities and limit theorems requires in general much more sophisticated tools from probability theory. Important progress has been made in recent years, but since it is often more due to modern probabilistic techniques than to refined discrete geometry, the presentation here will be briefer than the topic deserves.

A central limit theorem (CLT) for one of our series of random variables $\varphi(\mu, n)$ is an assertion of the form

$$\lim_{n\to\infty} \left| \mathbb{P}\left(\frac{\varphi(\mu,n) - \mathbb{E}\,\varphi(\mu,n)}{\sqrt{\operatorname{Var}\varphi(\mu,n)}} \le t \right) - \Phi(t) \right| = 0$$

for all $t \in \mathbb{R}$, where $\Phi(t) = (2\pi)^{-1/2} e^{-t^2/2}$ is the distribution function of the standard normal distribution in one dimension. Here, the values of the expectation and the variance Var may, or may not, be known explicitly for each n, or may be replaced by asymptotic values.

First we consider the planar case. The first central limit theorems for random variables $\varphi(K, n)$ were obtained in pioneering work of Groeneboom [Gro88]. For a convex polygon $P \subset \mathbb{R}^2$ with r vertices, he proved that

$$\frac{f_0(P,n) - \frac{2}{3} r \log n}{\sqrt{\frac{10}{27} r \log n}} \xrightarrow{\mathcal{D}} \gamma_1$$

for $n \to \infty$, where $\xrightarrow{\mathcal{D}}$ denotes convergence in distribution and γ_1 is the standard normal distribution in one dimension. Groeneboom also showed a CLT for $f_0(B^2, n)$, where B^2 is the circular disc (an asymptotic variance appearing here was determined by Finch and Hueter [FiH04]).

For a polygon P with r vertices, a result of Cabo and Groeneboom [CaG94], in a version suggested by Buchta [Buc05] (for further clarification, see also Groeneboom [Gro12]), says that

$$\frac{V_2(P)^{-1} D_2(P,n) - \frac{2}{3} r \frac{\log n}{n}}{\sqrt{\frac{28}{27} r \frac{\log n}{n^2}}} \xrightarrow{\mathcal{D}} \gamma_1 .$$

A result of Hsing [Hsi94] was made more explicit by Buchta [Buc05] and gives a CLT for $D_2(B^2, n)$.

An unpublished preprint by Nagaev and Khamdamov [NaK91] contains a central limit theorem for the joint distribution of $f_0(P, n)$ and $D_2(P, n)$, for a convex polygon P. This result was re-proved by Groeneboom [Gro12], based on his earlier work [Gro88].

Bräker, Hsing, and Bingham [BrHB98] investigated the asymptotic distribution of the Hausdorff distance between a planar convex body K (either smooth or a polygon) and the convex hull of n i.i.d. uniform points in K. A thorough study of the asymptotic properties of $D_2(\mu, n)$ and $D_1(\mu, n)$ was presented by Bräker and Hsing [BrH98], for rather general distributions μ (including the uniform distribution) concentrated on a convex body K in the plane, where K is either sufficiently smooth and of positive curvature, or a polygon.

For general convex bodies in the plane, central limit theorems for vertex number and area were finally proved by Pardon [Par11], [Par12], first for the Poisson model and then for the uniform model.

Turning to higher dimensions, we remark that Küfer [Küf94] studied the asymptotic behavior of $D_d(B^d, n)$ and showed, in particular, that its variance is at most of order $n^{-\frac{d+3}{d+1}}$, as $n \to \infty$. Schreiber [Schr02] determined the asymptotic orders of the moments of $D_1(B^d, n)$ and proved that $\lim_{n\to\infty} n^{\frac{2}{d+1}} D_1(B^d, n) = a_d$ in probability, with an explicit constant a_d.

Essential progress in higher dimensions began with two papers by Reitzner. For convex bodies $K \in \mathbb{R}^d$ of class C^2_+ he obtained in [Rei03], using the Efron–Stein jackknife inequality, optimal upper bounds for the orders of Var $V_d(K, n)$ and Var $f_0(K, n)$. He deduced corresponding strong laws of large numbers, for example,

$$\lim_{n\to\infty} n^{\frac{2}{d+1}} D_d(K, n) = \Gamma_d \Omega(K)$$

with probability one, where Γ_d is an explicit constant. Using a CLT for dependency graphs due to Rinott, Reitzner [Rei05b] showed central limit theorems for V_d and f_0 in the Poisson model (that is, for $\widetilde{K}_n$, the convex hull of the points of a Poisson point process with intensity measure $n\mu_K$). One of Reitzner's results says that, for a convex body K of class C^2_+,

$$\left| \mathbb{P}\left(\frac{V_d(\widetilde{K}_n) - \mathbb{E}V_d(\widetilde{K}_n)}{\sqrt{\mathrm{Var}\, V_d(\widetilde{K}_n)}} \le x \right) - \Phi(x) \right| \le c(K) n^{-\frac{1}{2}+\frac{1}{d+1}} \log^{2+\frac{2}{d+1}} n$$

with a constant $c(K)$. He has a similar result for $f_k(\widetilde{K}_n)$, and also central limit theorems for V_d and f_k in the uniform model, however, with the standard deviation of $V_d(K, n)$ replaced by that of $V_d(\widetilde{K}_n)$.

For the Poisson model in a given polytope, central limit theorems for the volume and the numbers f_k were achieved by Bárány and Reitzner [BáR10a].

Using martingale techniques, Vu [Vu05] obtained concentration inequalities for $V_d(K, n)$ and $f_0(K, n)$. Based on this, he obtained in [Vu06] for K of class C^2_+ tail estimates of type

$$\mathbb{P}\left(|D_d(K, n)| \ge \sqrt{\lambda n^{-\frac{d+3}{d+1}}} \right) \le 2\exp(-c\lambda) + \exp(-c'n) \tag{12.2.10}$$

for any $0 < \lambda < n^\alpha$, where c, c' and α are positive constants. This allowed him to determine the order of magnitude for each moment of $D_d(K, n)$. Using such tail estimates and Reitzner's CLT for the Poisson model, Vu [Vu06] finally proved central limit theorems for $V_d(K, n)$ and $f_k(K, n)$, for K of class C^2_+.

For a convex body K with a freely rolling ball, Böröczky, Fodor, Reitzner, and Vígh [BöFRV09] obtained lower and upper bounds for the variance of $V_1(K, n)$ and proved a strong law of large numbers for $D_1(K, n)$. For K of class C^2_+, Bárány, Fodor, and Vígh [BáFV10] obtained matching lower and upper bounds for the orders of the variances of $V_i(K, n)$ and proved strong laws of large numbers for $D_i(K, n)$. For a polytope P, Bárány and Reitzner [BáR10b] determined asymptotic lower and upper bounds (and lower bounds also for general convex bodies) for the variances of $V_d(P,n)$ and $f_k(P,n)$ and deduced corresponding strong laws of large numbers.

For the uniform model and the Poisson model in the ball B^d, Calka and Schreiber [CaS06] established large deviation results and laws of large numbers for the vertex number f_0. Schreiber and Yukich [ScY08] determined the precise asymptotic behavior of the variance of $f_0(B^d,n)$ and proved a CLT. Calka and Yukich [CaY14] were able, based on prior joint work with Schreiber, to determine the precise asymptotics of the variance of the vertex number in the case of a convex body $K \subset \mathbb{R}^d$ of class C^3_+: for $k \in \{0,\dots,d-1\}$, it is proved in [CaY14] that there exists a constant $F_{k,d} > 0$ such that

$$\lim_{\lambda\to\infty} \lambda^{-\frac{d-1}{d+1}} \operatorname{Var} f_k(\widetilde{K}_\lambda) = F_{k,d}\, \Omega(K)$$

and also that

$$\lim_{n\to\infty} n^{-\frac{d-1}{d+1}} \operatorname{Var} f_k(K,n) = F_{k,d}\, \Omega(K).$$

For the diameter of the random polytope K_n, in case $K = B^d$ (and for more general spherically symmetric distributions), a limit theorem was proved by Mayer and Molchanov [MaM07].

12.2.4 RANDOM POINTS ON CONVEX SURFACES

If μ is a probability distribution on the boundary ∂K of a convex body K and μ has density h with respect to the volume measure of ∂K, we write $\varphi(\partial K,h,n) := \varphi(\mu,n)$ and $D_j(\partial K,h,n) := V_j(K) - V_j(\partial K,h,n)$. Some references concerning $\mathbb{E}\,\varphi(\partial K,h,n)$ are given in [Sch88, p. 224]. Most of them are superseded by an investigation of Reitzner [Rei02a]. For K of class C^2_+ and for continuous $h > 0$ he showed that

$$\mathbb{E} D_j(\partial K,h,n) = b^{(j,d)} \int_{\partial K} h^{-\frac{2}{d-1}} \kappa^{\frac{1}{d-1}} H_{d-j}\, dS \cdot n^{-\frac{2}{d-1}} + o(n^{-\frac{2}{d-1}})$$

as $n \to \infty$ (with H_{d-j} as in (12.2.9)). Under stronger differentiability assumptions on K and h, an asymptotic expansion with more terms was established. Similar results for support functions were obtained earlier by Gruber [Gru96]. The case $j = d$ of Reitzner's result was strengthened by Schütt and Werner [ScWe03], who admitted convex bodies that satisfy bounds on upper and lower curvatures. A still more general result is due to Böröczky, Fodor and Hug [BöFH13], who extended Reitzner's relation to convex bodies K satisfying only the assumption that a ball rolls freely in K.

For a convex body K of class C^2_+, Reitzner [Rei03] obtained an upper variance estimate for $V_d(\partial K,h,n)$ and a law of large numbers for $D_d(\partial K,h,n)$. Under the same assumption, Richardson, Vu, and Wu [RiVW08] established a concentration inequality of type (12.2.10) for $V_d(\partial K,h,n)$, upper estimates for the moments of $V_d(\partial K,h,n)$, and a CLT for the convex hull of the points of a Poisson process on ∂K with intensity measure n times the normalized volume measure on the boundary of K.

The following results concern the Hausdorff metric δ. Let $(X_k)_{k\in\mathbb{N}}$ be an i.i.d. sequence of random points on the boundary ∂K of a convex body K, the distribution of which has a continuous positive density h with respect to the volume measure on ∂K. Let $P_n := \mathrm{conv}\{X_1, \dots, X_n\}$. For $h = 1$, Dümbgen and Walther [DüW96] showed that $\delta(K, P_n)$ is almost surely of order $O((\log n/n)^{1/(d-1)})$ for general K, and of order $O((\log n/n)^{2/(d-1)})$ under a smoothness assumption. Glasauer and Schneider [GlS96] proved for K of class C^3_+ that

$$\lim_{n\to\infty} \left(\frac{n}{\log n}\right)^{\frac{2}{d-1}} \delta(K, P_n) = \frac{1}{2}\left(\frac{1}{\kappa_{d-1}} \max \frac{\sqrt{\kappa}}{h}\right)^{\frac{2}{d-1}} \quad \text{in probability.}$$

It was conjectured that this holds with almost sure convergence. For $d = 2$, this is true, and similar results hold with the Hausdorff distance replaced by area or perimeter difference; this was proved by Schneider [Sch88].

Bárány, Hug, Reitzner, and Schneider [BáHRS17] investigated spherical convex hulls of i.i.d. uniform points in a closed halfsphere. This model exhibits some new phenomena, since the boundary of the halfsphere is totally geodesic.

12.2.5 GAUSSIAN RANDOM POINTS

Let γ_d denote the standard normal distribution on $\mathbb{R}^d$. For the expectations $\mathbb{E} f_k(\gamma_d, n)$ it is known that

$$\mathbb{E} f_k(\gamma_d, n) \sim \frac{2^d}{\sqrt{d}} \binom{d}{k+1} \beta_{k,d-1} (\pi \log n)^{\frac{d-1}{2}} \tag{12.2.11}$$

as $n \to \infty$, where $\beta_{k,d-1}$ is the interior angle of the regular $(d-1)$-dimensional simplex at one of its k-dimensional faces. This follows from [AfS92], where the Grassmann approach (see Section 12.4.3) was used, together with an equivalence of [BaV94] explained in Section 12.4.3. Relation (12.2.11) was proved by Hug, Munsonius, and Reitzner [HuMR04] in a more direct way, together with similar asymptotic relations for other functionals, such as the k-volume of the k-skeleton. A limit relation for $V_k(\gamma_d, n)$ similar to (12.2.11) was proved by Affentranger [Aff91]. For the random variables $f_k(\gamma_d, n)$ and $V_k(\gamma_d, n)$, Hug and Reitzner [HuR05] obtained variance estimates and deduced laws of large numbers.

For Gaussian polytopes (that is, convex hulls of i.i.d. points with distribution γ_d), Hueter [Hue94], [Hue99] was the first to obtain central limit theorems. In [Hue94], she proved a CLT for $\varphi(\gamma_2, n)$ in the plane, where $\varphi \in \{V_1, V_2, f_0\}$, and in [Hue99] for $f_0(\gamma_d, n)$ (an unjustified criticism in [BáV07] was revoked in [BáV08]). In the plane, Massé [Mas00] derived from [Hue94] that

$$\lim_{n\to\infty} \frac{f_0(\gamma_2, n)}{\sqrt{8\pi \log n}} = 1 \quad \text{in probability.}$$

Bárány and Vu [BáV07] succeeded with proving a CLT for $V_d(\gamma_d, n)$ and for $f_k(\gamma_d, n)$, $k = 0, \dots, d-1$.

12.2.6 SPHERICALLY SYMMETRIC AND OTHER DISTRIBUTIONS

The following setup has been studied repeatedly. For $0 \le p \le r+1 \le d-1$, one considers $r+1$ independent random points, of which the first p are uniform in the ball B^d and the last $r+1-p$ are uniform on the boundary sphere S^{d-1}. Precise information on the moments and the distribution of the r-dimensional volume of the convex hull is available; see the references in [Sch88, pp. 219, 224] and the work of Affentranger [Aff88].

Among spherically symmetric distributions, the Beta distributions are particularly tractable. For these, again, the r-dimensional volume of the convex hull of $r+1$ i.i.d. random points has frequently been studied. We refer to the references given in [Sch88] and Chu [Chu93]. Affentranger [Aff91] determined the asymptotic behavior, as $n \to \infty$, of the expectation $\mathbb{E}V_j(\mu, n)$, where μ is either the Beta type-1 distribution, the uniform distribution in B^d, or the standard normal distribution in $\mathbb{R}^d$. The asymptotic behavior of $\mathbb{E}f_{d-1}(\mu, n)$ was also found for these cases. Further information is contained in the book of Mathai [Mat99].

For more general spherically symmetric distributions μ, the asymptotic behavior of the random variables $\varphi(\mu, n)$ will essentially depend on the tail behavior of the distribution. Extending work of Carnal [Car70], Dwyer [Dwy91] obtained asymptotic estimates for $\mathbb{E}f_0(\mu, n)$, $\mathbb{E}f_{d-1}(\mu, n)$, $\mathbb{E}V_d(\mu, n)$, and $\mathbb{E}S(\mu, n)$. These investigations were continued, under more general assumptions, by Hashorva [Has11].

Aldous *et al.* [AlFGP91] considered an i.i.d. sequence $(X_k)_{k\in\mathbb{N}}$ in $\mathbb{R}^2$ with a spherically symmetric (or more general) distribution, and under an assumption of slowly varying tail, they determined a limiting distribution for $f_0(\mu, n)$.

Devroye [Dev91] and Massé [Mas99], [Mas00] constructed spherically symmetric distributions μ in the plane for which $f_0(\mu, n)$ has some unexpected properties. For example, Devroye showed that for any monotone sequence $\omega_n \uparrow \infty$ and for every $\epsilon > 0$, there is a radially symmetric distribution μ in the plane for which $\mathbb{E}f_0(\mu, n) \geq n/\omega_n$ infinitely often and $\mathbb{E}f_0(\mu, n) \leq 4 + \epsilon$ infinitely often.

For distributions μ on $\mathbb{R}^d$ with a density with respect to Lebesgue measure, Devroye [Dev91] showed that $\mathbb{E}f_0(\mu, n) = o(n)$ and $\lim_{n\to\infty} n^{-1} f_0(\mu, n) = 0$ almost surely.

12.3 RANDOM GEOMETRIC CONFIGURATIONS

The topic of this section is still finitely many i.i.d. random points, but instead of taking functionals of their convex hull, we consider qualitative properties, in different ways related to convexity.

12.3.1 SYLVESTER'S PROBLEM AND CONVEX POSITION

Sylvester's classical problem asked for $\mathbb{E}\psi_3(K, 4)$ for a convex body $K \subset \mathbb{R}^2$. More generally, one may ask for $\mathbb{E}\psi_{d+1}(K, n)$ for a convex body $K \subset \mathbb{R}^d$ and $n > d+1$, the probability that the convex hull of n i.i.d. uniform points in K is a simplex. (See [ScW08], Section 8.1, for some history, and the notes for Subsection 8.2.3 for more information.)

At the other end, $p(n, K) := \mathbb{E}\psi_n(K, n)$ is of interest, the probability that n i.i.d. uniform points in K are in convex position. Bárány and Füredi [BáF88] determined the limits $\lim_{d\to\infty} p(n(d), B^d)$ for suitable functions $n(d)$. Valtr [Val95] determined $p(n, K)$ if P is a parallelogram, and in [Val96] if P is a triangle. Marckert [Mar17] calculated the values of $p(n, B^2)$ for $n \leq 8$. For convex bodies $K \subset \mathbb{R}^2$ of area one, Bárány [Bár99] obtained the astonishing limit relation

$$\lim_{n\to\infty} n^2 \sqrt[n]{p(n, K)} = \frac{1}{4} e^2 A^3(K),$$

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where $A(K)$ is the supremum of the affine perimeters of all convex bodies contained in K. He also established a law of large numbers for convergence to a limit shape. There is a unique convex body $\tilde{K} \subset K$ with affine perimeter $A(K)$. If K_n denotes the convex hull of n i.i.d. uniform points in K and δ is the Hausdorff metric, then Bárány's result says that

$$\lim_{n\to\infty} \mathbb{P}(\delta(K_n, \tilde{K}) > \epsilon \mid f_0(K_n) = n) = 0$$

for every $\epsilon > 0$. For a convex body $K \subset \mathbb{R}^d$ of volume one, Bárány [Bár01] showed that

$$c_1 < n^{\frac{2}{d-1}} \sqrt[n]{p(n,K)} < c_2$$

for $n \geq n_0$, where the constants $n_0, c_1, c_2 > 0$ depend only on d.

12.3.2 FURTHER QUALITATIVE ASPECTS

For a given distribution μ on $\mathbb{R}^2$, let $P_1, \ldots, P_j, Q_1, \ldots, Q_k$ be i.i.d. points distributed according to μ. Let $p_{jk}(\mu)$ be the probability that the convex hull of $P_1, \ldots, P_j$ is disjoint from the convex hull of $Q_1, \ldots, Q_k$. Continuing earlier work of several authors, Buchta and Reitzner [BuR97a] investigated $p_{jk}(\mu)$. For the uniform distribution μ in a convex body K, they connected $p_{jk}(\mu)$ to equiaffine inner parallel curves of K, found an explicit representation in the case of polygons, and proved, among other results, that

$$\lim_{n\to\infty} \frac{p_{nn}(\mu)}{n^{3/2}4^{-n}} \geq \frac{8\sqrt{\pi}}{3},$$

with equality if K is centrally symmetric. The investigation was continued by Buchta and Reitzner in [BuR97b].

Of interest for the investigation of random points in planar polygons is the study of ***convex chains***. Let $X_1, \ldots, X_n$ be i.i.d. uniform random points in the triangle with vertices $(0,0), (0,1), (1,1)$, let $p_k^{(n)}$ be the probability that precisely k points of $X_1, \ldots, X_n$ are vertices of $T_n := \mathrm{conv}\{(0,0), X_1, \ldots, X_n, (1,1)\}$, and let N_n be the number of these vertices. Bárány, Rote, Steiger, and Zhang [BáRSZ00] calculated $p_n^{(n)}$, and Buchta [Buc06] found $p_k^{(n)}$ in general. Buchta [Buc12] determined explicitly the first four moments of N_n and proved that

$$\mathbb{E}N_n^m = \left(\frac{2}{3}\log n\right)^m + O\left(\log^{m-1} n\right)$$

as $n \to \infty$, for $m \in \mathbb{N}$. For the same model, Buchta [Buc13] obtained expectation and variance for the missed area.

A random convex n-chain is defined as the boundary of T_n minus the open segment connecting $(0,0)$ and $(1,1)$, conditional on the event that T_n has $n+2$ vertices. Bárány *et al.* [BáRSZ00] established a limit shape of random convex n-chains, as $n \to \infty$, and proved several related limit theorems, among them a CLT. Ambrus and Bárány [AmB09] introduced longest convex chains defined by $X_1, \ldots, X_n$ ('long' in terms of number of vertices), and they found the asymptotic behavior of their expected length, as $n \to \infty$, as well as a limit shape.

Related to k-sets (see Chapter 1 of this Handbook) is the following investigation of Bárány and Steiger [BáS94]. If X is a set of n points in general position in $\mathbb{R}^d$, a subset $S \subset X$ of d points is called a k-simplex if X has exactly k points on one side of the affine hull of S. The authors study the expected number of k-simplices for n i.i.d. random points, for different distributions.

An *empty convex polytope* in a finite point set in $\mathbb{R}^d$ is a convex polytope with vertices in the point set, but no point of the set in the interior. For i.i.d. uniform random points $X_1, \dots, X_n$ in a convex body $K \subset \mathbb{R}^d$ (mostly $d = 2$), empty polytopes in $\{X_1, \dots, X_n\}$ were investigated in [BáF87], [BaGS13], [BáMR13], [FaHM15].

Various elementary geometric questions can be asked, even about a small number of random points. To give one example: Let $X_1, \dots, X_m$, $m \geq 2$, be independent uniform random points in a convex body $K \subset \mathbb{R}^d$. The probability that the smallest ball containing $X_1, \dots, X_m$ is contained in K is maximal if and only if K is a ball; see [BaS95]. Many examples are treated in the book of Mathai [Mat99].

12.4 RANDOM FLATS AND HALFSPACES

Next to random points, randomly generated r-dimensional flats in $\mathbb{R}^d$ are the objects of study in stochastic geometry that are particularly close to discrete geometry. Like convex hulls of random points, intersections of random halfspaces yield random polytopes in a natural way.

GLOSSARY

$A(d, r)$: Affine Grassmannian of r-flats (r-dimensional affine subspaces) in $\mathbb{R}^d$, with its standard topology.

r-Flat process in $\mathbb{R}^d$: A point process in the space $A(d, r)$.

Isotropic: An r-flat process is isotropic if its distribution is rotation invariant.

12.4.1 POISSON FLAT PROCESSES

A basic model for infinite discrete random arrangements of r-flats in $\mathbb{R}^d$ is provided by a stationary Poisson process in $A(d, r)$. Fundamental work was done by Miles [Mil71] and Matheron [Math75]. We refer to [ScW08, Sec 4.4] for an introduction and references.

A natural geometric question involves intersections. For example, let X be a stationary Poisson hyperplane process in $\mathbb{R}^d$. For $k \in \{0, \dots, d-1\}$, a kth ***intersection density*** χ_k of X can be defined in such a way that, for any Borel set $A \subset \mathbb{R}^d$, the expectation of the total k-dimensional volume inside A of the intersections of any $d-k$ hyperplanes of the process is given by χ_k times the Lebesgue measure of A. Then, given the intensity χ_{d-1}, the maximal kth intersection density χ_k (for $k \in \{0, \dots, d-2\}$) is achieved if the process is isotropic. This result is due to Thomas (1984, see [ScW08, Sec. 4.4]).

For a modified intersection density, reverse inequalities have been proved by Hug and Schneider [HuS11]. For X as above, define Γ_k as the supremum of $\chi_k(\Lambda X)\chi_{d-1}(\Lambda X)^{-k}$ over all nondegenerate linear transformations Λ. For $k \in \{2, \dots, d\}$, the minimum of Γ_k is attained if and only if the hyperplanes of X attain almost surely only d directions.

Intersection densities can also be considered for stationary Poisson r-flats, for example for $d/2 \le r \le d-1$ and intersections of any two r-flats. Here nonisotropic extremal cases occur, such as in the case $r = 2$, $d = 4$, solved by Mecke [Mec88]. Various other cases have been treated; see Mecke [Mec91], Keutel [Keu91], and the references given there.

For stationary Poisson r-flat processes with $1 \le r < d/2$, instead of intersection densities one can introduce a notion of ***proximity***, measuring how close the flats come to each other in the mean. A sharp inequality for this proximity was proved by Schneider [Sch99]. Very thorough investigations of stationary Poisson flat processes, regarding these and many other aspects, including limit theorems, are due to Schulte and Thäle [ScT14] and to Hug, Thäle, and Weil [HTW15].

In Heinrich [Hei09] one finds, together with references to earlier related work, a CLT for the total number of intersection points of a stationary Poisson hyperplane process within expanding convex sampling windows. As an application of very general results on U-statistics for Poisson point processes, such CLTs were obtained by Reitzner and Schulte [ReiS13].

12.4.2 INTERSECTIONS OF RANDOM HALFSPACES

Intersections of random halfspaces appear as solution sets of systems of linear inequalities with random coefficients. Therefore, such random polyhedra play a role in the average case analysis of linear programming algorithms (see the book by Borgwardt [Bor87] and its bibliography).

There are two standard ways to generate a convex polytope, either as the convex hull of finitely many points or as the intersection of finitely many closed halfspaces. We consider here the second way, with random halfspaces.

The following model is, in a heuristic sense, dual to that of the convex hull of i.i.d. random points in a convex body. For a given convex body K let $K_1 := K + B^d$ and let $\mathcal{H}_K$ be the set of hyperplanes meeting K_1 but not the interior of K. Let $\mu^{(K)}$ be the distribution on $\mathcal{H}_K$ arising from the motion invariant measure on the space of hyperplanes by restricting it to $\mathcal{H}_K$ and normalizing. Let $H_1, \ldots, H_n$ be independent random hyperplanes with distribution $\mu^{(K)}$, and let H_i^- be the closed halfspace bounded by H_i and containing K. Choose a polytope Q with $K \subset \operatorname{int} Q$, $Q \subset \operatorname{int} K_1$ and set $K^{(n)} := \bigcap_{i=1}^n H_i^- \cap Q$ (the intersection with Q makes $K^{(n)}$ bounded, and the choice of Q does not affect the asymptotic results). The following contributions concern this model. For K of class C_+^3, Kaltenbach [Kal90] obtained an asymptotic formula (as $n \to \infty$) for $\mathbb{E}V_d(K^{(n)}) - V_d(K)$. Fodor, Hug, and Ziebarth [FHZ16] proved such a limit result for the same functional under the sole assumption that K slides freely in a ball. They also obtained an asymptotic upper estimate for the variance and a corresponding law of large numbers. Böröczky and Schneider [BöS10] considered the mean width for this model and showed the existence of constants c_1, c_2, independent of n, such that

$$c_1 n^{-1} \log^{d-1} n < \mathbb{E}V_1(K^{(n)}) - V_1(K) < c_2 n^{-\frac{2}{d+1}}$$

for all sufficiently large n. For a simplicial polytope P, they also obtained precise asymptotic relations for $\mathbb{E}V_1(P^{(n)}) - V_1(P)$ and for $\mathbb{E}f_0(P^{(n)})$ and $\mathbb{E}f_{d-1}(P^{(n)})$.

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For an arbitrary convex body K, Böröczky, Fodor, and Hug [BöFH10] proved that

$$\lim_{n\to\infty} n^{\frac{2}{d+1}}\mathbb{E}\left(V_1(K^{(n)}) - V_1(K)\right) = c_d \int_{\partial K} \kappa^{\frac{d}{d+1}}\, dS$$

with an explicit constant c_d, where κ is the generalized Gauss–Kronecker curvature, and a similar relation for $\mathbb{E} f_{d-1}(K^{(n)})$.

A different model for circumscribed random polytopes is obtained in the following way. Let $X_1, \dots, X_n$ be i.i.d. random points on the boundary of a smooth convex body K. Let $K'_{(n)}$ be the intersection of the supporting halfspaces of K at $X_1, \dots, X_n$, and let $K_{(n)}$ be the intersection of $K'_{(n)}$ with some fixed large cube (to make it bounded). For $K = B^d$ and the uniform distribution on ∂K, Buchta [Buc87] found $\mathbb{E}\, f_0(K'_{(n)})$, asymptotically and explicitly. Put $D_{(j)}(K, n) := V_j(K_{(n)}) - V_j(K)$. Under the assumption that the distribution of the X_i has a positive density h and that K and h are sufficiently smooth, Böröczky and Reitzner [BöR04] obtained asymptotic expansions, as $n \to \infty$, for $\mathbb{E}D_{(j)}(K, n)$ in the cases $j = d$, $d - 1$, and 1.

Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of i.i.d. random points on ∂K, and let $K_{(n)}$ and h be defined as above. If ∂K is of class C^2 and positive curvature and h is positive and continuous, one may ask whether $n^{2/(d-1)} D_{(j)}(K, n)$ converges almost surely to a positive constant, if $n \to \infty$. For $d = 2$ and $j = 1, 2$, this was shown by Schneider [Sch88]. Reitzner [Rei02b] proved such a result for $d \geq 2$ and $j = d$. He deduced that random approximation, in this sense, is very close to best approximation.

Still another model of circumscribed random polytopes starts with a stationary Poisson hyperplane process X in $\mathbb{R}^d$ with a nondegenerate directional distribution. Given a convex body K, let $P_{X,K}$ be the intersection of all closed halfspaces bounded by hyperplanes of X and containing K. Hug and Schneider [HuS14] studied the behavior of the Hausdorff distance $\delta(K, P_{X,K})$ as the intensity of X tends to infinity. This behavior depends heavily on relations between the directional distribution of X and the surface area measure of K.

An interesting class of parametric random polytopes with rotation invariant distribution is obtained from an isotropic Poisson hyperplane process in $\mathbb{R}^n$, which depends on a distance exponent and an intensity, by taking the intersection of the closed halfspaces that are bounded by a hyperplane of the process and contain the origin. Various aspects of these random polytopes were investigated by Hörrmann, Hug, Reitzner, and Thäle [HöHRT15].

12.4.3 PROJECTIONS TO RANDOM FLATS

For combinatorial problems about tuples of random points in $\mathbb{R}^d$, the following approach leads to a natural distribution. Every configuration of $N + 1 > d$ labeled points in general position in $\mathbb{R}^d$ is affinely equivalent to the orthogonal projection of the set of labeled vertices of a fixed regular simplex $T^N \subset \mathbb{R}^N$ onto a unique d-dimensional linear subspace of $\mathbb{R}^N$. This establishes a one-to-one correspondence between the (orientation-preserving) affine equivalence classes of such configurations and an open dense subset of the Grassmannian $G(N, d)$ of oriented d-dimensional subspaces of $\mathbb{R}^N$. The unique rotation-invariant probability measure on $G(N, d)$ thus leads to a probability distribution on the set of affine equivalence classes of $(N + 1)$-tuples of points in general position in $\mathbb{R}^d$. References for this

Grassmann approach, which was proposed by Vershik and by Goodman and Pollack, are given in Affentranger and Schneider [AfS92]. Baryshnikov and Vitale [BaV94] proved that an affine-invariant functional of $(N+1)$-tuples with this distribution is stochastically equivalent to the same functional taken at an i.i.d. $(N+1)$-tuple of standard normal points in $\mathbb{R}^d$.

In particular, every d-polytope with $N+1 \geq d+1$ vertices is affinely equivalent to an orthogonal projection of an N-dimensional regular simplex. Similarly, every centrally symmetric d-polytope with $2N \geq 2d$ vertices is affinely equivalent to an orthogonal projection of an N-dimensional regular cross-polytope. This leads to further geometrically natural models for d-dimensional random polytopes. For this, we take a regular simplex, cube, or cross-polytope in $\mathbb{R}^N$, $N > d$, and project it orthogonally to a uniform random d-subspace of $\mathbb{R}^N$. For more information on this approach, including formulas for the expected k-face numbers of these random polytopes and their asymptotics, and for references, we refer to [ScW08, Section 8.3]. Note 7 of that section contains hints to important applications by Donoho and Tanner. Here it plays a role, for example, to choose d and k in dependence on N so that the random projections are k-neighborly with high probability. See also the later work [DoT10], which describes a particular application to compressed sensing.

12.4.4 RANDOM FLATS THROUGH CONVEX BODIES

The notion of a uniform random point in a convex body K in $\mathbb{R}^d$ is extended by that of a uniform random r-flat through K. A random r-flat in $\mathbb{R}^d$ is a measurable map from some probability space into $A(d,r)$. It is a ***uniform*** (***isotropic uniform***) random r-flat through K if its distribution can be obtained from a translation invariant (resp. rigid-motion invariant) measure on $A(d,r)$, by restricting it to the r-flats meeting K and normalizing to a probability measure. (For details, see [WeW93, Section 2].)

A random r-flat E (uniform or not) through K generates the random section $E \cap K$, which has often been studied, particularly for $r = 1$. References are in [ScWi93, Section 7] and [ScW08, Section 8.6]. Finitely many i.i.d. random flats through K lead to combinatorial questions. Associated random variables, such as the number of intersection points inside K if $d = 2$ and $r = 1$, are hard to attack; for work of Sulanke (1965) and Gates (1984) see [ScWi93].

Of special interest is the case of $n \leq d$ i.i.d. uniform hyperplanes $H_1, \dots, H_n$ through a convex body $K \subset \mathbb{R}^d$. Let $p_n(K,\psi)$ denote the probability that the intersection $H_1 \cap \dots \cap H_n$ also meets K; here ψ (an even probability measure on S^{d-1}) is the directional distribution of the hyperplanes. If ψ is rotation invariant, then $p_n(K,\psi) = n!\kappa_n V_n(K)/(2V_1(K))^n$. For references on this equality and some inequalities for $p_n(K,\psi)$ in general, we refer to [BaS95].

If $N > d$ i.i.d. uniform hyperplanes through K are given, they give rise to a random cell decomposition of $\operatorname{int} K$. For $k \in \{0,\dots,d\}$, the expected number, $\mathbb{E}\nu_k$, of k-dimensional cells of this decomposition was determined in [Sch82],

$$\mathbb{E}\nu_k = \sum_{n=d-k}^{d} \binom{n}{d-k}\binom{N}{n} p_n(K,\psi).$$

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12.5 RANDOM MOSAICS

By a ***tessellation*** of $\mathbb{R}^d$, or a ***mosaic*** in $\mathbb{R}^d$, we understand a collection of d-dimensional convex polytopes such that their union is $\mathbb{R}^d$, the intersection of the interiors of any two of the polytopes is empty, and any bounded set meets only finitely many of the polytopes. (For tessellations in general, we refer to [OkBSC00].) Except in Section 12.5.5, all considered mosaics are 'face-to-face,' that is, the intersection of any two of its polytopes is either empty or a face of each of them. A ***random mosaic*** can be modeled by a point process in the space of convex polytopes, such that the properties above are satisfied almost surely. (Alternatively, a random mosaic is often modeled as the random closed set given by the union of its cell boundaries.) For an introduction, we refer to [ScW08, Section 10.1]. Further general references are the basic article by Møller [Møl89] and the surveys given in [MeSSW90, Chapter 3], [WeW93, Section 7], and [ChSKM13, Chapter 9].

NOTATION

X	stationary random mosaic in $\mathbb{R}^d$
$X^{(k)}$	process of its k-dimensional faces, $k = 0, \dots, d$
$d_j^{(k)}$	density of the jth intrinsic volume of the polytopes in $X^{(k)}$
$\gamma^{(k)}$	$= d_0^{(k)}$, k-face intensity of X
$Z^{(k)}$	typical k-face of X
n_{jk}	mean number of k-faces of the typical (j,k)-face star (j-face together with the set of its adjacent k-faces)
Z_0	zero cell (the a.s. unique cell containing 0) of X

Under a natural assumption on the stationary random mosaic X, the notions of 'density' and 'typical' exist with a precise meaning. Here, we can only convey the intuitive idea that one averages over expanding bounded regions of the mosaic and performs a limit procedure. Exact definitions can be found in [ScW08]; we also refer to Chapter 15 of that book for all references missing below.

12.5.1 GENERAL MOSAICS

For arbitrary stationary random mosaics, there are a number of identities relating averages of combinatorial quantities. Basic examples are:

$$\sum_{k=0}^{j}(-1)^k n_{jk} = 1, \qquad \sum_{k=j}^{d}(-1)^{d-k} n_{jk} = 1, \qquad \gamma^{(j)} n_{jk} = \gamma^{(k)} n_{kj},$$
$$\sum_{i=j}^{d}(-1)^i d_j^{(i)} = 0, \qquad \text{and in particular} \qquad \sum_{i=0}^{d}(-1)^i \gamma^{(i)} = 0.$$

If the random mosaic X is normal, meaning that every k-face ($k \in \{0, \dots, d-1\}$) is contained in exactly $d-k+1$ d-polytopes of X, then

$$(1-(-1)^k)\gamma^{(k)} = \sum_{j=0}^{k-1}(-1)^j \binom{d+1-j}{k-j} \gamma^{(j)}$$

for $k = 1, \ldots, d$. Lurking in the background are the polytopal relations of Euler, Dehn–Sommerville, and Gram; for these, see Chapters 15 and 17 of this Handbook. For many special relations between the numbers $\gamma^{(i)}$ and n_{jk} in two and three dimensions, we refer to [ScW08, Thms. 10.1.6, 10.1.7].

12.5.2 VORONOI AND DELAUNAY MOSAICS

A discrete point set in $\mathbb{R}^d$ induces a Voronoi and a Delaunay mosaic (see Chapter 27 for the definitions). Starting from a stationary Poisson point process $\widetilde{X}$ in $\mathbb{R}^d$, one obtains in this way a stationary ***Poisson–Voronoi mosaic*** and ***Poisson–Delaunay mosaic***. Both of these are completely determined by the intensity, $\widetilde{\gamma}$, of the underlying Poisson process $\widetilde{X}$. For a Poisson–Voronoi mosaic and for $k \in \{0, \ldots, d\}$, one has

$$d_k^{(k)} = \frac{2^{d-k+1}\pi^{\frac{d-k}{2}}}{d(d-k+1)!} \frac{\Gamma\left(\frac{d^2-kd+k+1}{2}\right)\Gamma\left(1+\frac{d}{2}\right)^{d-k+\frac{k}{d}}\Gamma\left(d-k+\frac{k}{d}\right)}{\Gamma\left(\frac{d^2-kd+k}{2}\right)\Gamma\left(\frac{d+1}{2}\right)^{d-k}\Gamma\left(\frac{k+1}{2}\right)} \widetilde{\gamma}^{\frac{d-k}{d}}.$$

In particular, the vertex density is given by

$$\gamma^{(0)} = \frac{2^{d+1}\pi^{\frac{d-1}{2}}}{d^2(d+1)} \frac{\Gamma\left(\frac{d^2+1}{2}\right)}{\Gamma\left(\frac{d^2}{2}\right)} \left[\frac{\Gamma\left(1+\frac{d}{2}\right)}{\Gamma\left(\frac{d+1}{2}\right)}\right]^d \widetilde{\gamma}.$$

For many other parameters, their explicit values in terms of $\widetilde{\gamma}$ are known, especially in small dimensions. We refer to [ScW08, Thm. 10.2.5] for a list in two and three dimensions. The lecture notes of Møller [Møl89] are devoted to random Voronoi tessellations.

If a Voronoi mosaic X and a Delaunay mosaic Y are induced by the same stationary Poisson point process, then the k-face intensities $\beta^{(k)}$ of Y and $\gamma^{(k)}$ of X are related by $\beta^{(j)} = \gamma^{(d-j)}$ for $j = 0, \ldots, d$, by duality. For stationary Poisson–Delaunay mosaics in the plane, some explicit parameter values are shown in [ScW08, Thm. 10.2.9].

A generalization of the Voronoi mosaics is provided by Laguerre mosaics, where the generating points are endowed with weights. A systematic study of Laguerre mosaics generated by stationary Poisson point processes is made by Lautensack and Zuyev [LaZ08].

OPEN PROBLEM

PROBLEM 12.5.1 *Find the k-face intensities $\gamma^{(k)}$ of stationary Poisson–Voronoi mosaics explicitly in all dimensions.*

12.5.3 HYPERPLANE TESSELLATIONS

A random mosaic X is called a ***stationary hyperplane tessellation*** if it is induced, in the obvious way, by a stationary hyperplane process (as defined in Section 12.4.1). Such random mosaics have special properties. Under an assumption of gen-

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eral position, they satisfy, for $0 \leq j \leq k \leq d$,

$$d_j^{(k)} = \binom{d-j}{d-k} d_j^{(j)}, \quad \text{in particular} \quad \gamma^{(k)} = \binom{d}{k} \gamma^{(0)},$$

and

$$n_{kj} = 2^{k-j} \binom{k}{j}$$

(see, e.g., [ScW08, Thm. 10.3.1]).

A stationary Poisson hyperplane process $\widehat{X}$, satisfying a suitable assumption of nondegeneracy, induces a stationary random mosaic X, called a ***Poisson hyperplane mosaic***. The process $\widehat{X}$ is determined, up to stochastic equivalence, by its spherical directional distribution $\widehat{\varphi}$, an even probability measure on the unit sphere S^{d-1}, and the intensity $\widehat{\gamma} > 0$. Several parameters of X can be expressed in terms of the ***associated zonoid*** $\Pi_{\widehat{X}}$, which is the convex body defined by its support function,

$$h(\Pi_{\widehat{X}}, u) = \frac{\widehat{\gamma}}{2} \int_{S^{d-1}} |\langle u, v \rangle| \, \widehat{\varphi}(dv), \quad u \in S^{d-1}.$$

Examples are

$$d_j^{(k)} = \binom{d-j}{d-k} \binom{d}{j} V_{d-j}(\Pi_{\widehat{X}}),$$

in particular,

$$\gamma^{(k)} = \binom{d}{k} V_d(\Pi_{\widehat{X}}).$$

In the isotropic case (where the distributions are invariant under rigid motions), the associated zonoid is a ball of radius depending only on $\widehat{\gamma}$ and the dimension. For the preceding relations and for further information, we refer to [ScW08, Sec10.3].

12.5.4 TYPICAL CELLS AND FACES

A stationary random mosaic gives rise to several interesting random polytopes. The almost surely unique cell containing the origin is called the ***zero cell*** (or Crofton cell, in the case of a hyperplane process). The ***typical cell*** is obtained, heuristically, by choosing at random, with equal chances, one of the cells of the mosaic within a large bounded region of the mosaic and translating it so that a suitable center lies at the origin. See also the explanations given in [Cal10, Sec. 5.1.2] and [Cal13, Sec. 6.1.2]. A precise definition avoiding ergodic theory can make use either of the grain distribution of a stationary particle process (see, e.g., [ScW08, Sec. 10.1]) or of Palm distributions (see, e.g., [Sch09]). Similarly the typical k-face can be defined, as well as the weighted typical k-face, where, heuristically, the chance to be chosen is proportional to the k-dimensional volume. The weighted d-cell is, up to translations, stochastically equivalent to the zero cell (e.g., [ScW08, Thm. 10.4.1]). In special cases, information on the distributions is available, so for the typical cell of a stationary Poisson–Delaunay mosaic ([ScW08, Thm. 10.4.4]) or of a stationary Poisson hyperplane mosaic ([ScW08, Thms. 10.4.5, 10.4.6]); less explicitly also for the weighted typical k-face ([Sch09, Thm. 1]).

For the typical k-face $Z^{(k)}$ and the weighted typical k-face $Z_0^{(k)}$ of a stationary hyperplane tessellation, we mention a few aspects of particular geometric interest.

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For a mosaic generated by a stationary Poisson hyperplane process $\widehat{X}$ and for $0 \le j \le k \le d$,

$$\mathbb{E}V_j(Z^{(k)}) = \frac{\binom{d-j}{d-k} V_{d-j}(\Pi_{\widehat{X}})}{\binom{d}{k} V_d(\Pi_{\widehat{X}})},$$

as follows from [ScW08, (10.3), (10.43), (10.44)].

For a mosaic generated by a stationary hyperplane process satisfying only some general position and finiteness conditions, one of the results mentioned in Section 12.5.3 says that

$$\mathbb{E}f_j(Z^{(k)}) = 2^{k-j}\binom{k}{j}, \quad 0 \le j \le k \le d,$$

independent of the distribution. Let X be a stationary Poisson hyperplane tessellation. Then the variance of the vertex number of the typical k-face of X satisfies

$$0 \le \operatorname{Var} f_0(Z^{(k)}) \le \left(2^k k! \sum_{j=0}^{k} \frac{\kappa_j^2}{4^j (k-j)!}\right) - 2^{2k}.$$

For $k \ge 2$, equality on the left side holds if and only if X is a parallel mosaic (i.e., the hyperplanes of X belong to d translation classes), and on the right side if (and for $k = d$ only if) X is isotropic with respect to a suitable scalar product on $\mathbb{R}^d$ ([Sch16]). For weighted faces one has, for $k \in \{2, \dots, d\}$,

$$2^k \le \mathbb{E}f_0(Z_0^{(k)}) \le 2^{-k} k! \kappa_k^2,$$

with equality on the left if and only if X is a parallel mosaic and on the right if X is isotropic ([Sch09]).

For the zero cell Z_0 of a stationary, isotropic Poisson hyperplane mosaic, there is a linear relation between $\mathbb{E}f_{d-k}(Z_0)$ and $\mathbb{E}V_k(Z_0)$ (see [Sch09, p. 693]), but more information on these expectations is desirable.

A well-known problem of D.G. Kendall, dating back to the 1940s, asked for the asymptotic shape of the zero cell of a planar isotropic Poisson line mosaic under the condition that is has large area. A solution was found by Kovalenko (1997). We mention here the first general higher-dimensional result of this type. Let $\widehat{X}$ be a stationary Poisson hyperplane process in $\mathbb{R}^d$ with intensity $\widehat{\gamma} > 0$ and spherical directional distribution $\widehat{\varphi}$. Let $B_{\widehat{X}}$ be the Blaschke body of $\widehat{X}$, that is, the unique origin symmetric convex body for which $\widehat{\varphi}$ is the area measure. The deviation of a convex body K from the homothety class of $B_{\widehat{X}}$ is measured by

$$r_B(K) := \inf\{s/r - 1 : rB + z \subset K \subset sB + z,\, z \in \mathbb{R}^d,\, r, s > 0\}.$$

Let Z_0 be the zero cell of the mosaic induced by $\widehat{X}$. Then there is a positive constant c_0 depending only on $B_{\widehat{X}}$ such that for $\varepsilon \in (0,1)$ and $a^{\frac{1}{d}}\widehat{\gamma} \ge \sigma_0 > 0$ one has

$$\mathbb{P}\left(r_B(Z_0) \ge \varepsilon \mid V_d(Z_0) \ge a\right) \le c \exp\left\{-c_0\, \varepsilon^{d+1} a^{\frac{1}{d}} \widehat{\gamma}\right\},$$

where c is a constant depending on $B_{\widehat{X}}, \varepsilon, \sigma_0$. In particular, the conditional probability $\mathbb{P}(r_B(Z_0) \ge \varepsilon \mid V_d(Z_0) \ge a\}$ tends to zero for $a \to \infty$. This was proved by Hug, Reitzner, and Schneider (2004). Many similar results have been obtained, for different size measures, also for Poisson–Voronoi and Poisson–Delaunay tessellations, and for typical and weighted typical faces of lower dimensions. For more information and for references, we refer to [ScW08, pp. 512–514] and to the more recent survey [Sch13, Sections 5–7].

12.5.5 STIT TESSELLATIONS

The two most prominent (because tractable) models of random mosaics, the Poisson–Voronoi mosaic and the Poisson hyperplane tessellation, have in recent years been supplemented by a third one, that of STIT tessellation (the name stands for 'stable under iteration'). A comparison of the three models in two and three dimensions was made by Redenbach and Thäle [ReT13]. We roughly indicate the idea behind the STIT tessellations. Let X and Y be two random mosaics in $\mathbb{R}^n$. For $k \in \mathbb{N}$, let Y_k be a random mosaic stochastically equivalent to Y such that $X, Y_1, Y_2, \dots$ are independent. Choose a (measurable) numeration of the cells of X and replace the kth cell of X by its intersections with the cells of Y_k, for $k \in \mathbb{N}$. This yields the cells of a new random mosaic (which is not face-to-face), denoted by $X \lhd Y$. Define

$$\underbrace{X \lhd \cdots \lhd X}_{m+1} := (\underbrace{X \lhd \cdots \lhd X}_{m}) \lhd X \quad \text{and} \quad I_m(X) := m \cdot (\underbrace{X \lhd \cdots \lhd X}_{m})$$

for $m = 2, 3, \dots$ A stationary random mosaic X is called ***stable with respect to iteration*** (***STIT***) if $I_m(X)$ has the same distribution as X, for $m = 2, 3, \dots$

According to Nagel and Weiss [NaW05], the starting point of their existence proof for stationary STIT tessellations can be described as follows. Let Λ be a nondegenerate, locally finite, translation invariant measure on $A(d, d-1)$, and let $W \subset \mathbb{R}^d$ be a compact set, let $[W] := \{H \in A(d, d-1) : H \cap W \neq \emptyset\}$ and assume that $0 < \Lambda([W]) < \infty$. The window W gets an exponentially distributed 'lifetime', with a parameter depending on $\Lambda([W])$. At the end of this lifetime, W is divided into two cells, by a random hyperplane with distribution $\Lambda(\cdot \cap [W])/\Lambda([W])$. With each of these cells, the splitting process starts again, independent of each other and of the previous splittings. This procedure of repeated cell division (or 'cracking') is stopped at a fixed time $a > 0$, yielding a random tessellation $R(a, W)$ of W (the cells of which are intersections of W with convex polytopes). It is proved in [NaW05] that for any $a > 0$ there exists a random tessellation $Y(a)$ of $\mathbb{R}^d$ such that $Y(a) \cap W$ is equal in distribution to $R(a, W)$, for all windows W satisfying the assumptions. It is also proved that $Y(a)$ is a stationary STIT tessellation, called the crack STIT tessellation determined by Λ and a.

Variants of the construction of STIT tessellations were described by Mecke, Nagel, and Weiss [MeNW08a], [MeNW08b]. In recent years, various aspects of STIT tessellations have been investigated thoroughly. By way of example for the extensive later developments, we refer to [ScT13a], [ScT13b] and the references given there.

12.6 SOURCES AND RELATED MATERIAL

SOURCES FOR STOCHASTIC GEOMETRY IN GENERAL

Matheron [Math75]: A monograph on basic models of stochastic geometry and applications of integral geometry.

Preliminary version (August 10, 2017). To appear in the Handbook of Discrete and Computational Geometry, J.E. Goodman, J. O'Rourke, and C. D. Tóth (editors), 3rd edition, CRC Press, Boca Raton, FL, 2017.

Santaló [San76]: The classical work on integral geometry and its applications to geometric probabilities.

Solomon [Sol78]: A selection of topics from geometric probability theory.

Ambartzumian [Amb90]: A special approach to stochastic geometry via factorization of measures, with various applications.

Weil and Wieacker [WeW93]: A comprehensive handbook article on stochastic geometry.

Klain and Rota [KlR97]: An introduction to typical results of integral geometry, their interpretations in terms of geometric probabilities, and counterparts of discrete and combinatorial character.

Mathai [Mat99]: A comprehensive collection of results on geometric probabilities, with extensive references.

Schneider and Weil [ScW08]: An introduction to the mathematical models of stochastic geometry, with emphasis on the application of integral geometry and functionals from convexity.

Chiu, Stoyan, Kendall, and Mecke [ChSKM13]: Third edition of a monograph on theoretical foundations and applications of stochastic geometry.

RELEVANT SURVEYS

Buchta [Buc85]: An early survey on random polytopes.

Schneider [Sch88], Affentranger [Aff92]: Surveys on approximation of convex bodies by random polytopes.

Bauer and Schneider [BaS95]: Information on inequalities and extremum problems for geometric probabilities.

Gruber [Gru97], Schütt [Schü02]: Surveys comparing best and random approximation of convex bodies by polytopes.

Bárány [Bár00], [Bár07], [Bár08]: Surveys containing (among other material) introductions to the floating body and cap covering method and their applications to random polytopes.

Hug [Hug07], Calka [Cal10], [Cal13]: Surveys on random mosaics.

Schneider [Sch08], Reitzner [Rei1], Hug [Hug13]: Surveys on random polytopes.

Schneider [Sch13]: Survey on extremal problems related to random mosaics.

RELATED CHAPTERS

Chapter 1: Finite point configurations
Chapter 2: Packing and covering
Chapter 15: Basic properties of convex polytopes
Chapter 17: Face numbers of polytopes and complexes
Chapter 22: Random simplicial complexes
Chapter 27: Voronoi diagrams and Delaunay triangulations
Chapter 44: Randomization and derandomization

REFERENCES

[Aff88] F. Affentranger. The expected volume of a random polytope in a ball. *J. Microscopy,* 151:277–287, 1988.

[Aff91] F. Affentranger. The convex hull of random points with spherically symmetric distributions. *Rend. Sem. Mat. Univ. Politec. Torino*, 49:359–383, 1991.

[Aff92] F. Affentranger. Aproximación aleatoria de cuerpos convexos. *Publ. Mat.*, 36:85–109, 1992.

[AfS92] F. Affentranger and R. Schneider. Random projections of regular simplices. *Discrete Comput. Geom.*, 7:219–226, 1992.

[AfW91] F. Affentranger and J.A. Wieacker. On the convex hull of uniform random points in a simple d-polytope. *Discrete Comput. Geom.*, 6:291–305, 1991.

[AlFGP91] D.J. Aldous, B. Fristedt, P.S. Griffin, and W.E. Pruitt. The number of extreme points in the convex hull of a random sample. *J. Appl. Probab.*, 28:287–304, 1991.

[Amb90] R.V. Ambartzumian. *Factorization Calculus and Geometric Probability.* Volume 33 of *Encyclopedia Math. Appl.,* Cambridge University Press, 1990.

[AmB09] G. Ambrus and I. Bárány. Longest convex chains. *Random Structures Algorithms*, 35:137–162, 2009.

[BaGS13] J. Balogh, H. González-Aguilar, and G. Salazar. Large convex holes in random point sets. *Comput. Geom.*, 46:725–733, 2013.

[Bár89] I. Bárány. Intrinsic volumes and f-vectors of random polytopes. *Math. Ann.*, 285:671–699, 1989.

[Bár92] I. Bárány. Random polytopes in smooth convex bodies. *Mathematika*, 39:81–92, 1992.

[Bár99] I. Bárány. Sylvester's question: the probability that n points are in convex position. *Ann. Probab.*, 27:2020–2034, 1999.

[Bár00] I. Bárány. The technique of M-regions and cap covering: a survey. *Rend. Circ. Mat. Palermo, Ser. II, Suppl.*, 65:21–38, 2000.

[Bár01] I. Bárány. A note on Sylvester's four-point problem. *Studia Sci. Math. Hungar.*, 38:73–77, 2001.

[Bár07] I. Bárány. Random polytopes, convex bodies, and approximation. In: A.J. Baddeley, I. Bárány, R. Schneider, and W. Weil, *Stochastic Geometry. C.I.M.E. Summer School*, vol. 1892 of *Lecture Notes in Math.*, page 77–118, Springer, Berlin, 2007.

[Bár08] I. Bárány. Random points and lattice points in convex bodies. *Bull. Amer. Math. Soc.*, 45:339–365, 2008.

[BáB93] I. Bárány and C. Buchta. Random polytopes in a convex polytope, independence of shape, and concentration of vertices. *Math. Ann.*, 297:467–497, 1993.

[BáFV10] I. Bárány, F. Fodor, and V. Vígh. Intrinsic volumes of inscribed random polytopes in smooth convex bodies. *Adv. Appl. Prob. (SGSA)*, 42:605–619, 2010.

[BáF87] I. Bárány and Z. Füredi. Empty simplices in Euclidean space. *Canad. Math. Bull.*, 30:436–445, 1987.

[BáF88] I. Bárány and Z. Füredi. On the shape of the convex hull of random points. *Probab. Theory Related Fields*, 77:231–240, 1988.

[BáHRS17] I. Bárány, D. Hug, M. Reitzner, and R. Schneider. Random points in halfspheres. *Random Structures Algorithms*, 50:3–22, 2017.

Preliminary version (August 10, 2017). To appear in the Handbook of Discrete and Computational Geometry, J.E. Goodman, J. O'Rourke, and C. D. Tóth (editors), 3rd edition, CRC Press, Boca Raton, FL, 2017.

[BáL88] I. Bárány and D.G. Larman. Convex bodies, economic cap coverings, random polytopes. *Mathematika*, 35:274–291, 1988.

[BáMR13] I. Bárány, J.-F. Marckert, and M. Reitzner. Many empty triangles have a common edge. *Discrete Comput. Geom.*, 50:244–252, 2013.

[BáR10a] I. Bárány and M. Reitzner. Poisson polytopes. *Ann. Probab.*, 38:1507–1531, 2010.

[BáR10b] I. Bárány and M. Reitzner. On the variance of random polytopes. *Adv. Math.*, 225:1986–2001, 2010.

[BáRSZ00] I. Bárány, G. Rote, W. Steiger, and C.-H. Zhang. A central limit theorem for convex chains in the square. *Discrete Comput. Geom.*, 23:35–50, 2000.

[BáS94] I. Bárány and W. Steiger. On the expected number of k-sets. *Discrete Comput. Geom.*, 11:243–263, 1994.

[BáV93] I. Bárány and R.A. Vitale. Random convex hulls: floating bodies and expectations. *J. Approx. Theory*, 75:130–135, 1993.

[BáV07] I. Bárány and V. Vu. Central limit theorems for Gaussian polytopes. *Ann. Probab.*, 35:1593–1621, 2007.

[BáV08] I. Bárány and V. Vu. Correction: "Central limit theorems for Gaussian polytopes." *Ann. Probab.*, 36:1998, 2008.

[BaV94] Y.M. Baryshnikov and R.A. Vitale. Regular simplices and Gaussian samples. *Discrete Comput. Geom.*, 11:141–147, 1994.

[BaS95] C. Bauer and R. Schneider. Extremal problems for geometric probabilities involving convex bodies. *Adv. Appl. Prob.*, 27:20–34, 1995.

[BeR15] M. Beermann and M. Reitzner. Beyond the Efron–Buchta identities: Distributional results for Poisson polytopes. *Discrete Comput. Geom.*, 53:226–244, 2015.

[Bor87] K.H. Borgwardt. *The Simplex Method: A Probabilistic Analysis*. Springer-Verlag, Berlin, 1987.

[BöFH10] K.J. Böröczky, F. Fodor, and D. Hug. The mean width of random polytopes circumscribed around a convex body. *J. London Math. Soc. (2)*, 81:499–523, 2010.

[BöFH13] K.J. Böröczky, F. Fodor, and D. Hug. Intrinsic volumes of random polytopes with vertices on the boundary of a convex body. *Trans. Amer. Math. Soc.*, 365:785–809, 2013.

[BöFRV09] K.J. Böröczky, F. Fodor, M. Reitzner, and V. Vígh. Mean width of inscribed random polytopes in a reasonably smooth convex body. *J. Multivariate Anal.*, 100:2287–2295, 2009.

[BöHH08] K.J. Böröczky, L.M. Hoffmann, and D. Hug. Expectation of intrinsic volumes of random polytopes. *Periodica Math. Hungar.*, 57:143–164, 2008.

[BöR04] K.J. Böröczky and M. Reitzner. Approximation of smooth convex bodies by random circumscribed polytopes. *Ann. Appl. Prob.*, 14:239–273, 2004.

[BöS10] K.J. Böröczky and R. Schneider. The mean width of circumscribed random polytopes. *Canad. Math. Bull.*, 53:614–628, 2010.

[BrH98] H. Bräker and T. Hsing. On the area and perimeter of a random convex hull in a bounded convex set. *Probab. Theory Related Fields*, 111:517–550, 1998.

[BrHB98] H. Bräker, T. Hsing, and N.H. Bingham. On the Hausdorff distance between a convex set and an interior random convex hull. *Adv. Appl. Prob.*, 30:295–316, 1998.

[Buc84a] C. Buchta. Zufallspolygone in konvexen Vielecken. *J. Reine Angew. Math.*, 347:212–220, 1984.

[Buc84b] C. Buchta. Das Volumen von Zufallspolyedern im Ellipsoid. *Anz. Österreich. Akad. Wiss. Math.-Natur. Kl.*, 121:1–4, 1984.

[Buc85] C. Buchta. Zufällige Polyeder—Eine Übersicht. In: E. Hlawka, editor, *Zahlentheoretische Analysis*, vol. 1114 of *Lecture Notes in Math.*, pages 1–13, Springer-Verlag, Berlin, 1985.

[Buc87] C. Buchta. On the number of vertices of random polyhedra with a given number of facets. *SIAM J. Algebraic Discrete Methods*, 8:85–92, 1987.

[Buc90] C. Buchta. Distribution-independent properties of the convex hull of random points. *J. Theoret. Probab.*, 3:387–393, 1990.

[Buc05] C. Buchta. An identity relating moments of functionals of convex hulls. *Discrete Comput. Geom.*, 33:125–142, 2005.

[Buc06] C. Buchta. The exact distribution of the number of vertices of a random convex chain. *Mathematika*, 53:247–254, 2006.

[Buc12] C. Buchta. On the boundary structure of the convex hull of random points. *Adv. Geom.*, 12:179–190, 2012.

[Buc13] C. Buchta. Exact formulae for variances of functionals of convex hulls. *Adv. Appl. Prob. (SGSA)*, 45:917–924, 2013.

[BuM84] C. Buchta and J. Müller. Random polytopes in a ball. *J. Appl. Probab.*, 21:753–762, 1984.

[BuR97a] C. Buchta and M. Reitzner. Equiaffine inner parallel curves of a plane convex body and the convex hulls of randomly chosen points. *Probab. Theory Related Fields*, 108:385–415, 1997.

[BuR97b] C. Buchta and M. Reitzner. On a theorem of G. Herglotz about random polygons. *Rend. Circ. Mat. Palermo, Ser. II, Suppl.*, 50:89–102, 1997.

[BuR01] C. Buchta and M. Reitzner. The convex hull of random points in a tetrahedron: Solution of Blaschke's problem and more general results. *J. Reine Angew. Math.*, 536:1–29, 2001.

[CaG94] A.J. Cabo and P. Groeneboom. Limit theorems for functionals of convex hulls. *Probab. Theory Related Fields*, 100:31–55, 1994.

[Cal10] P. Calka. Tessellations. In: W.S. Kendall and I. Molchanov, editors, *New Perspectives in Stochastic Geometry*, pages 145–169, Oxford University Press, 2010.

[Cal13] P. Calka. Asymptotic methods for random tessellations. In: E. Spodarev, editor, *Stochastic Geometry, Spatial Statistics and Random Fields: Asymptotic Methods*, vol. 2068 of *Lecture Notes in Math.*, pages 183–204, Springer, Berlin, 2013.

[CaS06] P. Calka and T. Schreiber. Large deviation probabilities for the number of vertices of random polytopes in the ball. *Adv. Appl. Prob. (SGSA)*, 38:47–58, 2006.

[CaY14] P. Calka and J.E. Yukich. Variance asymptotics for random polytopes in smooth convex bodies. *Probab. Theory Relat. Fields*, 158:435–463, 2014.

[Car70] H. Carnal. Die konvexe Hülle von n rotations-symmetrisch verteilten Punkten. *Z. Wahrscheinlichkeitsth. Verwandte Geb.*, 15:168–178, 1970.

[ChSKM13] S.N. Chiu, D. Stoyan, W.S. Kendall, and J. Mecke. *Stochastic Geometry and its Applications.* 3rd ed., John Wiley & Sons, Chichester, 2013.

[Chu93] D.P.T. Chu. Random r-content of an r-simplex from beta-type-2 random points. *Canad. J. Statist.*, 21:285–293, 1993.

[Cow07] R. Cowan. Identities linking volumes of convex hulls. *Adv. Appl. Prob. (SGSA)*, 39:630–644, 2007.

[Cow10] R. Cowan. Recurrence relationships for the mean number of faces and vertices for random convex hulls. *Discrete Comput. Geom.*, 43:209–220, 2010.

[DeGG+13] O. Devillers, M. Glisse, X. Goaoc, G. Moroz, and M. Reitzner. The monotonicity of f-vectors of random polytopes. *Electron. Commun. Probab.*, 18:23, 2013.

[Dev91] L. Devroye. On the oscillation of the expected number of extreme points of a random set. *Statist. Probab. Lett.*, 11:281–286, 1991.

[DoT10] D.L. Donoho and J. Tanner. Counting the faces of randomly-projected hypercubes and orthants, with applications. *Discrete Comput. Geom.*, 43:522–541, 2010.

[DüW96] L. Dümbgen and G. Walther. Rates of convergence for random approximations of convex sets. *Adv. Appl. Prob.*, 28:384–393, 1996.

[Dwy91] R.A. Dwyer. Convex hulls of samples from spherically symmetric distributions. *Discrete Appl. Math.*, 31:113–132, 1991.

[Efr65] B. Efron. The convex hull of a random set of points. *Biometrika*, 52:331–343, 1965.

[FaHM15] R. Fabila-Monroy, C. Huemer, and D. Mitsche. Empty non-convex and convex fourgons in random point sets. *Studia Sci. Math. Hungar.*, 52:52–64, 2015.

[FiH04] S. Finch and I. Hueter. Random convex hulls: a variance revisited. *Adv. Appl. Prob. (SGSA)*, 36:981–986, 2004.

[FHZ16] F. Fodor, D. Hug, and I. Ziebarth. The volume of random polytopes circumscribed araound a convex body. *Mathematika*, 62:283–306, 2016.

[GlS96] S. Glasauer and R. Schneider. Asymptotic approximation of smooth convex bodies by polytopes. *Forum Math.*, 8:363–377, 1996.

[Gro88] P. Groeneboom. Limit theorems for convex hulls. *Probab. Theory Related Fields*, 79:327–368, 1988.

[Gro12] P. Groeneboom. Convex hulls of uniform samples from a convex polygon. *Adv. Appl. Prob. (SGSA)*, 44:330–342, 2012.

[Gru96] P.M. Gruber. Expectation of random polytopes. *Manuscripta Math.*, 91:393–419, 1996.

[Gru97] P.M. Gruber. Comparisons of best and random approximation of convex bodies by polytopes. *Rend. Circ. Mat. Palermo, Ser. II, Suppl.*, 50:189–216, 1997.

[Has11] E. Hashorva. Asymptotics of the convex hull of spherically symmetric samples. *Discrete Appl. Math.*, 159:201–211, 2011.

[Hei09] L. Heinrich. Central limit theorems for motion-invariant Poisson hyperplanes in expanding convex bodies. *Rend. Circ. Mat. Palermo, Ser. II, Suppl.*, 81:187–212, 2009.

[HöHRT15] J. Hörrmann, D. Hug, M. Reitzner, and Ch. Thäle. Poisson polyhedra in high dimensions. *Adv. Math.* 281:1–39, 2015.

[Hsi94] T. Hsing. On the asymptotic distribution of the area outside a random convex hull in a disk. *Ann. Appl. Probab.*, 4:478–493, 1994.

[Hue94] I. Hueter. The convex hull of a normal sample. *Adv. Appl. Prob.*, 26:855–875, 1994.

[Hue99] I. Hueter. Limit theorems for the convex hull of random points in higher dimensions. *Trans. Amer. Math. Soc.*, 351:4337–4363, 1999.

[Hug07] D. Hug. Random mosaics. In: A.J. Baddeley, I. Bárány, R. Schneider, and W. Weil, *Stochastic Geometry. C.I.M.E. Summer School*, vol. 1892 of *Lecture Notes in Math.*, pages 247–266, Springer, Berlin, 2007.

Preliminary version (August 10, 2017). To appear in the Handbook of Discrete and Computational Geometry, J.E. Goodman, J. O'Rourke, and C. D. Tóth (editors), 3rd edition, CRC Press, Boca Raton, FL, 2017.

[Hug13] D. Hug. Random polytopes. In: E. Spodarev, editor, *Stochastic Geometry, Spatial Statistics and Random Fields: Asymptotic Methods*, vol. 2068 of *Lecture Notes in Math.*, pages 205–238, Springer, Berlin, 2013.

[HuMR04] D. Hug, G.O. Munsonius, and M. Reitzner. Asymptotic mean values of Gaussian polytopes. *Beiträge Algebra Geom.*, 45:531–548, 2004.

[HuR05] D. Hug and M. Reitzner. Gaussian polytopes: variance and limit theorems. *Adv. Appl. Prob. (SGSA)*, 37:297–320, 2005.

[HuS11] D. Hug and R. Schneider. Reverse inequalities for zonoids and their application. *Adv. Math.*, 228:2634–2646, 2011.

[HuS14] D. Hug and R. Schneider. Approximation properties of random polytopes associated with Poisson hyperplane processes. *Adv. Appl. Prob. (SGSA)*, 46:919–936, 2014.

[HTW15] D. Hug, C. Thäle, and W. Weil. Intersection and proximity of processes of flats. *J. Math. Anal. Appl.*, 426:1–42, 2015.

[Kal90] F.J. Kaltenbach. Asymptotisches Verhalten zufälliger konvexer Polyeder. Dissertation, Univ. Freiburg i. Br., 1990.

[Keu91] J. Keutel. Ein Extremalproblem für zufällige Ebenen und für Ebenenprozesse in höherdimensionalen Räumen. Dissertation, Univ. Jena, 1991.

[KlR97] D.A. Klain and G.-C. Rota. *Introduction to Geometric Probability.* Cambridge University Press, 1997.

[Küf94] K.-H. Küfer. On the approximation of a ball by random polytopes. *Adv. Appl. Prob.*, 26:876–892, 1994.

[LaZ08] C. Lautensack and S. Zuyev. Random Laguerre tessellations. *Adv. Appl. Prob.*, 40:630–650, 2008.

[Mar17] J.-F. Marckert. Probability that n random points in a disk are in convex position. *Braz. J. Probab. Stat.*, 31:320–337, 2017.

[Mas99] B. Massé. On the variance of the number of extreme points of a random convex hull. *Statist. Probab. Lett.*, 44:123–130, 1999.

[Mas00] B. Massé. On the LLN for the number of vertices of a random convex hull. *Adv. Appl. Prob.*, 32:675–681, 2000.

[Mat99] A.M. Mathai. *An Introduction to Geometrical Probability: Distributional Aspects with Applications.* Gordon and Breach, Singapore, 1999.

[Math75] G. Matheron. *Random Sets and Integral Geometry.* John Wiley & Sons, Chichester, 1975.

[MaM07] M. Mayer and I. Molchanov. Limit theorems for the diameter of a random sample in the ball. *Extremes*, 10:129–150, 2007.

[Mec88] J. Mecke. An extremal property of random flats. *J. Microscopy*, 151:205–209, 1988.

[Mec91] J. Mecke. On the intersection density of flat processes. *Math. Nachr.*, 151:69–74, 1991.

[MeNW08a] J. Mecke, W. Nagel, and V. Weiss. A global construction of homogeneous random planar tessellations that are stable under iterations. *Stochastics*, 80:51–67, 2008.

[MeNW08b] J. Mecke, W. Nagel, and V. Weiss. The iteration of random tessellations and a construction of a homogeneous process of cell divisions. *Adv. Appl. Prob. (SGSA)*, 40:49–59, 2008.

[MeSSW90] J. Mecke, R. Schneider, D. Stoyan, and W. Weil. *Stochastische Geometrie.* Volume 16 of *DMV Sem.*, Birkhäuser, Basel, 1990.

[Mil70] R.E. Miles. On the homogeneous planar Poisson point process. *Math. Biosci.*, 6:85–127, 1970.

[Mil71] R.E. Miles. Poisson flats in Euclidean spaces. II: Homogeneous Poisson flats and the complementary theorem. *Adv. Appl. Prob.*, 3:1–43, 1971.

[MiP89] V.D. Milman and A. Pajor. Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed n-dimensional space. In: *Geometric Aspects of Functional Analysis*,pages 64–104, vol. 1376 of *Lecture Notes in Math.*, Springer, Berlin, 1989.

[Møl89] J. Møller. Random tessellations in $\mathbb{R}^d$. *Adv. Appl. Prob.*, 21:37–73, 1989.

[Møl89] J. Møller. *Lectures on Random Voronoi Tessellations.* Lecture Notes in Statistics 87, Springer, New York, 1994.

[NaK91] A.V. Nagaev and I.M. Khamdamov. Limit theorems for functionals of random convex hulls (in Russian). Preprint, Institute of Mathematics, Academy of Sciences of Uzbekistan, Tashkent, 1991.

[NaW05] W. Nagel and V. Weiss. Crack STIT tessellations: characterization of stationary random tessellations stable with respect to iteration. *Adv. Appl. Prob. (SGSA)*, 37:859–883, 2005.

[OkBSC00] A. Okabe, B. Boots, K. Sugihara, and S.N. Chiu. *Spatial Tessellations: Concepts and Applications of Voronoi Diagrams*, 2nd edition. John Wiley & Sons, Chichester, 2000.

[Par11] J. Pardon. Central limit theorems for random polygons in an arbitrary convex set. *Ann. Probab.*, 39:881–903, 2011.

[Par12] J. Pardon. Central limit theorems for uniform model random polygons. *J. Theor. Prob.*, 25:823–833, 2012.

[Rad12] L. Rademacher. On the monotonicity of the expected volume of a random simplex. *Mathematika*, 58:77–91, 2012.

[ReT13] C. Redenbach and Ch. Thäle. Second-order comparison of three fundamental tessellation models. *Statistics*, 47:237–257, 2013.

[Rei02a] M. Reitzner. Random points on the boundary of smooth convex bodies. *Trans. Amer. Math. Soc.*, 354:2243–2278, 2002.

[Rei02b] M. Reitzner. Random polytopes are nearly best approximating. *Rend. Circ. Mat. Palermo, Ser. II, Suppl.*, 70:263–278, 2002.

[Rei03] M. Reitzner. Random polytopes and the Efron-Stein jackknife inequality. *Ann. Probab.*, 31:2136–2166, 2003.

[Rei04] M. Reitzner. Stochastical approximation of smooth convex bodies. *Mathematika*, 51:11–29, 2004.

[Rei05a] M. Reitzner. The combinatorial structure of random polytopes. *Adv. Math.*, 191:178–208, 2005.

[Rei05b] M. Reitzner. Central limit theorems for random polytopes. *Probab. Theory Related Fields*, 133:483–507, 2005.

[Rei1] M. Reitzner. Random polytopes. In: W.S. Kendall and I. Molchanov, editors, *New Perspectives in Stochastic Geometry*, pages 45–76, Oxford University Press, 2010.

[ReiS13] M. Reitzner and M. Schulte. Central limit theorems for U-Statistics of Poisson point processes. *Ann. Probab.*, 41:3879–3909, 2013.

[RiVW08] R.M. Richardson, V. Vu, and L. Wu. An inscribing model for random polytopes. *Discrete Comput. Geom.*, 39:469–499, 2008.

Preliminary version (August 10, 2017). To appear in the Handbook of Discrete and Computational Geometry,
J.E. Goodman, J. O'Rourke, and C. D. Tóth (editors), 3rd edition, CRC Press, Boca Raton, FL, 2017.

[San76] L.A. Santaló. *Integral Geometry and Geometric Probability.* Volu. 1 of *Encyclopedia of Mathematics*, Addison–Wesley, Reading, 1976.

[Sch82] R. Schneider. Random hyperplanes meeting a convex body. *Z. Wahrsch. Verw. Gebiete*, 61:379–387, 1982.

[Sch87] R. Schneider. Approximation of convex bodies by random polytopes. *Aequationes Math.*, 32:304–310, 1987.

[Sch88] R. Schneider. Random approximation of convex sets. *J. Microscopy*, 151:211–227, 1988.

[Sch99] R. Schneider. A duality for Poisson flats. *Adv. Appl. Prob. (SGSA)*, 331:63–68, 1999.

[Sch08] R. Schneider. Recent results on random polytopes. *Boll. Un. Math. Ital. (9)*, 1:17–39, 2008.

[Sch09] R. Schneider. Weighted faces of Poisson hyperplane tessellations. *Adv. Appl. Prob. (SGSA)*, 41:682–694, 2009.

[Sch13] R. Schneider. Extremal properties of random mosaics. In: I. Bárány, K.J. Böröczky, G. Fejes Tóth, and J. Pach, editors, *Geometry — Intuitive, Discrete, and Convex: A Tribute to László Fejes Tóth*, pages 301–330, vol. 24 of *Bolyai Soc. Math. Studies*, Springer, Berlin, 2013.

[Sch14] R. Schneider. *Convex Bodies: The Brunn-Minkowski Theory*, second expanded edition. Vol. 151 of *Encyclopedia of Mathematics and Its Applications*, Cambridge University Press, 2014.

[Sch16] R. Schneider. Second moments related to Poisson hyperplane tessellations. *J. Math. Anal. Appl.*, 434:1365–1375, 2016.

[ScW08] R. Schneider and W. Weil. *Stochastic and Integral Geometry.* Springer, Berlin, 2008

[ScWi80] R. Schneider and J.A. Wieacker. Random polytopes in a convex body. *Z. Wahrsch. Verw. Gebiete*, 52:69–73, 1980.

[ScWi93] R. Schneider and J.A. Wieacker. Integral geometry. In P.M. Gruber and J.M. Wills, editors, *Handbook of Convex Geometry*, pages 1349–1390, Elsevier, Amsterdam, 1993.

[Schr02] T. Schreiber. Limit theorems for certain functionals of unions of random closed sets. *Teor. Veroyatnost. i Primenen.*, 47:130–142, 2002; *Theory Probab. Appl.*, 47:79–90, 2003.

[ScT13a] T. Schreiber and Ch. Thäle. Geometry of iteration stable tessellations: Connection with Poisson hyperplanes. *Bernoulli*, 19:1637–1654, 2013.

[ScT13b] T. Schreiber and Ch. Thäle. Limit theorems for iteration stable tessellations. *Ann. Probab.*, 41:2261–2278, 2013.

[ScY08] T. Schreiber and J.E. Yukich. Variance asymptotics and central limit theorems for generalized growth processes with applications to convex hulls and maximal points. *Ann. Probab.*, 36:363–396, 2008.

[ScT14] M. Schulte and Ch. Thäle. Distances between Poisson k-flats. *Methodol. Comput. Appl. Probab.*, 16:311–329, 2014.

[Schü94] C. Schütt. Random polytopes and affine surface area. *Math. Nachr.*, 170:227–249, 1994.

[Schü02] C. Schütt. Best and random approximation of convex bodies by polytopes. *Rend. Circ. Mat. Palermo, Ser. II, Suppl.*, (part II) 70:315-334, 2002.

[ScWe03] C. Schütt and E. Werner. Polytopes with vertices chosen randomly from the boundary of a convex body. In: GAFA Seminar Notes, vol. 1807 of *Lecture Notes in Math*, pages 241–422, Springer, Berlin, 2003.

[Sol78] H. Solomon. *Geometric Probability*. SIAM, Philadelphia, 1978.

[Val95] P. Valtr. Probability that n random points are in convex position. In: I. Bárány and J. Pach, editors, *The László Fejes Tóth Festschrift*. *Discrete Comput. Geom.*, 13:637–643, 1995.

[Val96] P. Valtr. The probability that n random points in a triangle are in convex position. *Combinatorica*, 16:567–573, 1996.

[Vu05] V. Vu. Sharp concentration of random polytopes. *Geom. Funct. Anal.*, 15:1284–1318, 2005.

[Vu06] V. Vu. Central limit theorems for random polytopes in a smooth convex set. *Adv. Math.*, 207:221–243, 2006.

[WaW01] U. Wagner and E. Welzl. A continuous analogue of the upper bound theorem. *Discrete Comput. Geom.*, 26:205–219, 2001.

[WeW93] W. Weil and J.A. Wieacker. Stochastic geometry. In P.M. Gruber and J.M. Wills, editors, *Handbook of Convex Geometry*, pages 1391–1438, Elsevier, Amsterdam, 1993.

[Zin03] A. Zinani. The expected volume of a tetrahedron whose vertices are chosen at random in the interior of a cube. *Monatsh. Math.*, 139:341–348, 2003.

Preliminary version (August 10, 2017). To appear in the Handbook of Discrete and Computational Geometry,
J.E. Goodman, J. O'Rourke, and C. D. Tóth (editors), 3rd edition, CRC Press, Boca Raton, FL, 2017.