On Crossings in Geometric Proximity Graphs

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Abstract

We study the number of crossings among edges of some higher order proximity graphs of the family of the Delaunay graph. That is, given a set P of n points in the Euclidean plane, we give lower and upper bounds on the minimum and the maximum number of crossings that these geometric graphs defined on P have.

1 Introduction

Let P be a set of n points in the plane in general position (no three are collinear). A *geometric graph on P* is a graph with vertex set P and such that its edges are drawn as straight-line segments. When two edges share an interior point we say that they give rise to a *crossing*.

The number of crossings is a parameter that has been attracting extensive studies in the context of combinatorial graphs. Given a graph G, the *crossing number of G*, denoted by $cr(G)$, is the minimum number of crossings in any drawing of G, i.e., in any non-degenerate representation of the graph in the plane. The *rectilinear crossing number of G*, denoted by $\overline{cr}(G)$, is the smallest number of crossings in any drawing of G in which the edges are represented by straight-line segments.

There are several classes of graphs for which the number of crossings has been computed or approximated. There are also several results sensitive to the size of the graph -particulary the famous crossing lemma [5, 12, 6]- and to the exclusion of some configurations [6, 13, 15, 19, 8].

The most famous problems in this setting are to obtain the exact values of the crossing number and the rectilinear crossing number of the complete graph K_n and the complete bipartite graph $K_{m,n}$. Recently some of these problems have received a great amount of attention and a continous chain of improvements has led to the bounds shown in Table 1.

	K_n	$K_{m,n}$
$cr \geq$	$0.8594\,\frac{1}{4}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n-2}{2}\rfloor\lfloor\frac{n-3}{2}\rfloor$ (for $n \to \infty$) [11]	$0.8594\,\frac{m}{m-1}\lfloor\frac{m-1}{2}\rfloor\lfloor\frac{m}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n}{2}\rfloor$ (for $n \to \infty$) [11]
$cr \leq$	$\frac{1}{4}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n-2}{2}\rfloor\lfloor\frac{n-3}{2}\rfloor$ [17]	$\lfloor\frac{m-1}{2}\rfloor\lfloor\frac{m}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n}{2}\rfloor$ [20]
$\overline{cr} \geq$	$0.379972\binom{n}{4} + \Theta(n^3)$ [4]	$0.8594\,\frac{m}{m-1}\lfloor\frac{m-1}{2}\rfloor\lfloor\frac{m}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n}{2}\rfloor$ (for $n \to \infty$) [11]
$\overline{cr} \leq$	$0.380488\binom{n}{4} + \Theta(n^3)$ [2]	$\lfloor\frac{m-1}{2}\rfloor\lfloor\frac{m}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n}{2}\rfloor$ [20]

Table 1: Current bounds on the crossing number and the rectilinear crossing number of K_n and $K_{m,n}$.

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In this paper the graphs under study are some higher order proximity graphs of the family of the Delaunay graph. In a proximity graph of a set of points in the plane two points are connected by a straight-line segment if some proximity rule is satisfied. There are several rules that have been proposed in the literature leading to different proximity graphs (see the survey [10]); here we focus on the k-nearest neighbour graph, the k-relative neighbourhood graph, the k-Gabriel graph and the k-Delaunay graph.

If P is a set of points in the plane, each of the specified proximity graphs is a geometric graph on P that has some number of crossings that will be denoted by $\boxtimes(\ )$. For example, consider the k-nearest neighbour graph of P (k-$\mathsf{NNG}(P)$ for short). We define the *lowest crossing number of* k-$\mathsf{NNG}(n)$ and, respectively, the *worst crossing number of* k-$\mathsf{NNG}(n)$ as

$$\begin{aligned}\mathsf{lcr}(k\text{-}\mathsf{NNG}(n)) &= \min_{|P|=n} \boxtimes(k\text{-}\mathsf{NNG}(P)),\\ \mathsf{wcr}(k\text{-}\mathsf{NNG}(n)) &= \max_{|P|=n} \boxtimes(k\text{-}\mathsf{NNG}(P)).\end{aligned}$$

We define analogous parameters for all the mentioned proximity graphs and give upper and lower bounds for all cases. We are only interested in $k > 1$ ($k > 2$ for k-$\mathsf{NNG}(P)$), because otherwise we recover the simplest versions of the graphs, which are known to be planar.

Observe that there is a subtle difference between the lowest crossing number and the rectilinear crossing number, as the former does not deal with purely combinatorial graphs, but with geometric ones. In fact, the lowest crossing number of a proximity graph equals the rectilinear crossing number of the drawings of all graphs that can be represented as this proximity graph.

An important part of our research has been devoted to look at particular cases in which k is small (see Sections 2 and 3). This decision is well-grounded, as it has been explained in the literature (for example, in [1] and [9]) that these are the most interesting cases from an applications point of view. In Section 4 we look at the number of crossings for large values of k. We point out that our results hold if, besides not containing three collinear points, P satisfies two more non-degeneracy assumptions: no four points are concyclic and, for each $p \in P$, the set of its k nearest points in P is well-defined.

Full proofs and examples can be found in [3].

2 Bounds on $\mathsf{lcr}(k\text{-}\mathsf{NNG}(n))$ for small values of k

We define the *k-nearest neighbour graph of P*, denoted by k-$\mathsf{NNG}(P)$, as the geometric graph on P in which two points p_i, p_j are joined by an edge if either p_j is one of the k nearest neighbours in P of p_i or viceversa.

In this section we provide bounds on the lowest crossing number of the k-nearest neighbour graph for small values of k. Because of the hierarchical relation satisfied by the graphs we investigate (see Section 4), the lower bounds also hold for the lowest crossing number of the other proximity graphs if we shift the value of k one unit down.

The results we have obtained are summarized in Table 2. Refer to [3] for a more detailed analysis. Due to space limitations, here we only explain the case $k = 10$.

The lower bound for $\mathsf{lcr}(10\text{-}\mathsf{NNG}(n))$ follows from the following result, proved in [14]:

Theorem 2.1. *The crossing number of any graph G with $v(G) \geq 3$ vertices and $e(G)$ edges satisfies*

$$cr(G) \geq \frac{7}{3}e(G) - \frac{25}{3}(v(G) - 2).$$

Indeed we know that the number of edges of the 10-nearest neighbour graph of P is greater than or equal to $5n$, as each vertex has degree at least 10. So we have

$$\boxtimes(10\text{-}\mathsf{NNG}(P)) \geq cr(10\text{-}\mathsf{NNG}(P)) \geq \frac{7}{3}e(10\text{-}\mathsf{NNG}(P)) - \frac{25}{3}\left(v(10\text{-}\mathsf{NNG}(P)) - 2\right) = \frac{10n}{3} + \frac{50}{3}.$$

k	$\mathsf{lcr}(k\text{-}\mathsf{NNG}(n))$
1	0
2	0
3	0
4	0, for $n \geq 14$
5	0, for $n \geq 44$
6	≤ 66, for $n \geq 84$
7	$n/2 + \Theta(1)$
8	$n + \Theta(1)$
9	$13n/6 + 50/3 \leq \mathsf{lcr} \leq 31n/13 + \Theta(1)$
10	$10n/3 + 50/3 \leq \mathsf{lcr} \leq 4n + \Theta(1)$

Table 2: $\mathsf{lcr}(k\text{-}\mathsf{NNG}(n))$ for the first values of k.

As for the upper bound, consider the configuration in Figure 1 (a precise description of the point coordinates can be found in [3]). An easy assignment of crossings to points shows that each point that is not in the boundary of the set or in the next layer causes four crossings. The points in the boundary and the next layer add $\Theta(\sqrt{n})$ crossings. Observe that the point positions can be slightly perturbed without modifying the ten nearest neighbours of each point. Thus, if we arrange the points in circular strips and each strip contains exactly the minimum number of points ensuring that the adjacencies in the 10-nearest neighbour graph do not change, the number of points in the boundary of the set becomes constant. Consequently, the 10-nearest neighbour graph of the resulting set of points has $4n + \Theta(1)$ crossings.

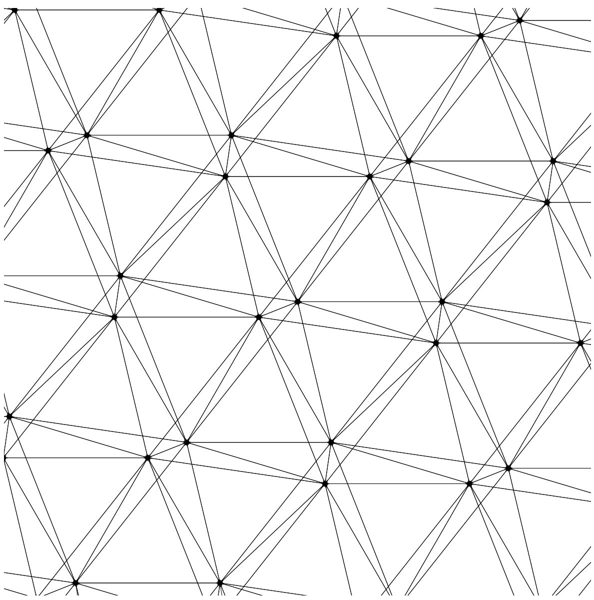

Figure 1: $10\text{-}\mathsf{NNG}(P)$ with $4n + \Theta(\sqrt{n})$ crossings.

3 1-Delaunay graphs

The *k-Delaunay graph* of P, which was introduced in [1] and will be denoted by $k\text{-}\mathsf{DG}(P)$, is the geometric graph with vertex set P, and an edge between points p_i and p_j if there exists a disk through p_i and p_j with at most k points of P in its interior.

We have already remarked that, from the viewpoint of applications, the most significant k-Delaunay graphs are those where the value of k is very small. In this section we carry out a detailed analysis of the 1-Delaunay graph.

As in [1], we have studied the general case but we have also devoted some attention to the situation where all points are in convex position. Our contributions are presented in Table 3. As all the proofs for the general case are quite long, we only describe here the value of lcr in the convex case. Specifically, we prove that, if P is in convex position, $\mathsf{lcr}(\text{1-}\mathsf{DG}(n)) = 6n - 3\lfloor \frac{n}{2} \rfloor - 19$.

	general case	convex case
lcr	$n - 4$	$6n - 3\lfloor \frac{n}{2} \rfloor - 19$
wcr	$n^2 + \Theta(n) \leq \mathsf{wcr} \leq 4n^2 + \Theta(n)$	$\frac{n^2}{2} + \Theta(n) \leq \mathsf{wcr} \leq \frac{7n^2}{8} + \Theta(n)$

Table 3: 1-Delaunay graphs.

Let $e_b = e\,(\text{0-}\mathsf{DG}(P)) = 2n - 3$ and $e_r = e\,(\text{1-}\mathsf{DG}(P)) - e\,(\text{0-}\mathsf{DG}(P))$. In [1] it is shown that $e_r \geq \lceil \frac{3n}{2} \rceil - 5 = 2n - \lfloor \frac{n}{2} \rfloor - 5$. We make use of this result to prove the lower bound for $\mathsf{lcr}(\text{1-}\mathsf{DG}(n))$:

Theorem 3.1. *For every set P in convex position,* $\boxtimes\,(\text{1-}\mathsf{DG}(P)) \geq 6n - 3\lfloor \frac{n}{2} \rfloor - 19$.

Proof. Note that all edges forming an ear in P are in $\text{1-}\mathsf{DG}(P)$, and that the number of crossings between two edges of this type is n. Let H be the graph obtained from $\text{1-}\mathsf{DG}(P)$ by removing these edges and the ones in the convex hull of P. Since the size of $\text{1-}\mathsf{DG}(P)$ is $e_b + e_r$, H contains at least $2n - \lfloor \frac{n}{2} \rfloor - 8$ edges. Each of them induces two crossings with the ear edges.

Let H_p be a maximal planar subgraph of H. It is easy to see that H_p contains at most $n - 5$ edges. Thus there are at least $n - \lfloor \frac{n}{2} \rfloor - 3$ edges in H but not in H_p, each of which induces at least one crossing with an edge of H_p.

Adding everything up, we can conclude that $\text{1-}\mathsf{DG}(P)$ has no less than $6n - 3\lfloor \frac{n}{2} \rfloor - 19$ crossings. □

The point set in Figure 2, which is carefully described in [3], shows that this bound can be attained and, consequently, is best possible.

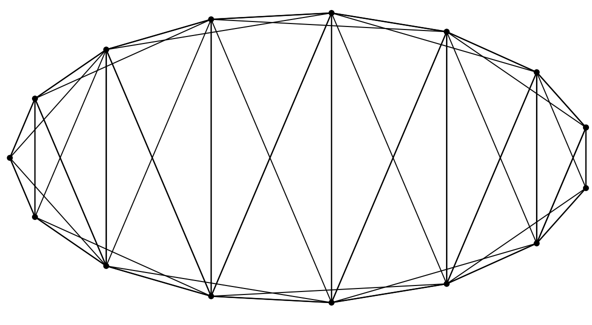

Figure 2: $\text{1-}\mathsf{DG}(P)$ with $6n - 3\lfloor \frac{n}{2} \rfloor - 19$ crossings.

4 General bounds

In this section we are interested in the asymptotic behavior when k is large of the number of crossings in the graphs under study. First we define the two remaining graphs.

For every pair of points $p_i, p_j \in P$, let $\mathsf{O\text{-}LENS}(p_i, p_j) = \{x \in \mathbb{R}^2 : |p_i x| < |p_i p_j| \text{ and } |p_j x| < |p_i p_j|\}$. The *$k$-relative neighbourhood graph* of P, denoted by $k\text{-}\mathsf{RNG}(P)$, is the geometric graph on P where two vertices p_i, p_j are adjacent if $\mathsf{O\text{-}LENS}(p_i, p_j)$ contains at most k points in P. The graph k-$\mathsf{RNG}(P)$ was introduced in [7].

Now let $\mathsf{C\text{-}DISC}(p_i, p_j)$ be the closed disc centered at the midpoint of the segment $\overline{p_i p_j}$ with both p_i and p_j on its boundary. The *k-Gabriel graph* of P, denoted by k-$\mathsf{GG}(P)$, is the geometric graph with vertex set P, and an edge between points p_i and p_j if $\mathsf{C\text{-}DISC}(p_i, p_j)$ contains at most k points in P different from p_i, p_j. The first definition of the *k-Gabriel graph*, which is slightly different from ours, was given in [18].

We have derived bounds for both the lowest crossing number and the worst crossing number of all graphs (see Table 4). In some cases, we have used the fact that there exists a relation of containment among the different classes of proximity graphs we have presented:

$$(k+1)\text{-}\mathsf{NNG}(P) \subseteq k\text{-}\mathsf{RNG}(P) \subseteq k\text{-}\mathsf{GG}(P) \subseteq k\text{-}\mathsf{DG}(P).$$

	k-$\mathsf{NNG}(n)$	k-$\mathsf{RNG}(n)$	k-$\mathsf{GG}(n)$	k-$\mathsf{DG}(n)$
$\mathsf{lcr} \geq$	$\frac{128}{31827}k^3n$	$\frac{128}{31827}k^3n$	$\frac{128}{31827}k^3n$	$\frac{128}{31827}k^3n$
$\mathsf{lcr} \leq$	$\frac{2}{27\pi^2}k^3n$	$\frac{128}{27\pi^2}k^3n$	$\frac{128}{27\pi^2}k^3n$	$\frac{128}{27\pi^2}k^3n$
$\mathsf{wcr} \geq$	$\frac{1}{3}k^3n$	$\frac{1}{3}k^3n$	$\frac{1}{4}k^2n^2$	$\frac{1}{2}k^2n^2$
$\mathsf{wcr} \leq$	k^3n	$9k^3n$	$\frac{9}{2}k^2n^2$	$\frac{9}{2}k^2n^2$

Table 4: General bounds.

The lower bounds for the lowest crossing numbers follow from an improved version of the crossing lemma proved in [14]. As for the upper bounds, we use a construction given in [16] of a graph with fixed size and few crossings, and show that it can be seen as a higher order proximity graph.

In contrast, our approach to bound from above the worst crossing numbers relies strongly on the geometric properties of each particular graph. As a representative example, we sketch the proof of the upper bound for $\mathsf{wcr}(k\text{-}\mathsf{RNG}(n))$.

Lemma 4.1. *In any angular sector with apex $p \in P$ and amplitude $\alpha \leq \pi/3$ there are at most $k+3$ points in P that are connected to p in the graph k-$\mathsf{RNG}(P)$.*

Theorem 4.2. *For every point set P, $\boxtimes(k\text{-}\mathsf{RNG}(P)) \leq 9k^3n + o(k^3n)$.*

Proof. We use a charging scheme that assigns every crossing in k-$\mathsf{RNG}(P)$ to each of the two involved edges e satisfying that at least one of the endpoints of the other edge is contained in the lens associated to e.

Each crossing defines a quadrilateral that has at least one obtuse angle at some vertex p_i. If the diagonal opposite to this obtuse angle is the edge p_jp_k, p_i is contained in the lens associated to p_jp_k, so the crossing is assigned to (at least) the edge p_jp_k.

Now we bound the maximum number of crossings that may be assigned to an edge e. The lens associated to e contains at most k points in P. By Lemma 4.1, each of these points is adjacent to no more than $3k+9$ other points in P such that the edge that connects them crosses e. Consequently, at most $3k^2+9k$ crossings may be assigned to e.

Since each vertex in P has degree at most $6k+18$ (see Lemma 4.1), the number of edges of k-$\mathsf{RNG}(P)$ is bounded by $3kn + o(kn)$, which yields the theorem. $\square$

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