In questions 1 through 3 write the letter from **(a)** to **(d)** in the space provided. No partial credit will be awarded

1. The series $1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} \cdots + (-1)^n \frac{\pi^{2n}}{(2n)!} + \cdots$ converges. What is its sum?

   (a) -1.2113528
   (b) $-3/\pi$
   (c) $\cos(3)$
   (d) -1

Answer:D

2. Suppose that the function $f(x)$ is approximated near $x = 0$ by a sixth degree Taylor Polynomial

$$P_6(x) = 1 - 2x + 8x^3 - \frac{1}{24}x^6$$

   Which of the following is the only correct statement?

   (a) $f'(0) = 0$, $f''(0) = 8$, and $f^{(6)}(0) = -1/24$.
   (b) $f'(0) = -2$, $f''(0) = 8$, and $f^{(6)}(0) = -30$.
   (c) $f'(0) = 0$, $f''(0) = 0$, and $f^{(6)}(0) = -1/24$.
   (d) $f'(0) = -2$, $f''(0) = 0$, and $f^{(6)}(0) = -30$.

Answer:D

3. Suppose that the power series $\sum_{n=0}^{\infty} C_n (x+5)^n$ diverges when $x = 2$ and converges when $x = -8$. Which of the following is the only correct statement?

   (a) The power series converges for $x = -3$ and diverges for $x = -6$.
   (b) The power series converges for $x = -3$ and diverges for $x = 6$.
   (c) The power series converges for $x = 3$ and diverges for $x = -6$.
   (d) The power series converges for $x = 3$ and diverges for $x = 6$.

Answer:B

4. Determine if the series $\sum_{n=2}^{\infty} \frac{1}{n\left(\ln(n)\right)^2}$ converges or not. (Hint: you may want to use the integral test)

Notice that

$$\int_2^{\infty} \frac{1}{x\left(\ln x\right)^2} dx = \lim_{b\to\infty} \int_2^{b} \frac{1}{x\left(\ln x\right)^2} dx.$$

Let $u = \ln x$, then $du = \frac{1}{x} dx$ and

$$\begin{aligned}
\int_2^{\infty} \frac{1}{x\left(\ln x\right)^2} dx &= \lim_{b\to\infty} \int_{x=2}^{x=b} \frac{du}{u^2} = \lim_{b\to\infty} \int_{x=2}^{x=b} u^{-2} du \\
&= \lim_{b\to\infty} \left(-\frac{1}{u}\right)\Bigg|_{x=2}^{x=b} = \lim_{b\to\infty} \left(-\frac{1}{\ln x}\right)\Bigg|_{x=2}^{x=b} \\
&= \lim_{b\to\infty} \left(-\frac{1}{\ln b} + \frac{1}{\ln 2}\right) = \frac{1}{\ln 2}.
\end{aligned}$$

Since $\ln b \to \infty$ when $b \to \infty$. Thus the integral converges and as a consequence the series converges as well by the integral test.

5. Determine if the following series are convergent or divergent. Make sure you indicate which test you are using.

(a) $\sum_{n=1}^{\infty} \frac{n+7}{3n-1}$

$$\lim_{n\to\infty} \frac{n+7}{3n-1} = \lim_{n\to\infty} \frac{1+\frac{7}{n}}{3-\frac{1}{n}} = \frac{1}{3} \neq 0$$

So the series diverges by the divergence test.

(b) $\sum_{n=1}^{\infty} \frac{n^2}{n^4+1}$

Since $\frac{n^2}{n^4+1} \sim \frac{n^2}{n^4} = \frac{1}{n^2}$ then we will compare to $\sum_{n=1}^{\infty} \frac{1}{n^2}$. This series is convergent since it is a p-series with $p = 2 > 1$. Now observe that

$$\begin{aligned}
\frac{n^2}{n^4+1} &< \frac{1}{n^2} \Leftrightarrow \\
n^4 &< n^4 + 1
\end{aligned}$$

and this last inequality is clearly true. Thus by comparison test the series converges. Also, we can use the limit comparison test. In this case

$$\begin{aligned}
\lim_{n\to\infty} \frac{n^2}{n^4+1} \cdot \frac{n^2}{1} &= \lim_{n\to\infty} \frac{n^4}{n^4+1} \\
&= \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^4}} = 1 \neq 0, \infty.
\end{aligned}$$

So the two series behave the same and then the required series converges.

(c) $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ Apply the ratio test:

$$\begin{aligned}
\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} &= \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} \\
&= \lim_{n\to\infty} \frac{2}{(n+1)} = 0 < 1
\end{aligned}$$

So by the ratio test, the series converges.

6. Consider the power series $\sum_{n=0}^{\infty} \frac{(x-2)^n}{3^n(n+1)}$

(a) Find the radius of convergence
Apply the ratio test:

$$\begin{aligned}
\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} &= \lim_{n\to\infty} \frac{|x-2|^{n+1}}{3^{n+1}(n+2)} \cdot \frac{3^n(n+1)}{|x-2|^n} \\
&= |x-2| \lim_{n\to\infty} \frac{(n+1)}{3(n+2)} = |x-2| \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)}{3\left(1+\frac{2}{n}\right)} \\
&= \frac{|x-2|}{3} \begin{cases} <1 & \text{convergent} \\ =1 & \text{inconclusive} \\ >1 & \text{divergent} \end{cases}.
\end{aligned}$$

So if $|x-2| < 3$ the series converges, and if $|x-2| > 3$ it diverges. Thus the radius of convergence $R = 3$.

(b) Find the interval of convergence.
The center of the power series is $a = 2$, so the endpoints are $x = -1$ and $x = 5$.
If $x = 5$ then

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(x-2)^n}{3^n(n+1)} &= \sum_{n=0}^{\infty} \frac{(5-2)^n}{3^n(n+1)} = \sum_{n=0}^{\infty} \frac{3^n}{3^n(n+1)} \\
&= \sum_{n=0}^{\infty} \frac{1}{n+1} = \sum_{n=1}^{\infty} \frac{1}{n}
\end{aligned}$$

which is the harmonic series, so it diverges.
If $x = -1$ then

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(x-2)^n}{3^n(n+1)} &= \sum_{n=0}^{\infty} \frac{(-1-2)^n}{3^n(n+1)} = \sum_{n=0}^{\infty} \frac{(-3)^n}{3^n(n+1)} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}.
\end{aligned}$$

Now we apply the alternating test. First, clearly

$$\lim_{n\to\infty} \frac{1}{n+1} = 0$$

and

$$\begin{array}{rcl} a_{n+1} & < & a_n \Leftrightarrow \\ \dfrac{1}{n+2} & < & \dfrac{1}{n+1} \Leftrightarrow \\ n+1 & < & n+2 \end{array}$$

where the last is clearly true. Since the series satisfies properties 1 and 2 in the alternating test, then the series is convergent.

The interval of convergence is

$$[-1,5) = \{x : -1 \leq x < 5\}.$$

(a) Find the Taylor Polynomial of degree 4 for the function $f(x) = \sqrt{1+2x}$ around $a = 0$.

We calculate the derivatives of f as well as plugging in $x = 0$:

functions	$x = 0$
$f(x) = (1+2x)^{1/2}$	1
$f'(x) = (1+2x)^{-1/2}$	1
$f''(x) = -(1+2x)^{-3/2}$	-1
$f^{(3)}(x) = 3(1+2x)^{-5/2}$	3
$f^{(4)}(x) = -15(1+2x)^{-7/2}$	-15

Thus

$$\begin{aligned} P_4(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\ &= 1 + x - \frac{1}{2!}x^2 + \frac{3}{3!}x^3 - \frac{15}{4!}x^4 \\ &= 1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{5}{8}x^4. \end{aligned}$$

(b) The function $f(x)$ and the Tayor Polynomial $P_4(x)$ are very close to each other when x is close to $a = 0$. Use $P_4(x)$ from part (a) to get an approximation for $f(0.25) = \sqrt{1.5}$

$$f(1) \approx P_4(1) = 1 + 1 - \frac{1}{2} + \frac{1}{2} - \frac{5}{8} = \frac{11}{8} = 1.375$$

7. (Extra) Let $a_n > 0$. Prove that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} \ln(1+a_n)$ are either both convergent or both divergent.

If $\lim_{n\to\infty} a_n \neq 0$ then $\lim_{n\to\infty} \ln(1+a_n) \neq \ln(1) = 0$. In this case both series are divergent. If $\lim_{n\to\infty} a_n = 0$ then for n large enough we have that $a_n < 1$. Then from the Taylor expansion for $\ln(1+x)$ we get that

$$\ln(1+a_n) = a_n - \frac{a_n^2}{2} + \frac{a_n^3}{3} - \frac{a_n^4}{4} + \cdots + (-1)^{k+1}\frac{a_n^k}{k} + \cdots,$$

then if we divide by a_n we get that

$$\frac{\ln(1+a_n)}{a_n} = 1 - \frac{a_n}{2} + \frac{a_n^2}{3} - \frac{a_n^3}{4} + \cdots + (-1)^{k+1}\frac{a_n^{k-1}}{k} + \cdots$$

Then we take limits when $n \to \infty$ and we get

$$\begin{aligned}
\lim_{n\to\infty} \frac{\ln(1+a_n)}{a_n} &= \lim_{n\to\infty} \left(1 - \frac{a_n}{2} + \frac{a_n^2}{3} - \frac{a_n^3}{4} + \cdots + (-1)^{k+1}\,\frac{a_n^{k-1}}{k} + \cdots\right) \\
&= \lim_{n\to\infty} 1 - \lim_{n\to\infty} \frac{a_n}{2} + \lim_{n\to\infty} \frac{a_n^2}{3} - \lim_{n\to\infty} \frac{a_n^3}{4} + \cdots + \lim_{n\to\infty} (-1)^{k+1}\,\frac{a_n^{k-1}}{k} + \cdots \\
&= 1 \text{ (since } \lim_{n\to\infty} a_n = 0).
\end{aligned}$$

Thus, since $1 \neq 0, \infty$ then by Limit Comparison Test, $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} \ln(1+a_n)$ behave the same.