Solutions to Exam #2, Math 150B, Fall 2004. Professor Bernardo M. Ábrego

In questions 1 through 3 write the letter from **(a)** to **(d)** in the space provided. No partial credit will be awarded

1. The integral that represents the length of the curve $y = \sqrt{x^2+1}$ between $x = 0$ and $x = 3$, is:

   (a) $\int_0^3 (2+x^2)\, dx$

   (b) $\int_0^3 \sqrt{1+\frac{1}{2(x^2+1)}}\, dx$

   (c) $\int_0^3 \sqrt{1+\frac{x^2}{x^2+1}}\, dx$

   (d) $\int_0^3 \left(1+\frac{1}{2(x^2+1)}\right) dx$

Answer:C

We know that the arc-length is given by $\int_a^b \sqrt{f'(x)^2+1}dx = \int_0^3 \sqrt{\frac{d}{dx}\left(\sqrt{x^2+1}\right)^2+1}dx$
$= \int_0^3 \sqrt{\left((1/2)\,2x\,(x^2+1)^{-1/2}\right)^2+1}dx = \int_0^3 \sqrt{x^2(x^2+1)^{-1}+1}dx = \int_0^3 \sqrt{1+\frac{x^2}{x^2+1}}\, dx.$

2. Consider the following assertions:

   (I) If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$ then $\lim_{x\to a} f(x)g(x) = 0$

   (II) If $\lim_{x\to\infty} \ln(f(x)) = 2$ then $\lim_{x\to\infty} f(x) = e^2$

   Which of the following is the only correct statement?

   (a) I is true and II is true.

   (b) I is true and II is false.

   (c) I is false and II is true.

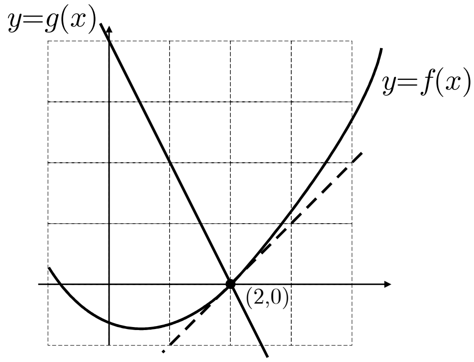
   (d) I is false and II is false.

Answer:C

To see that (I) is false consider $a = 0$, $f(x) = x^2$ and $g(x) = 1/x^2$. Clearly $\lim_{x\to 0} f(x) = 0$ and $\lim_{x\to 0} g(x) = \infty$ but $\lim_{x\to 0} f(x)g(x) = \lim_{x\to 0} 1 = 1$. (II) is true since e^x is a continuous function, then

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{\ln f(x)} = e^{\lim_{x\to\infty}\ln(f(x))} = e^2.$$

3. The figure below shows the graphs of $y = f(x)$ and its tangent line at (2,0), and also the graph of $y = g(x)$ which is a line going through (2,0) and (1,2). What is the value of $\lim_{x\to 2} \frac{f(x)}{g(x)}$?



(a) 0 (b) $-1/2$ (c) $-2/5$ (d) does not exist

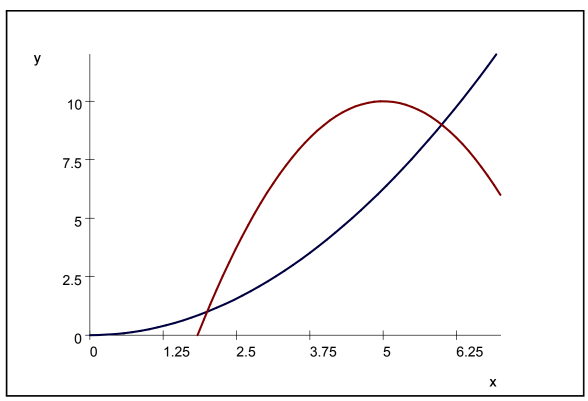
Answer:B

From the figure we know that $f(2) = g(2) = 0$, and $g'(0) = -2, f'(0) = 1$. Thus by L'hôpital's rule (since we have $0/0$) we get

$$\lim_{x \to 2} \frac{f(x)}{g(x)} = \lim_{x \to 2} \frac{f'(x)}{g'(x)} = \frac{1}{-2}.$$

4. Let R be the region bounded by the curves $y = x^2/4$ and $y = 10 - (x-5)^2$.

 (a) Find an integral that represents the volume obtained by rotating R about the x-axis. Indicate which method you are using. DO NOT evaluate the integral.
 $x^2/4$



 Need to first find out where these two curves intersect. We solve $x^2/4 = y = 10 - (x-5)^2$, we get $x^2 = 40 - 4x^2 + 40x - 100$. Then $5x^2 - 40x + 60 = 0$ which factors as $5(x-2)(x-6) = 0$. So the region is bounded by the points $(2,1)$ and $(6,9)$.
 Washer or disk method. Volume(Slice)$= \pi \left(r_2^2 - r_1^2\right) \Delta x = \pi \left(\left(10 - (x-5)^2\right)^2 - \left(x^2/4\right)^2 \right) \Delta x$

$$\text{Volume} = \int_2^6 \pi \left(\left(10 - (x-5)^2\right)^2 - \left(x^2/4\right)^2 \right) dx$$

 (b) Find an integral that represents the volume obtained by rotating R about the y-axis. Indicate which method you are using. DO NOT evaluate the integral.

Shell method. Volume(Slice)$= 2\pi x\,(r_2 - r_1)\,\Delta x = 2\pi x\left(\left(10-(x-5)^2\right)-\left(x^2/4\right)\right)\Delta x$

$$\text{Volume} = \int_2^6 2\pi x\left(\left(10-(x-5)^2\right)-\left(x^2/4\right)\right)dx$$

(c) Find an integral that represents the volume obtained by rotating R about the line $x = 8$. Indicate which method you are using. DO NOT evaluate the integral
Shell method. Volume(Slice)$= 2\pi\,(8-x)\,(r_2 - r_1)\,\Delta x = 2\pi x\left(\left(10-(x-5)^2\right)-\left(x^2/4\right)\right)\Delta x$

$$\text{Volume} = \int_2^6 2\pi\,(8-x)\left(\left(10-(x-5)^2\right)-\left(x^2/4\right)\right)dx$$

5. Solve the following integrals: or show that it diverges

(a) $\displaystyle\int_2^\infty xe^{-x^2}\,dx.$

$$\int_2^\infty xe^{-x^2}dx = \lim_{b\to\infty}\int_2^b xe^{-x^2}dx,$$

let $u = -x^2, du = -2xdx$, then

$$\begin{aligned}\int_2^\infty xe^{-x^2}dx &= \lim_{b\to\infty} -\frac{1}{2}\int_{x=2}^{x=b} e^u du = \lim_{b\to\infty} -\frac{1}{2}e^u\Big|_{x=2}^{x=b}\\ &= \lim_{b\to\infty} -\frac{1}{2}e^{-x^2}\Big|_{x=2}^{x=b} = \lim_{b\to\infty}\left(-\frac{1}{2}e^{-b^2}+\frac{1}{2}e^{-4}\right) = \frac{1}{2}e^{-4}.\end{aligned}$$

(b) $\displaystyle\int_{-2}^{-1}\frac{dx}{(x+1)^{4/3}}.$

$$\begin{aligned}\int_{-2}^{-1}\frac{dx}{(x+1)^{4/3}} &= \lim_{b\to -1^-}\int_{-2}^{b}(x+1)^{-4/3}dx = \lim_{b\to -1^-} -3(x+1)^{-1/3}\Big|_{x=-2}^{x=b}\\ &= \lim_{b\to -1^-}\frac{-3}{(x+1)^{1/3}}\Bigg|_{x=-2}^{x=b} = \lim_{b\to -1^-}\frac{-3}{(b+1)^{1/3}}+\frac{3}{(-2+1)^{1/3}}\\ &= \frac{-3}{0^-}-3=\infty.\ \text{So the integral diverges.}\end{aligned}$$

6. A cable weighing 2 lb/ft is used to haul a 200 lb load to the top of a shaft that is 500 ft deep. How much work is done?

The work required to pull the load from a height of x to a height of $x+\Delta x$ is:

$$\text{Work}\,(\text{from } x \text{ to } x+\Delta x) = \text{Weight}\cdot\text{Distance}$$

The distance is Δx and the weight is the weight of the load (200lb) plus the weight of the $500-x$ feet of cable, which weighs $(500-x)$2lb. Thus

$$\text{Work}\,(\text{from } x \text{ to } x+\Delta x) = (200+(500-x)\,2)\cdot\Delta x.$$

Then the total work is

$$\begin{aligned}\text{Work} &= \int_0^{500}(200+(500-x)\,2)\,dx = \int_0^{500}(1200-2x)\,dx\\ &= 1200x - x^2\Big|_{x=0}^{x=500} = 1200\,(500)-500^2 = 350000\text{ft-lb}.\end{aligned}$$

7. Find the following limits or indicate if they are ∞ or $-\infty$

(a) $\lim_{x\to 0} \dfrac{2^x - 1}{3^x - 1}$

We have an indeterminate of the form 0/0. Then by L'hôpital's rule we get

$$\lim_{x\to 0} \frac{2^x - 1}{3^x - 1} = \lim_{x\to 0} \frac{(2^x - 1)'}{(3^x - 1)'} = \lim_{x\to 0} \frac{(\ln 2)\, 2^x}{(\ln 3)\, 3^x} = \frac{\ln 2}{\ln 3}$$

(b) $\lim_{x\to 0^+} (\cos x)^{1/x}$

We have an indeterminate of the form 1^∞. Let $y = (\cos x)^{1/x}$,then $\ln y = \frac{1}{x} \ln(\cos x)$ and by L'hôpital's rule we get

$$\begin{aligned} \lim_{x\to 0^+} \ln y &= \lim_{x\to 0^+} \frac{\ln(\cos x)}{x} \text{ (0/0 indeterminate) } \stackrel{L}{=} \\ &= \lim_{x\to 0^+} \frac{(\ln(\cos x))'}{x'} = \lim_{x\to 0^+} \frac{\frac{1}{\cos x}(-\sin x)}{1} = 0. \end{aligned}$$

Therefore

$$\lim_{x\to 0^+} (\cos x)^{1/x} = \lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\ln y} = e^{\lim_{x\to 0^+} \ln y} = e^0 = 1.$$