LECTURE 5.

Addition and subtraction of rational expressions

Two rational expressions in general have different denominators, therefore if you want to add or subtract them you need to equate the denominators first. The common denominator with the smallest possible degree is the Least Common Multiple of the original ones. On this lecture we will consider the procedure of finding the Least Common Multiple and solve some examples on addition and subtraction of rational expressions.

5.1. Addition and subtraction of rational expressions

The rules for adding and subtracting rational expressions are the same as rules for adding and subtracting fractions.

$$\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b}, \qquad \frac{a}{b}-\frac{c}{b}=\frac{a-c}{b}.$$

In this example both the denominators are equal and are not zero.

Example 5.1. (*3.4 ex.13*)

$$\frac{x^2}{x^2+4}-\frac{x^2+1}{x^2+4}=\frac{x^2-x^2-1}{x^2+4}=\frac{1}{x^2+4}.$$

(*3.4 ex.16*)

$$\begin{aligned}\frac{4x+5}{x-4}-\frac{2x+4}{4-x}&=\frac{4x+5}{x-4}-\frac{2x+4}{-(x-4)}\\&=\frac{4x+5}{x-4}+\frac{2x+4}{x-4}=\frac{6x+9}{x-4}.\end{aligned}$$

If denominators of two rational expressions are not equal, we can use following general rule for adding and subtracting quotients

$$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}, \qquad b\neq 0, \quad d\neq 0.$$

Let us prove this formula

$$\frac{a}{b}+\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{d}+\frac{c}{d}\cdot\frac{b}{b}=\frac{ad}{bd}+\frac{cb}{db}=\frac{ad+cb}{bd}.$$

Similarly,

$$\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}.$$

Example 5.2. (*3.4 ex.22*)

$$\frac{x^2}{3} + \frac{x}{x+1} = \frac{x^2(x+1) + 3x}{3(x+1)} = \frac{x^3 + x^2 + 3x}{3(x+1)} = x\frac{(x^2 + x + 3}{3(x+1)}.$$

(*3.4 ex.36*)

$$\begin{aligned}\frac{2x-1}{x-1} - \frac{2x+1}{x+1} &= \frac{(2x-1)(x+1) - (x-1)(2x+1)}{(x-1)(x+1)} \\ &= \frac{2x^2 - x + 2x - 1 - 2x^2 + 2x - x + 1}{(x-1)(x+1)} \\ &= \frac{2x}{(x-1)(x+1)}.\end{aligned}$$

(*3.4 ex.41*)

$$\begin{aligned}&\frac{x}{x+1} + \frac{(x-2)}{x-1} + \frac{x+1}{x-2} \\ &= \frac{x(x-1)(x-2) + (x-2)(x-2)(x-1) + (x+1)(x+1)(x-1)}{(x+1)(x-1)} \\ &= \frac{x(x^2 - 3x + 3) + (x+1)(x^2 - 4x + 4) + (x^2 + 2x + 1)(x-1)}{(x+1)(x-1)(x-2)} \\ &= \frac{x^3 - 3x^2 + 3x + x^3 - 4x^2 + 4x + x^2 - 4x + 4 + x^3 + 2x^2 + x - x^2 - 2x - 1}{(x+1)(x-1)(x-2)} \\ &= \frac{3x^3 - 5x^2 + 2x + 3}{(x+1)(x-1)(x-2)} = \frac{(x-1)x(3x-2) + 3}{(x+1)(x-1)(x-2)}.\end{aligned}$$

One can check that there is no extra canceling by trying long division of the numerator by the factors $(x+1)$, $(x-1)$, and $(x-2)$ of the denominator.

5.2. Least common multiple

The general formula for summing up two rational expressions will work in every situation. But not in every situation the general formula is the best. Let us consider following example

Example 5.3. (*3.4 ex.69*)

$$\frac{x+4}{x^2 - x - 2} - \frac{2x+3}{x^2 + 2x - 8} \underset{\text{formula}}{\overset{\text{general}}{=}} \frac{(x+4)(x^2 + 2x - 8) - (x^2 - x - 2)(2x+3)}{(x^2 - x - 2)(x^2 + 2x - 8)}.$$

A little bit smarter way to do it is to factor the denominators, first:

$$\frac{(x+4)}{(x-2)(x+1)} - \frac{(2x+3)}{(x+4)(x-2)}.$$

Then, obviously, we can equate the denominators by multiplying the first rational expression by $\frac{(x+4)}{(x+4)}$ and the second one by $\frac{(x+1)}{(x+1)}$:

$$\begin{aligned}&\frac{(x+4)(x+4)-(2x+3)(x+1)}{(x-2)(x+1)(x+4)}\\ &\qquad= \frac{x^2+8x+16-2x^2-3x-2x-3}{(x-2)(x+1)(x+4)} = \frac{-x^2+3x-13}{(x-2)(x+1)(x+4)}.\end{aligned}$$

Let us summarize what we did. We wanted to equate the denominators and we wanted to keep our work simple. The work will be kept simple if a common denominator has the smallest possible degree. In this example a polynomial $(x-2)(x+1)(x+4)$ is the one and it is called the Least Common Multiple of polynomials $(9x-2)(x-1)$ and $(x-2)(x+4)$.

Definition 5.1. The polynomial $P(x)$ is the least common multiple of polynomials $P_1(x), \cdots P_n(x)$ if

a) each of p_i is a factor of $P(x)$;

b) there is no polynomial $q(x)$ with degree less than degree of $P(x)$ such that a) is satisfied.

Remark 5.1. Part b) says that $P(x)$ has the least possible degree for property a).

Question 5.1. *Let $P(x)$ be the least common multiple of polynomials $P_1(X)$, $P_2(X)$, $P_3(x)$.*

a) Consider $\frac{P(x)}{P_1(x)}$, will $P_1(x)$ - cancel?

b) Consider $\frac{P(x)}{P_2(x)}$, will $P_2(x)$ - cancel?

c) Consider $\frac{P(x)}{P_1(x)P_2(x)}$, will $P_1(x)P_2(x)$ - cancel?

Let us discuss the process of finding the least common multiple of two or more polynomials. The first step is to factor each polynomial completely. Then to construct the least common multiple we successively combine prime factors of the original polynomials. For each original polynomial we add only those of it's factor which are missing in the least common multiple. In particular, let polynomials $M(x)$, $N(x)$, $P(x)$ have factoring, correspondingly

$$M(x) = a(x)b(x)c(x), \qquad N(x) = b(x)c(x)d(x), \qquad P(x) = b(x)d(x)e(x),$$

then the least common multiple for $M_{(x)}$, $N_{(x)}$, and $P(x)$ will be

$$LCM = a(x)b(x)c(x)d(x)b(x)e(x).$$

Here initially we added to LCM all factors from $M(x)$ since LCM should have all factors of $M(x)$ and had nothing initially. On the second step we added $d(x)$ from $N(x)$ since other two multipliers already were LCM. On the third step we added $e(x)$ and $b(x)$ from $P(x)$. Although LCM already had one $b(x)$, still $P(x)$ had two of them so one was missing, thus, we added it.

Example 5.4. (*3.4 ex.51*) Find the LCM to $x^3 - x$, $x^3 - 2x^2 + x$, and $x^3 - 1$. First, we will factor three polynomials

$$\begin{aligned} x^3 - x &= x(x+1)(x-1), \\ x^3 - 2x^2 + x &= x(x-1)(x-1), \\ x^3 - 1 &= (x-1)(x^2 + x + 1). \end{aligned}$$

The LCM is then given by

$$\text{LCM} = x(x+1)(x-1)(x-1)(x^2 + x + 1).$$

(*3.4 ex.72*)

$$\begin{aligned} &\frac{x}{(x-1)^2} + \frac{2}{x} - \frac{x+1}{x^3 - x^2} \\ &\quad = \frac{x}{(x-1)^2} + \frac{2}{x} - \frac{x+1}{x^2(x-1)} \\ &\quad = \frac{x \cdot x^2 + 2 \cdot x \cdot (x-1)^2 - (x+1)(x-1)}{(x-1)^2 x^2} \\ &\quad = x^3 + 2x^3 - 4x^2 + 2x - x^2 + 1 = \frac{3x^3 - 5x^2 + 2x + 1}{(x-1)^2 x^2}. \end{aligned}$$

By doing long division of the numerator over the $(x-1)$ one can see that $(x-1)$ does not cancels.

LECTURE 6.

Mixed quotients

When the numerator or the denominator of a quotient contains combinations of rational functions we call it a mixed quotient. On this lecture we will consider how to simplify mixed quotients.

6.1. Mixed quotients

Definition 6.1. When sums and/or differences of rational expressions appear as the numerator and/or denominator of a quotient, the quotient is called a *mixed quotient.*

To simplify a mixed quotient means to write it as a rational expression reduced to lowest terms.

Example of Mixed Quotient

$$\frac{\dfrac{a}{b}+\dfrac{c}{d}}{\dfrac{e}{f}+\dfrac{k}{l}}, \text{ where } a, b, c, d, e, f, k, l \text{ are polynomials.}$$

There are two favorite methods for simplifying mixed quotients. The first methods is to simplify mixed quotient step by step by considering rational expressions is the numerator, then in the denominator and, finally, the whole thing.

In the second approach you will need to compute the least common multiplied of the denominators of all rational expressions entering the mixed quotient. Then the mixed quotient is simplified in one step by multiplying both numerator and denominator with the least common multiple.

We will consider some examples for each of two methods.

6.2. Simplifying by parts

Method Summary: Consider numerator, simplify; consider denominator, simplify, etc.

$$\frac{\dfrac{a}{b}+\dfrac{c}{d}}{\dfrac{e}{f}+\dfrac{k}{l}}=\frac{\dfrac{ad+bc}{bd}}{\dfrac{el+kf}{fl}}=\frac{(ad+bc)\cdot fl}{(el+fk)\cdot bd}$$

Example 6.1. (*3.5 ex.1*)

$$\frac{\dfrac{x}{x+1}+\dfrac{4}{x+1}}{\dfrac{x+4}{2}}=\frac{\dfrac{x+4}{x+1}}{\dfrac{x+4}{2}}=\frac{2(x+4)}{(x+1)(x+4)}=\frac{2}{x+1}.$$

6.3. Simplifying by one strike!

Method Summary: Take LCM of all "little" denominators ($b\ d\ f\ e$) and multiply both the numerator and the denominator of the mixed quotient by LCM.

$$\frac{\dfrac{a}{b}+\dfrac{c}{d}}{\dfrac{e}{f}+\dfrac{k}{l}}\cdot\frac{LCM}{LCM}=\frac{\dfrac{a}{b}\cdot LCM+\dfrac{c}{d}\cdot LCM}{\dfrac{e}{f}\cdot LCM+\dfrac{k}{e}\cdot LCM}.$$

Example 6.2. (*3.5 ex.16*)

$$\frac{1-\dfrac{x}{x+1}}{2-\dfrac{x-1}{x}}\qquad\qquad LCM=(x+1)x,$$

Multiplying both the numerator and the denominator by LCM

$$\frac{1-\dfrac{x}{x+1}}{2-\dfrac{x-1}{x}}\cdot\frac{(x+1)x}{(x+1)x}=\frac{(x+1)x-x^2}{2(x+1)x-1(x-1)(x+1)}=\frac{x^2+x-x^2}{2x^2+2x-x^2+1}=\frac{x}{(x+1)^2}.$$

(*3.5 ex.22*)

$$\frac{\dfrac{(2x+5)}{x}-\dfrac{x}{x-3}}{\dfrac{x^2}{x-3}-\dfrac{(x+1)^2}{x+3}}$$

$$=\frac{\dfrac{(2x+5)(x-3)-x^2}{x(x-3)}}{\dfrac{x^2(x+3)-(x+1)^2x-3}{(x-3)(x+3)}}=\frac{((2x+5)(x-3)-x^2)(x-3)(x+3)}{x(x-3)(x^2(x+3)-(x+1)^2(x-3))}$$

$$=\frac{(2x^2+5x-6x-15-x^2)(x+3)}{x(x^3+3x^2)-(x^2+2x+1)(x-3)}=\frac{(x^2-x-15)(x+3)}{x(x^3+3x^2-x^3-2x^2-x+3x^2+6x+3)}$$

$$=\frac{(x^2-x-15)(x+3)}{x(4x^2+5x+3)}.$$

(*3.5 ex.27*)

$$1 - \frac{1}{1 - \dfrac{1}{x}} = 1 - \frac{1}{\dfrac{x-1}{x}} = 1 - \frac{x}{x-1} = \frac{x-1-x}{x-1} = \frac{-1}{x-1} = \frac{1}{1-x}.$$

Solving the same example by the Method 2 looks a little more straightforward

$$1 - \frac{1}{1 - \dfrac{1}{x}} = 1 - \frac{1}{1 - \dfrac{1}{x}} \cdot \frac{x}{x} = 1 - \frac{x}{x-1} = \frac{x-1-x}{x-1} = \frac{1}{1-x}.$$

LECTURE 7.

Negative exponents. Scientific notation. Square root

On this lecture we will discuss the negative exponents. Properties of negative exponent turn out to be the same as that of usual positive exponent. Therefore, both negative and positive exponents represent the same object. We will also consider the scientific notations and how to convert numbers into scientific notation and will discuss properties of square root. In the end of the lecture we will discuss rationalizing of mathematical expressions.

7.1. Negative exponents

Let us start from some simple remarks. If n is a counting number then a raised to the power n is defined by

$$a^n = \underbrace{a \cdot \ldots \cdot a}_{n \text{ factors.}}$$

The number n is also can be referred as the exponents of a. Let us derive simple properties for the exponents

$$a^n \cdot a^m = \overbrace{\underbrace{a \cdot \ldots \cdot a}_{n} \cdot \underbrace{a \cdot \ldots \cdot a}_{m}}^{n+m} = a^{n+m} \ ;$$

$$(ab)^n = \underbrace{(ab)(ab) \cdot \ldots \cdot (ab)}_{n \text{ times}} = a^n b^n \ ;$$

$$\left(\frac{a}{b}\right)^n = \left(\frac{a}{b}\right) \cdot \left(\frac{a}{b}\right) \cdot \ldots \cdot \left(\frac{a}{b}\right) = \frac{a^n}{b^n} \ ;$$

$$(a^n)^m = \underbrace{\overbrace{\underbrace{(a \cdot \ldots \cdot a)}_{n} \cdot \underbrace{(a \cdot \ldots \cdot a)}_{n} \cdot \ldots \cdot \underbrace{(a \cdot \ldots \cdot a)}_{n}}^{n \times m}}_{m} = a^{n \cdot m} \ ;$$

$$\frac{a^n}{a^m} = \frac{\overbrace{a \cdot \ldots \cdot a}^{n}}{\underbrace{a \cdot \ldots \cdot a}_{m}} = a^{n-m} \ , \quad m < n \ .$$

Let us take a better look at the last formula: $m < n$ is an obvious the limitation for it. The formula is not working for $m \geq n$ simply because it is undefined for that cases. From the other side what if we could extend existing definitions to new cases? If this is possible, we will improve the performance of our old formula.

The first bad case is $m = n$ for which a^0 is undefined. Obviously, the case $m = n$ can be captured by defining

$$a^0 = 1.$$

You can check that the whole set of five formulas will still fit together.

Similarly, to make the last formula meaningful for $m > n$ we need to define

$$a^{-1} = \frac{1}{a}, \qquad a^{-2} = \frac{1}{a^2}, \qquad a^{-3} = \frac{1}{a^3}, \qquad \text{etc.}$$

Eventually, we arrive to the definition of the negative exponent

$$a^{-n} = \frac{1}{a^n}.$$

It is easy do check that properties of exponents are the same for negative exponents and we can improve the last formula:

$$\frac{a^m}{a^n} = a^m \cdot a^{-n} = a^{m+(-n)} = a^{m-n} \qquad \text{for any numbers } m \text{ and } n.$$

Example 7.1.

(*4.1 ex.11*)

$$3^{-6} \cdot 3^4 = 3^{-6+4} = 3^{-2} = \frac{1}{9}.$$

(*4.1 ex.13*)

$$\frac{8^2}{2^3} = \frac{8^2}{8} = 8^2 \cdot 8^{-1} = 8^{2-1} = 8.$$

As an exercise, let us derive one more property of the exponents

$$\left(\frac{a}{b}\right)^{-n} = \frac{1}{\left(\frac{a}{b}\right)^n} = \frac{1}{\frac{a^n}{b^n}} = \frac{b^n}{a^n} = \left(\frac{b}{a}\right)^n$$

(*4.1 ex.44*)

$$\frac{4x^{-2}(yz)^{-1}}{(-5)^2x^4y^2z^{-2}} = 4 \cdot (-5)^{-2}x^{-2}y^{-1}z^{-1}x^{-4}y^{-2}z^{-2} = 4(-5)^{-2}x^{-6}y^{-3}x^{-1} = \frac{4z}{25x^6y^3}.$$

(*4.1 ex.52*)

$$\frac{(3xy^{-1})^2}{(2x^{-1}y)^3} = \frac{3^2x^2y^{-2}}{2^3x^{-3}y^3} = \frac{3^2x^2x^3}{2^3y^3y^2} = \frac{3^2x^5}{2^3y^5}.$$

7.2. Scientific Notation

Numbers you meet in applications may very from very large to very small. As an example let us take

$$1378000000000000 \text{ or } 0.00000000000000013$$

these numbers are tedious to write and difficult to read. Therefore people use exponents to make a better representation of such numbers.

Recall if you have some number in decimals

$$\sqcup\sqcup.\sqcup\sqcup\sqcup$$

the multiplication by 10 will move the point to the right and division by 10 will move comma to the left. Therefore,

$$\sqcup.\sqcup\sqcup\sqcup\sqcup\times 10 = (\sqcup\sqcup.\sqcup\sqcup\sqcup\times\frac{1}{10})\times 10 = \sqcup\sqcup.\sqcup\sqcup\sqcup\times\frac{10}{10} = (\sqcup\sqcup.\sqcup\sqcup\sqcup\times 10)\times\frac{1}{10} = \sqcup\sqcup\sqcup.\sqcup\sqcup\times 10^{-1}$$

Definition 7.1. When a number has been written as a product of

$$a \times 10^S \ , \ 1 \leq a \leq 10$$

it is said to be written in scientific notation.

Example 7.2. (*4.1 ex.69*) Write in scientific notation:

$$454.2 = 4.542 \times 10^2 \, .$$

(*4.1 ex.81*) Write as decimal

$$1.1 \times 18^8 = 110000000 \, .$$

Question 7.1. *Try to put the following number in any readable form:*

$$318\ 000\ 000\ 000\ 001.$$

7.3. Square roots

A real number is said to be squared when it is raised to the power 2. Taking square root is the operation inverse to squaring.

Definition 7.2. The square root of a number a is number b such that

$$b^2 = a$$

Question 7.2. *If b is a square root of a, what could be said about $-b$? How many roots has a if a is negative?*

Let us list some properties of square root

$$\begin{array}{lll} \text{if} & a > 0 & \text{there exist two square roots;} \\ \text{if} & a = 0 & \text{the square root of 0 is 0;} \\ \text{if} & a < 0 & \text{there are no square roots.} \end{array}$$

The positive square root is called the Principal square root and the symbol $\sqrt{\cdot}$ called radical sign is used to denote the principle (or non-negative) square root. So, $\sqrt{a} > 0$; another square root is given by $-\sqrt{a}$.

Definition 7.3. In some cases an expression can be presented as a square of something not ended with the radical sign. Such expression are called *perfect squares*. Examples:

$$(a+x)^2\ ,\ (\sqrt{x}+a)^2 \qquad \text{are perfect squares,}$$

$$(\sqrt{3})^2 \qquad \text{is not a perfect square.}$$

Remark 7.1. Taking square root or perfect squares is easy and pleasant. Note nevertheless, that some care still should be taken. You should keep in mind that $\sqrt{a^2} = |a|$. As an exercise, try to prove why absolute value is need to be putted.

Example 7.3. (*4.2 ex.6*) Evaluate

$$\sqrt{9} + \sqrt{16} = \sqrt{3^2} + \sqrt{4^2} = 3 + 4.$$

Some more properties of square roots

$$\sqrt{ab} = \sqrt{a}\sqrt{b}, \qquad \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

to prove the above identities we need to recall the definition of square root. The definition says that $\sqrt{a}\sqrt{b} = \sqrt{ab}$ if and only if $\left(\sqrt{a}\sqrt{b}\right)^2 = ab$. Let us check it

$$\left(\sqrt{a}\sqrt{b}\right)^2 = \left(\sqrt{a}\right)^2\left(\sqrt{b}\right)^2 = ab\,.$$

Example 7.4. (*4.2 ex.16*)

$$\sqrt{\frac{25}{4}} = \frac{\sqrt{25}}{\sqrt{4}} = \frac{5}{2}.$$

Next we will consider some examples on sums and differences of square roots. The idea is to combine the like terms.
(*4.2. ex.27*)

$$\begin{aligned} &2\sqrt{12} - 3\sqrt{6} + 5\sqrt{27} \\ &\quad = 2\sqrt{4\cdot 3} - 3\sqrt{3\cdot 2} + 5\sqrt{9\cdot 3} = 2\sqrt{4}\sqrt{3} - 3\sqrt{3}\cdot\sqrt{2} + 5\sqrt{9}\cdot\sqrt{3} \\ &\quad = \sqrt{3}(2\sqrt{4} - 3\sqrt{2} + 5\sqrt{9}) = \sqrt{3}(2\cdot 2 - 3\sqrt{2} + 5\cdot 3) = \sqrt{3}(19 - 3\sqrt{2}). \end{aligned}$$

(*4.2. ex.36*)

$$(3-\sqrt{2})(3+\sqrt{2}) = 3^2 - (\sqrt{2})^2 = 9 - 2 = 7.$$

7.4. Rationalizing

There are several conventions in mathematics describing how to write mathematical expressions. In particular, there is a rule saying that the denominator of a quotient should not contain radicals. The process of removing the radicals from the denominator is called rationalizing.

Example 7.5. Rationalize the following expressions. (*4.2. ex.43*)

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2};$$

(*4.2. ex.56*)

$$\frac{2}{\sqrt{5}+\sqrt{3}} = \frac{2}{\sqrt{5}+\sqrt{3}}\cdot\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}-\sqrt{3}} = \frac{2(\sqrt{5}-\sqrt{3})}{5-3} = \sqrt{5}-\sqrt{3};$$

(*4.2. ex.62*)

$$\begin{aligned} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} &= \frac{\frac{\sqrt{x}-\sqrt{x+h}}{\sqrt{x+h}\sqrt{x}}}{h} = \frac{h(\sqrt{x}-\sqrt{x+h})}{\sqrt{x+h}\sqrt{x}} = \frac{h\sqrt{x+h}\sqrt{x}(\sqrt{x}-\sqrt{x+h})h}{(x+h)x} \\ &= \frac{h(x\sqrt{x+h}-(x+h)\sqrt{x})}{(x+h)x} = h\frac{(x\sqrt{x+h}-(x+h)\sqrt{x})}{(x+h)x} \\ &= h\Big(\frac{\sqrt{x+h}}{x+h} - \frac{\sqrt{x}}{x}\Big); \end{aligned}$$

Solution 2:

$$\frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} = \frac{\frac{1}{\sqrt{x+h}}}{h} - \frac{\frac{1}{\sqrt{x}}}{h} = \frac{h}{\sqrt{x+h}} - \frac{h}{\sqrt{x}} = h\left(\frac{\sqrt{x+h}}{(x+h)} - \frac{\sqrt{x}}{x}\right).$$

Let us discuss the properties of the n-th root. It will take some advanced analysis to

First, let us summarize some facts about the exponents

$$b^n \text{ is } \begin{cases} > 0, & \text{if } b > 0 \text{ and } n \text{ is even} \\ > 0, & \text{if } b < 0 \text{ and } n \text{ is even} \\ > 0, & \text{if } b > 0 \text{ and } n \text{ is odd} \\ < 0, & \text{if } b < 0 \text{ and } n \text{ is odd} \end{cases}$$

Lecture 8.

Radicals

On this lecture we will define root of degree n and discuss it's properties. In the end we will add some review examples on square roots since techniques for working with n-th roots and with square roots are very similar.

8.1. Radicals

Definition 8.1. An n-th root of a number a is a number b such that $b^n = a$.

$$\sqrt[n]{a} = b \quad \Leftrightarrow \quad b^n = a.$$

Note, that the square root is a special case of this definition.

Let us discuss the properties of the n-th root. By doing some not very advanced analysis one can prove that there exists at least one n-th root of a if $a > 0$. Here we will accept this fact without a proof and will summarize other information which is available for radicals.

First, if the number n is even then with any root b there will be a root $-b$ since

$$(-b)^n = (b)^n = a, \qquad \text{if } n \text{ is even}.$$

Second, if the number n is odd then if a is negative then its n-th root will be negative and if a is positive then its n-th root will be positive. We summarize this information in the next table

$$\text{the number of } n\text{-th roots of } a \text{ is exactly } \begin{cases} \text{two}, & >0, <0 \quad \text{if } a > 0 \text{ and } n \text{ is even} \\ \text{none}, & \qquad\quad\;\; \text{if } a < 0 \text{ and } n \text{ is even} \\ \text{one}, & >0 \qquad\; \text{if } a > 0 \text{ and } n \text{ is odd} \\ \text{one}, & <0 \qquad\; \text{if } a < 0 \text{ and } n \text{ is odd} \end{cases}$$

Definition 8.2. The principal n-th root of a number a denoted by the radical sign $\sqrt{\cdot}$ is

$$\sqrt[n]{a} = \begin{cases} \text{positive } n\text{-th root if } n \text{ is even}, \\ n\text{-th root if } n \text{ is odd}. \end{cases}$$

Example 8.1. (*4.3 ex.1*)

$$\sqrt[4]{16} = \sqrt[4]{2^4} = 2.$$

(*4.3 ex.2*)

$$\sqrt[4]{1} = 1.$$

(*4.3 ex.4*)

$$\sqrt[3]{125} = \sqrt[3]{5 \cdot 25} = \sqrt[3]{5 \cdot 5 \cdot 5} = \sqrt[3]{5^3} = 5.$$

(*4.3 ex.10*)

$$\sqrt[3]{27x^6} = \sqrt[3]{3^3(x^2)^3} = \sqrt[3]{(3x^2)^3} = 3x^2.$$

8.2. Properties of radicals

The first property follows from the definition of radicals and the definition of the principal root

$$\sqrt[n]{a^n} = \begin{cases} a & \text{if } n \text{ is odd} \\ |a| & \text{if } n \text{ is even} \end{cases}$$

Example 8.2.

$$\sqrt[3]{(-3)^3} = -3, \qquad \sqrt[2]{(-2)^2} = 2 = |-2|$$

Another properties follow from the definition of radicals and properties of exponents

$$\begin{aligned} \sqrt[n]{ab} &= \sqrt[n]{a}\,\sqrt[n]{b}, \\ \sqrt[n]{\frac{a}{b}} &= \frac{\sqrt[n]{a}}{\sqrt[n]{b}}, \\ \sqrt[n]{a^m} &= \left(\sqrt[n]{a}\right)^m, \\ \sqrt[m]{\sqrt[n]{a}} &= \sqrt[mn]{a}. \end{aligned}$$

Remark 8.1. Note that these identities are only well-defined when both a and b are positive. For example, the first one is meaningless for $a < 0$, $b < 0$, and n even.

Question 8.1. *How to prove these identities?*

Let us prove the last one. When we raise both sides to degree nm. Using the properties of integer exponents and the definition of radicals we rewrite the left as

$$\left(\sqrt[m]{\sqrt[n]{a}}\right)^{m \cdot n} = \left(\left(\sqrt[m]{\sqrt[n]{a}}\right)^m\right)^n = \left(\sqrt[n]{a}\right)^n = a.$$

At the same time, according to the definition of the n-th root the right side is

$$\left(\sqrt[nm]{a}\right)^{mn} = a.$$

The identity is proved.

Example 8.3. (*4.3 ex.27*)

$$\sqrt{3x^2}\sqrt{12x} = \sqrt{3}|x|\sqrt{12} = 2\sqrt{3}\sqrt{3}|x|\sqrt{x} = 6|x|\sqrt{x}$$

(*4.3 ex.31*)

$$\sqrt{\frac{4}{9x^2y^4}} = \frac{\sqrt{4}}{\sqrt{9}\sqrt{x^2}\sqrt{y^4}} = \frac{2}{3|x|y^2}.$$

(*4.3 ex.30*)

$$\frac{\sqrt[3]{x^2y}\sqrt[3]{125x^3}}{\sqrt[3]{8x^3y^4}} = \frac{\sqrt[3]{5^3x^5y}}{\sqrt[3]{2^3x^3y^4}} = \frac{5x\sqrt[3]{x^2}\sqrt[3]{y}}{2xy\sqrt[3]{y}} = \frac{5\sqrt[3]{x^2}}{2y}.$$

(*4.3 ex.47*)

$$\sqrt{8x^3} - 3\sqrt{50x} + \sqrt{2x^5} = 2|x|\sqrt{2x} - 3\cdot 5\sqrt{2x} - x^2\sqrt{2x} = (2|x| - 15 - x^2)\sqrt{2x}.$$

(*4.3 ex.51*)

$$(3\sqrt[3]{6})(2\sqrt[3]{9}) = 3(\sqrt[3]{3\cdot 2})2(\sqrt[3]{3\cdot 3}) = 6\sqrt[3]{3^3}\sqrt[3]{2} = 18\sqrt[3]{2}.$$

(*4.3 ex.54*)

$$(\sqrt[4]{4} - 2)(\sqrt[4]{4} + 2) = (\sqrt[4]{4})^2 - (2)^2 = \sqrt[4]{16} - 4 = 2 - 4 = -2.$$

8.3. Review examples

Rationalize the denominator in each expression
(*4.2 ex.55*)

$$\frac{3}{\sqrt{3}-\sqrt{2}} = \frac{3}{\sqrt{3}-\sqrt{2}}\cdot\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}} = \frac{3(\sqrt{3}+\sqrt{2})}{3-2} = 3(\sqrt{3}+\sqrt{2}).$$

(*4.2 ex.57*)

$$\frac{\sqrt{3}-\sqrt{2}}{2\sqrt{5}-\sqrt{7}} = \frac{(\sqrt{3}-\sqrt{2})(2\sqrt{5}+\sqrt{7})}{4\cdot 5-7} = \frac{2\sqrt{15}-2\sqrt{10}+\sqrt{21}-\sqrt{14}}{13}.$$

Rationalize the numerator: (*4.2 ex.59*)

$$\frac{2-\sqrt{5}}{3+2\sqrt{5}} = \frac{(2-\sqrt{5})(2+\sqrt{5})}{(3-2\sqrt{5})(2+\sqrt{5})} = \frac{1-5}{6-4\sqrt{5}+3\sqrt{5}-2\cdot 5} = \frac{-1}{-4-\sqrt{5}} = \frac{1}{4+\sqrt{5}}$$

Solve equation

$$\begin{aligned}\sqrt{x+3} - \sqrt{2x-5} &= 0,\\ \sqrt{x+3} &= \sqrt{2x-5},\\ x+3 &= 2x-5,\\ x &= 8.\end{aligned}$$

LECTURE 9.

Rational exponents

On this lecture we will give the definition of rational exponent and will discuss its properties which are, in fact, similar to the properties of integer exponent.

9.1. Rational exponents

Let us compare two identifies, one for exponents and one for radicals

$$(a^m)^n = a^{mn} \qquad \sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$$

Since the similarity is obvious it give us a hope that both the natural exponents and radicals represent the same but more general object. The relation is even more obvious if we consider the following identity for radicals

$$\left(\sqrt[n]{a}\right)^n = a = a^{\left(\frac{1}{n}\cdot n\right)} = a^{\left(n\cdot\frac{1}{n}\right)} = a = \left(\sqrt[n]{a^n}\right)$$

Finally, the gap between integer exponents and radicals will be eliminated if we define

$$a^{\frac{1}{n}} = \sqrt[n]{a}\ ,$$

or, more general, for any rational number $\frac{m}{n}, n > 0$ define

$$a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m$$

Example 9.1. *(4.4 ex.6)*

$$(64)^{\frac{3}{2}} = (4\cdot 16)^{\frac{3}{2}} = (2^6)^{\frac{3}{2}} = \sqrt[2]{(2^6)^3} = \sqrt[2]{2^{18}} = 2^9.$$

(4.4 ex.8)

$$(25)^{\frac{-5}{2}} = (5^2)^{\frac{-5}{2}} = \sqrt[2]{(5^2)^{-5}} = \frac{1}{\sqrt[2]{(5^2)^5}} = \frac{1}{5^5}.$$

Let us list basic identifies for the rational exponents. These identifies will combine together properties of integer exponents and radicals.

$$\begin{aligned} a^r a^s &= a^{r+s}, \\ (a^r)^s &= a^{rs}, \\ (ab)^r &= a^r b^r, \\ \left(\frac{a}{b}\right)^r &= \frac{a^r}{b^r}, \\ a^{-r} &= \frac{1}{a^r}, \\ \frac{a^r}{a^s} &= a^{r-s}. \end{aligned}$$

As an exercise let us check the second one

$$\begin{aligned} a^r a^s = a^{\frac{m}{n}} a^{\frac{p}{q}} &= \left(\sqrt[n]{a}\right)^m \left(\sqrt[q]{a}\right)^p \\ &= \left(\sqrt[n]{(\sqrt[q]{a})^q}\right)^m \left(\sqrt[q]{(\sqrt[n]{a})^n}\right)^p \\ &= \left(\sqrt[n]{\sqrt[q]{a}}\right)^{qm} \left(\sqrt[q]{\sqrt[n]{a}}\right)^{pn} \\ &= \left(\sqrt[nq]{a}\right)^{qm} \left(\sqrt[nq]{a}\right)^{pn} \\ &= \left(\sqrt[nq]{a}\right)^{qm+pn} = a^{\frac{qm+pn}{nq}} = a^{r+s}. \end{aligned}$$

Again, the identifies are well-defined only for positive a and b. When a and b are negative some identifies will fail.

Example 9.2. (*4.4 ex.32*)

$$\left(\frac{x^{\frac{-3}{2}}}{y^{\frac{5}{2}}}\right)^{\frac{4}{3}} \left(\frac{x^{\frac{1}{3}}}{y^{\frac{1}{3}}}\right)^{-3} = \left(\frac{x^{\frac{-3}{2}\cdot\frac{4}{3}}}{y^{\frac{5}{2}\cdot\frac{4}{3}}}\right)\left(\frac{x^{\frac{1}{3}(-3)}}{y^{\frac{1}{3}(-3)}}\right) = \frac{x^{-2}}{y^{\frac{10}{3}}} \cdot \frac{x^{-1}}{y^{-1}} = \frac{1}{x^3 y^{\frac{7}{3}}} = \frac{1}{x^3\ \sqrt[3]{y^7}}.$$

(*4.4 ex.16*)

$$\left(\frac{1}{9}\right)^{1.5} = \left(\frac{1}{9}\right)^{\frac{15}{10}} = \left(\frac{1}{9}\right)^{\frac{3}{2}} = \sqrt[2]{\left(\frac{1}{9}\right)^3} = \left(\frac{1}{3}\right)^3 = \frac{1}{27}.$$

(*4.4 ex.28*)

$$\left(4x^{-1}y^{\frac{1}{3}}\right)^{\frac{3}{2}} = 4^{\frac{3}{2}}(x^{-1})^{\frac{3}{2}}\left(y^{\frac{1}{3}}\right)^{\frac{3}{2}} = 8x^{\frac{-3}{2}}y^{\frac{1}{2}} = 8\sqrt{\frac{y}{x^3}}$$

Question 9.1. *What*

$$(x^2)^{\frac{3}{2}}$$

is equal to? Answer:

$$(x^2)^{\frac{3}{2}} = |x|^3$$

LECTURE 10.

Geometry topics

On this lecture we will consider some area formulas, prove the Pythagorean theorem, and state some extra formulas which will be used later.

10.1. Area formulas

For a rectangle of a length l and width w area is defined to be

$$\text{Area} = lw.$$

For the right triangle with the base l and altitude the area according to the picture will be

$$\text{Area} = \frac{1}{2}lh.$$

For an arbitrary triangle:

$$\text{Area} = \frac{1}{2}l_1 h + \frac{1}{2}l_2 h = \frac{1}{2}lh.$$

Let us use area formulas to prove the Pythagorean Theorem.

10.2. The Pythagorean theorem

Definition 10.1. A right triangle is one that contains a right angle - that is, an angle of $90°$. The side of the triangle opposite to $90°$ angle is called the hypotenuse; the remaining two sides are called legs.

Theorem 10.1. ***Pythagorean Theorem.*** *In a right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the legs.*

$$c^2 = a^2 + b^2$$

Proof. The side of the big square is $a + b$. Note that all four triangles are right and congruent. The inside figure is square with the side c. Obviously, the area of the big square equals to the sum of areas of triangle and the small square.

$$(a+b)^2 = 4\Big(\frac{1}{2}\Big)ab + c^2$$
$$a^2 + 2ab + b^2 = 2ab + c^2$$
$$a^2 + b^2 = c^2$$

the Pythagorean Theorem is proved. □

Example 10.1. (*4.6 ex.34*) The area of square $ABCD$ is $\mathcal{S}_{ABCD} = 100$ square feet, the area of square $BEFG$ is $\mathcal{S}_{BEFG} = 16$ square feet What is the area of the triangle CGF.

Solution. Since $\mathcal{S}_{ABCD} = 100$ then side $BC = 10$. Since $\mathcal{S}_{BEFG} = 16$ then side $BG = GF = 4$, therefore $GC = BC - BG = 6$ and

$$\mathcal{S}_{CGF} = \frac{1}{2} 6 \cdot 4 = 12.$$

(*4.6 ex.8*) Check if the triangle with the sides below is a right one:

$$6, 8, 10.$$

Take $a = 6$, $b = 8$, and $c = 10$, then

$$a^2 + b^2 = 36 + 64 = 100 = c^2.$$

Answer: the triangle is right.
(*4.6 ex.14*) Check if the triangle with the sides below is a right one:

$$5, 4, 7.$$

Take $a = 5$, $b = 4$, and $c = 7$, then

$$a^2 + b^2 = 25 + 16 \neq 49 = c^2.$$

Answer: the triangle is not right.

Question 10.1. *Why the hypotenuse is always the longest side of the triangle. Prove using Pythagorean theorem.*

10.3. Some more geometrical formulas

Volume of a rectangular box of length l, width w, and height h:

$$\text{Volume} = lwh$$

For the circle with radius r (diameter $d = 2r$)

$$\text{Area} = \pi r^2, \qquad \text{Circumference} = 2\pi r = \pi d$$

Perimeter of a rectangular of length l and width w is given by

$$\text{Perimeter} = 2l + 2w.$$

Example 10.2. (*4.6 ex.38*) How far a person can see.
R is the radius of the Earth.
h is the height of a tower.
d is the distance the person can see.

According to Pythagorean theorem $d^2 + R^2 = (R + h)^2$ or $d = \sqrt{(R + h)^2 - R^2}$.