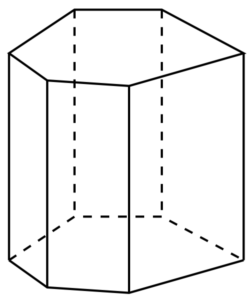
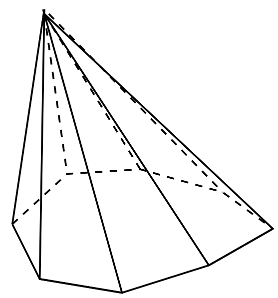
Euler's Formula: Applications

Platonic solids A *convex polygon* may be described as a finite region of the plane enclosed by a finite number of lines, in the sense that its interior lies entirely on one side of each line. Analogously, a convex polyhedron is a finite region of space enclosed by a finite number of planes. The part of the plane that is cut off by other planes is called a *face*. Any common side of two sides is called an *edge*. A point common to two or more edges is called a *vertex*.

The most common polyhedra are prisms and pyramids.



Prism



Pyramid

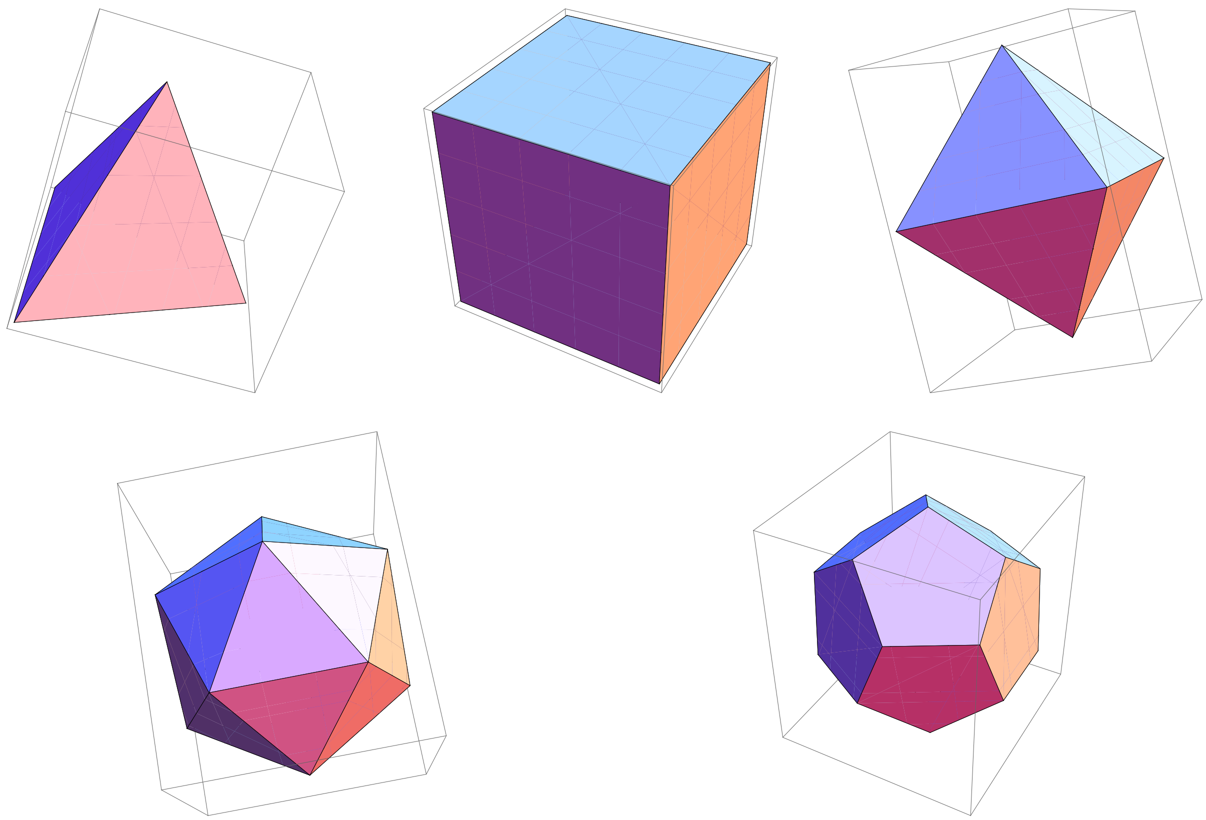
If you round out the corners and smooth out the edges, you can think of a polyhedra as the surface of a spherical body. An so you may also think of the polyhedra, with its faces, edges, and vertices, as determining a decomposition of the surface of the sphere. If the polyhedron has F faces, E edges, and V vertices, then you can apply Euler's formula to obtain

$$\boxed{V - E + F = 2}$$

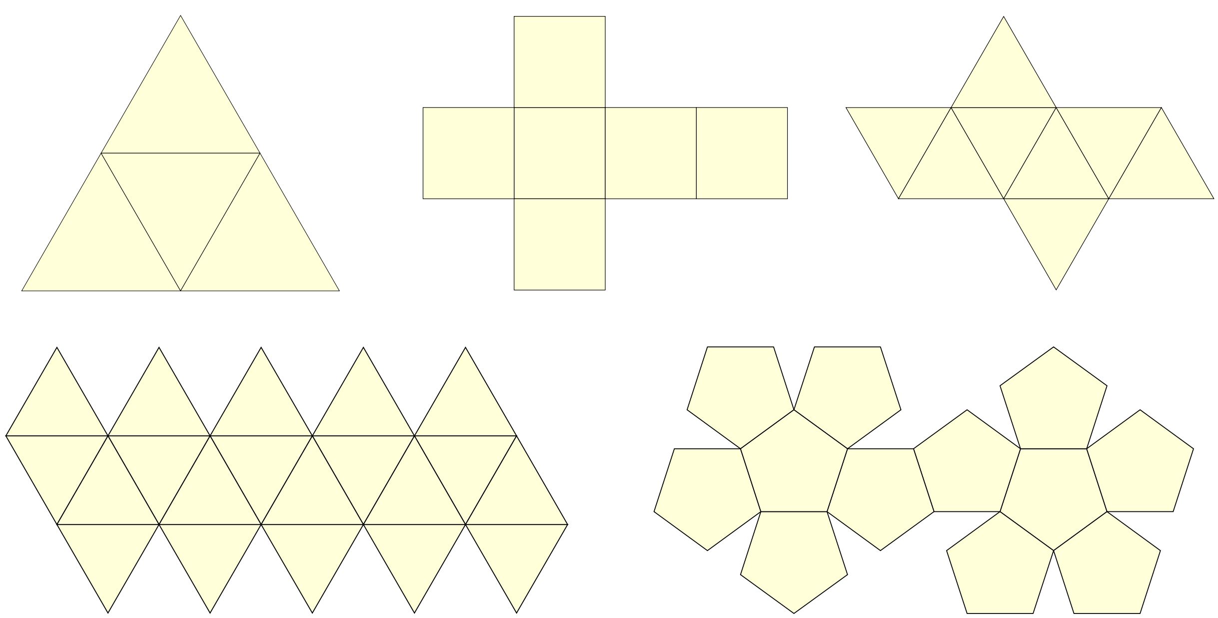
A (convex) polyhedron is called a *regular convex polyhedron* if all its faces are congruent to a regular polygon, and all its vertices are surrounded alike. Plain experimentation with sticks will allow you to easily construct 5 regular polyhedra: tetrahedron, cube, octahedron, icosahedron and dodecahedron.

These were know to the Greeks; they were regarded by Plato as symbolizing the four basic elements: fire, earth, air, and water. The fifth polyhedron, the dodecahedron, he assigned to the whole universe. Today, these polyhedra are also called *Platonic Solids*.

Here are pictures of the five Platonic solids.



When you cut apart a polyhedron along its edges and laid it flat on the plane, you obtain the net of the polyhedron. Here are the nets of the Platonic solids:



§ 1. We use Euler's formula $F - E + V = 2$ for the surface of the sphere to prove that there are only five regular convex polyhedra. Suppose that a regular polyhedron has F faces, each of which is a regular polygon with p sides, and that exactly q faces meet at every vertex. For example, for the cube we have $F = 6$ faces, each is a square (so $p = 4$) and $q = 3$ ssquares meet at each vertex.

(a) If E is the number of edges, then $2E = pF$ because each each edge belongs to two faces, so it is counted twice in the product pF.

(b) If V is the number of vertices, then $2E = qV$ because each edge has two vertices.

(c) Conclude from (1) and (2) and Euler's formula that

$$\frac{2E}{p} + \frac{2E}{q} - E = 2$$

(d) Conclude from (3) that

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2} + \frac{1}{E}$$

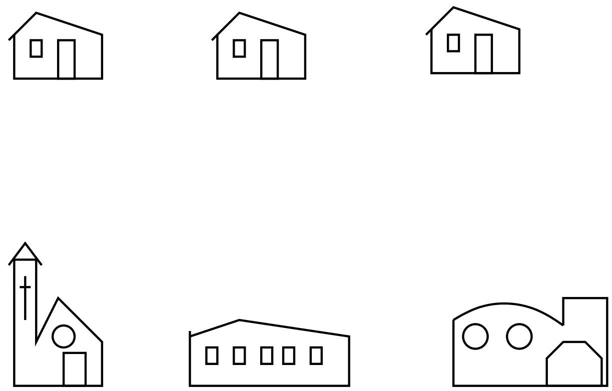
(e) Explain why we always have that $p \geq 3$ and $q \geq 3$ (Hint. A polygon has at least three sides, and at least three sides must meet at polygonal angle.)

(f) Explain why p and q cannot be both ≥ 4.

(g) List all the possibilities for p and q.

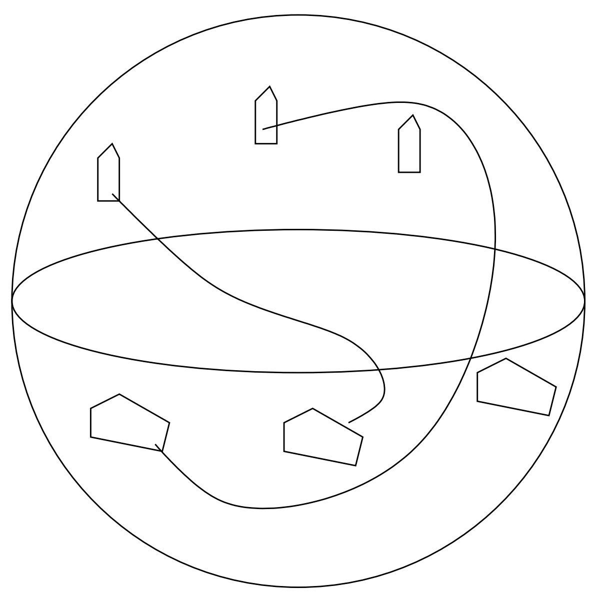
p	q

Three utilities problem This is an old problem of recreational mathematics that illustrates the use of Euler's formula. There are three houses in a county, and there is a church, a school, and a supermarket. The owners of the houses want to build roads from their properties to each the church, school, and supermarket, and want to do that in a way as to avoid crossings. Is that possible?



§ 2. To prove that the utilities problem has no solution, we will apply Euler's formula for the surface of a sphere. After doing some counting, we will arrive at a contradiction.

Imagine that a solution to the utilities problem was possible and imagine that you could actually picture it on the surface of the earth! You will see this surface divided into polygonal regions.



We do not know how many regions (faces F) there may be, but we know that each of them will be delimited by some of the roads between buildings, and their vertices will be the buildings. Thus there is a total of $V = 6$ vertices and a total of $E = 9$ edges.

(a) Because we are on the surface of a sphere, we can use Euler's formula $V - E + F = 2$ to obtain:

$F =$

(b) The faces of the division can be grouped into families $F_2, F_3, \ldots$, according to the number of edges that they have: if F_2 is the number of faces with exactly 2 edges, F_3 the number of faces with exactly three edges (triangle-like regions), F_4 the number of faces with exactly three edges (rectangle-like regions), and so on, what is the relation between F and the sum $F_2 + F_3 + F_4 + \ldots$?

(c) Imaging that you cut the surface apart along the edges, so that you end up with a total of F_2 faces with 2 sides each, F_3 faces with 3 sides each, and so on. How many edges are there in total?

(d) How does this relate to the number of edges that you see on the sphere? (Hint: when you put the pieces back, each edge is common to exactly 2 faces.)

(e) In this formula, why is $F_2 = 0$? why is $F_3 = 0$?

(f) Explain why $2E > 4F$. Is this in accordance with Euler's formula in 1?

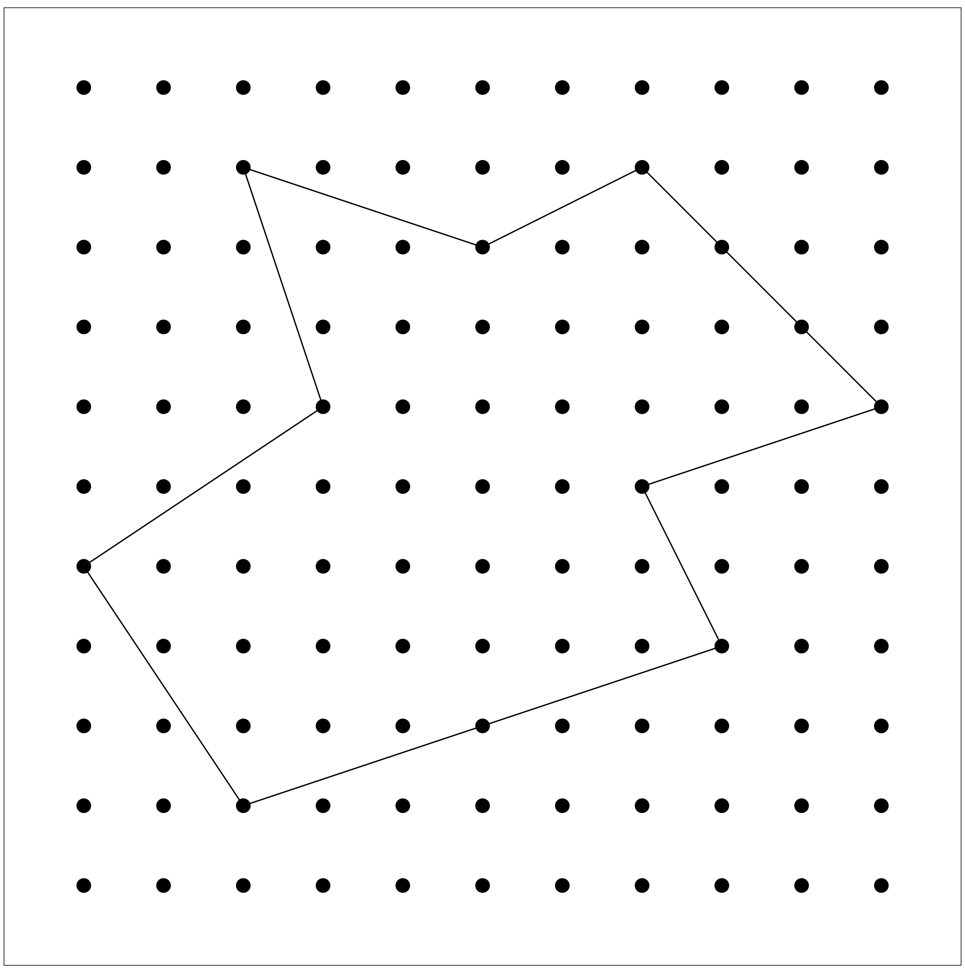
Areas on the Geoboard Points in the plane whose coordinates are both integers are called lattice points. Lattice points arise in a variety of problems. Here we apply Euler's formula to prove a surprising formula, discovered by G. Pick, for the area of a lattice polygon P (a polygon whose vertices are lattice points).

The formula is

$$\text{Area}\, P = V_I + \frac{1}{2}V_B - 1$$

where V_I is the number of lattice points inside the polygon P, and V_B is the number of lattice points on the boundary of P.

§ 3. According to Pick's formula, what is the area of the lattice polygon in this picture?



§ 4. A lattice triangle that contains no lattice points in its interior is called a *simple* lattice triangle. Note that any lattice polygon can be divided into simple lattice triangles.

Divide the lattice polygon in the figure into the least possible number of simple lattice triangles.

§ 5. To verify Pick's formula we need the fact that the area of a simple lattice triangle is $1/2$.

Divide your lattice polygon into simple lattice triangles, and suppose that you have T triangles in total. Cut out two copies of the polygon and identify their boundaries. The result is a sphere which is triangulated by copies of the simple lattice triangles that triangulated the original polygon. Let V, E, F be the vertices, edges, and faces of this triangulation of the surface of the sphere.

(a) Explain why we have $V = 2V_I + V_B$

(b) Explain why we have $F = 2T$

(c) Explain why $2E = 6T$. (Hint. All $F = 2T$ triangles provide a total of $6T$ edges. But there are repeats, because each edge is common to exactly 2 triangles.)

(d) Use Euler's formula $V - E + F = 2$ for the sphere to prove that $T = 2V_I + V_B - 2$.

(e) Use the fact that each simple lattice triangle has area $\frac{1}{2}$ to deduce Pick's formula.