Math 280: Final Study Guide, Fall 2019

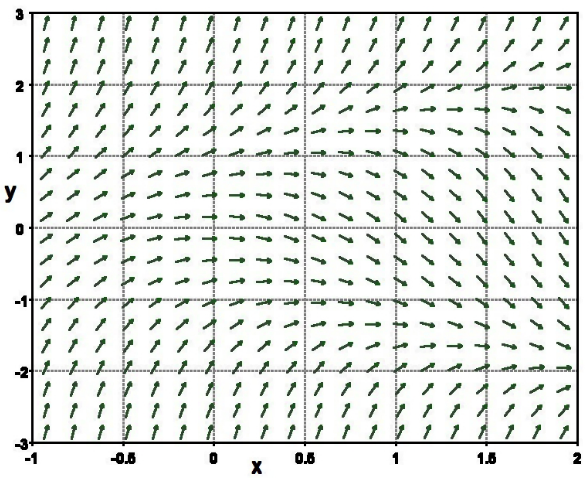
1.2: 3-6, 15-16, 17-24, 25-28, 29-34

1) Find all (x_0, y_0) such that the initial value problem with $y(x_0) = y_0$ has a unique solution $y(x)$ on a region with x near x_0. Do not solve.

   (a) $y\frac{dy}{dx} + y^2 = \tan(x)$

   (b) $(x^2 - 1)\frac{dy}{dx} = (y - 2)^{\frac{2}{3}}$.

2.1: 1-4, 19-20, 21-27, 29-30, 38-40



2) The direction field to $y' = y^2 - 2x$ is plotted above. Sketch the solutions to the initial value problems.

   (a) $y' = y^2 - 2x$, $y(0) = -2$.

   (b) $y' = y^2 - 2x$, $y(0) = 0$.

   (c) $y' = y^2 - 2x$, $y(0) = 1$.

3) Sketch the general solutions to the Autonomous ODE's by finding the critical points and identifying where the solutions are increasing/decreasing. Do not solve.

   (a) $\frac{dy}{dx} = 5y^2 - y^3$

   (b) $\frac{dy}{dx} = \frac{y^2 - y - 2}{y^2 + 1}$

4) Sketch the solutions to the initial value problems. Do not solve. (Hint: It may help to sketch the general solution first.)

(a) $\frac{dy}{dx} = y^2 - 6y + 5, \quad y(0) = 4.$

(b) $\frac{dy}{dx} = (y-2)^2, \quad y(1) = 0.$

2.2: 1-11, 15-18, 21-22, 23-26, 29-30, 31-33, 36, 43

5) (a) Find an explicit solution to the IVP. $x\frac{dy}{dx} - y^2 = 1, \quad y(1) = -1.$

   (b) Find an implicit solution to the IVP. $(xy + x)\frac{dy}{dx} = y\ln(x), \quad y(e) = 1.$

6) Find an explicit solution to the initial value problem. Then determine the maximum interval on which the solution is defined.

   (a) $y' = y^2, \ y(2) = 1.$

   (b) $xy' = -2y, \ y(2) = 1.$

7) Solve the initial value problem in terms of a **definite integral**.

   (a) $xy' = e^{2x}, \ y(1) = 3.$

   (b) $\frac{dy}{dx} = \frac{e^{x^2}}{y^2}, \ y(2) = 3.$

2.3: 1-24, 25-36, 43-48

8) Find the general solutions.

   (a) $\frac{dy}{dx} - 2xy = x.$

   (b) $x\frac{dy}{dx} + (x-2)y = x^3.$

   (c) $(x^2+1)\frac{dy}{dx} + xy = x.$

2.2, 2.3: Be able to recognize if an equation is linear or separable.

9) Find explicit solutions to the initial value problems.

   (a) $e^x\frac{dy}{dx} + 2xy^2 = 0, \quad y(0) = 1.$

   (b) $\cos(x)\frac{dy}{dx} + \sin(x)y = 1, \quad y(0) = 2.$

2.5: 1-8, 11-12, 15-19, 21-22

10) Solve the IVP's using the substitution $y = xu$.

   (a) $\frac{dy}{dx} = 1 - \frac{y}{x} + \left(\frac{y}{x}\right)^2, \quad y(1) = 3.$

   (b) $\frac{dy}{dx} = \frac{2x^2+y^2}{xy}, \quad y(1) = 2.$

3.1: 1-8, 11-12, 13-18, 21-25, 27-28, 29-30, 35-38

11) A frozen potato of temperature $25^\circ\,F$ is placed in an oven of temperature $325^\circ\,F$.

    (a) Using Newton's Law of Cooling/Heating, solve for the temperature $T(t)$.

    (b) Suppose you check that the temperature of the potato is 125° after 20 minutes. What is k for this potato? (Write an exact answer.)

    (c) Sketch a graph of the temperature $T(t)$.

12) A tank contains 100 gallons of water and 20 pounds of salt. Water containing .1-lb/gal of salt enters the tank at a rate of 3 gal/min, while water drains from the tank at the same rate.

    (a) Solve for the amount of salt in the tank, $x(t)$.

    (b) Sketch a graph of $x(t)$.

13) A tank contains 100 gallons of water and 20 pounds of salt. Water containing .1-lb/gal of salt enters the tank at a rate of 3 gal/min, while water drains from the tank at a rate of 1 gal/min. Solve for the amount of salt in the tank, $x(t)$.

14) A series RL-circuit has $L = 2\,H$ and $R = 10\,\Omega$

    (a) Find the current $\mathbf{i}(t)$ through the circuit if $I(0) = 1\,A$ and the input voltage is $E(t) = 100t\,V$.

    (b) Sketch the graph of $\mathbf{i}(t)$.

15) A package of mass $m = 100\,kg$ is dropped from an airplane. Suppose it is attached to a parachute, subject to air resistance, such that the downward velocity satisfies $m\frac{dv}{dt} = mg - kv$, where k is a positive constant and $g = 9.8 m/s^2$.

    (a) Suppose you designed the parachute such that the velocity would approach $10\,m/s$ if $t \to \infty$. What is k?

    (b) Solve for $v(t)$, assuming $v(0) = 55\,m/s$.

    (c) Sketch the solution that satisfies parts a) and b).

4.1: 1-6, 9-10, 15-19, 21, 23-28

16) Determine whether the given set of functions is linearly dependent on $(-\infty, \infty)$.

    (a) $\{e^{2x}\cos(x), e^{2x}\sin(x)\}$

    (b) $\{x^3, x^2, x\}$.

    (c) $\{1, \sin^2(x), \cos(2x)\}$

17) Verify that the given set of functions forms a fundamental solution set on the given interval.

(a) $x^2y'' - 5xy' + 9y = 0, \quad \{x^3, x^3\ln(x)\}, \quad$ on $(0, \infty)$.

(b) $(x-2)y'' - xy' + 2y = 0, \quad \{e^x, x^2 - 2x + 2\}, \quad$ on $(-\infty, 2)$.

4.2: 1-4, 6, 9-12, 16

18) Verify that $y_1(x)$ solves the ODE. Then find the general solution by reduction of order.

(a)

(b) $xy'' - y' + (1-x)y = 0, \quad y_1 = e^x$.

4.3: 1-11, 15-24, 29-31, 35-36, 49-58

19) Find the general solution to the higher order constant coefficient ODE's.

(a) $y''' + 4y'' + 5y' = 0$.

(b) $y''' + 3y'' + 3y' + y = 0$. (Hint: $(m+1)^3$.)

(c) $y'''' + 2y'' + y = 0$. (Hint: $(m^2+1)^2$.)

20) Find a constant-coefficient homogeneous ODE with the given general solution.

(a) $y = c_1e^{-3x}\cos(x) + c_2e^{-3x}\sin(x)$.

(b) $y = c_1e^{2x} + c_2xe^{2x} + c_3x^2e^{2x}$.

(c) $y = c_1 + c_2x + c_3\cos(5x) + c_4\sin(5x)$.

4.4: 1-14, 19, 21, 27-34; 4.5: 35-50, 55-58

21) Solve the initial value problems by undetermined coefficients.

(a) $y'' - 2y' = 4x + 3e^{-x}, \quad y(0) = 1,\ y'(0) = 2$.

(b) $y'' - 2y' + y = 2e^x - e^{2x}, \quad y(0) = 0,\ y'(0) = 1$.

22) Solve the initial value problems.

(a) $y'' + \omega^2y = \sin(\gamma x) \quad (\gamma \neq \omega), \quad y(0) = 0,\ y'(0) = 0$.

(b) $y'' + \omega^2y = \sin(\omega x), \quad y(0) = 0,\ y'(0) = 0$.

4.6: 1-6, 9, 11-12, 15, 23-26, 27-28; 4.7: 19-24

23) Find a particular solution by variation of parameters. Show that you get the same answer that you would by undetermined coefficients. $y'' - y' - 2y = e^{3x}$.

24) (a) Find a particular solution to $y'' + y = \sec^3(x)$.

(b) Find the general solution to $y'' + y = \sec^3(x)$.

(c) Solve the initial value problem $y'' + y = \sec^3(x), \quad y(\pi) = 0,\ y'(\pi) = 1$.

25) $\{x, x^3\}$ is a fundamental solution set to $x^2y'' - 3xy' + 3y = 0$. Use variation of parameters to find the general solution to $x^2y'' - 3xy' + 3y = x^5$.

26) $\{x, x\ln(x)\}$ is a fundamental solution set to $x^2y'' - xy' + y = 0$. Use variation of parameters to find a particular solution to $x^2y'' - xy' + y = x\ln(x)$.

27) Find a particular solution, using **definite integrals**. $y'' - y = \frac{1}{x^4+1}$.

4.4, 4.6: Know when to use undetermined coefficients and when to use variation of parameters.

28) Find the general solutions.

(a) $y'' + 2y' + y = \frac{e^{-x}}{x^3}$

(b) $y'' + 2y' + 2y = xe^{-2x}$

4.9: 1-7, 21-22

29) Solve the systems by substitution/elimination.

(a)

$$\begin{aligned} \frac{dx}{dt} &= 3x - 2y + t \qquad x(0) = 1 \\ \frac{dy}{dt} &= 2x - y + 3 \qquad y(0) = 0 \end{aligned}$$

(b)

$$\begin{aligned} \frac{dx}{dt} &= x - y \qquad x(0) = 1 \\ \frac{dy}{dt} &= x + y + 2e^{2t} \qquad y(0) = 2 \end{aligned}$$

5.1: 1-6, 8, 21-24, 25-31, 33-37, 41-42, 49-53, 56-57

30) An object of mass $m = 5\,kg$ is attached to an unforced spring ($F(t) = 0$), with spring constant $k = 25\,N/m$ and a damping force of 10 times the velocity. The object starts at 1 meter below equilibrium, and has an initial velocity $1\,m/s$ towards equilibrium (upwards).

(a) Solve for the position $x(t)$.

(b) Give a sketch of the solution.

(c) Is the spring overdamped, critically damped, or underdamped?

31) A ball of mass $m = 1\,kg$ is attached to an unforced spring ($F(t) = 0$), with spring constant $k = 9\,N/m$ and a damping force of of 6 times the velocity. The object starts at equilibrium, with initial velocity $3\,m/s$ upwards.

(a) Solve for the position of the ball.

(b) Is the spring overdamped, critically damped, or underdamped?

(c) Show that the maximum displacement of the ball from equilibrium is $\approx \frac{1}{2.7}$ meters.

(d) Sketch the solution.

32) An object of mass $m = 3\,kg$ is attached to a forced spring, with spring constant $k = 6\,N/m$ and a damping force of 9 times the velocity. The object starts at equilibrium with no initial velocity.

(a) Solve the position of the spring if the forcing function is $F(t) = 30\cos(t)\,N$.

(b) Identify the transient and steady state solutions.

7.1: 1-16, 19-36, 37-38

33) Find the Laplace transforms **directly from the definition**.

(a) $f(t) = te^{2t}$.

(b) $f(t) = \begin{cases} e^t & 0 \le t < 3 \\ 0 & t \ge 3. \end{cases}$

(c) $f(t) = \begin{cases} t & 0 \le t < 2 \\ 2 & t \ge 2. \end{cases}$

7.2: 1-30, 35-44, 45-46

34) Find $\mathcal{L}^{-1}\{F(s)\}$.

(a) $F(s) = \frac{6s+4}{s^3(s+2)}$.

(b) $F(s) = \frac{8s^2}{s^4-16}$.

(c) $F(s) = \frac{8s-24}{(s^2+1)(s^2+9)}$.

35) Solve the initial value problems using Laplace transforms.

(a) $y'' - 9y = 18$, $y(0) = 0$, $y'(0) = 0$.

(b) $y'' + y = 5e^{2t}$, $y(0) = 0$, $y'(0) = 3$.

(c) $y'' + 2y' = 5\cos(t)$, $y(0) = 1$, $y'(0) = 0$.

7.3: 1-8, 11-18, 21-30, 37-48, 49-54, 55-57, 59, 61-62, 63-65

36) Solve the initial value problems by Laplace transforms.

(a) $y' + 2y = 5e^{-t}\cos(2t), \quad y(0) = -1$.

(b) $y'' - y' - 2y = 4t, \quad y(0) = 2,\ y'(0) = 3$.

(c) $y'' - y = 2e^{-t}, \quad y(0) = 0,\ y'(0) = 3$.

(d) $y'' + 4y = 25te^{-t}, \quad y(0) = 0,\ y'(0) = 0$.

37) Write in terms of unit step functions. Then find the Laplace transform.

(a) $f(t) = \begin{cases} 1 & 0 \le t < 2 \\ 2 & 2 \le t < 3 \\ 4 & t \ge 3. \end{cases}$

(b) $f(t) = \begin{cases} 0 & 0 \le t < 3 \\ 1 & 3 \le t < 5 \\ 0 & t \ge 5. \end{cases}$

(c) $f(t) = \begin{cases} t & 0 \le t < 3 \\ 0 & t \ge 3 \end{cases}$

7.4: 1-8, 45, 49-50, 51-52

38) Find $\mathcal{L}\{f\}$

(a) $f(t) = t^2 \sin(2t)$.

(b) $f(t) = te^t \cos(t)$.

39) Solve by Laplace transforms.

(a) $y + \int_0^t y(\tau)d\tau = 5\cos(2t)$.

(b) $y' + \int_0^t y(\tau)d\tau = t + 2, \quad y(0) = 4$.

7.5: 1-12

40) Solve the initial value problems.

(a) $y'' + 2y' + 5y = 3\delta(t)\, y(0) = 1, y'(0) = 0$.

(b) $y'' - y = 2\delta(t-3),\ y(0) = 3, y'(0) = -1$.

8.1: 1-6, 7-10, 11-16, 17-20, 21-23

41) Verify that the vectors are a fundamental solution set of the system.

(a) $\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 1 & 2 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$. $\quad \vec{X}_1(t) = \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix}, \vec{X}_2(t) = \begin{bmatrix} (2t+1)e^{-t} \\ -2te^{-t} \end{bmatrix}$.

(b) $\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 1 & -2 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$. $\quad \vec{X}_1(t) = \begin{bmatrix} 2\cos(3x) \\ \cos(3x) + 3\sin(3x) \end{bmatrix}, \vec{X}_2(t) = \begin{bmatrix} 2\sin(3x) \\ \sin(3x) - 3\cos(3x) \end{bmatrix}$.

8.2: 1-6, 13, 21-24, 31, 35-40, 48

42) $\frac{dx}{dt} = x - y$.
$\frac{dy}{dt} = 4x + 5y$.

(a) Write the system in matrix form.

(b) Find a vector solution by eigenvalues/eigenvectors.

(c) Using the vector solution, write the solutions $x(t)$ and $y(t)$.

43) Do the same as the previous question, for the initial value problem
$\frac{dx}{dt} = 8x + 6y, \quad x(0) = -1.$
$\frac{dy}{dt} = -3x - y, \quad y(0) = 2.$

41) Do the same as the previous question, for the initial value problem
$\frac{dx}{dt} = x - 2y, \quad x(0) = 1.$
$\frac{dy}{dt} = 5x - y, \quad y(0) = 2.$

Transforms and equations.

$f(t)$	$F(s)$
$\delta(t)$	1
1	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
e^{at}	$\frac{1}{s-a}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$e^{at}\sin(\omega t)$	$\frac{\omega}{(s-a)^2+\omega^2}$
$e^{at}\cos(\omega t)$	$\frac{s-a}{(s-a)^2+\omega^2}$
$e^{at}f(t)$	$F(s-a)$
$\delta(t-a)$	e^{-a}
$u(t-a)$	$\frac{1}{s}e^{-as}$
$f(t-a)u(t-a)$	$F(s)e^{-as}$
$f(t)u(t-a)$	$\mathcal{L}\{f(t+a)\}e^{-as}$
$\frac{d^n}{dt^n}f(t)$	$s^nF(s) - s^{n-1}f(0) - ... - sf^{(n-2)}(0) - f^{(n-1)}(0)$

- $$\frac{dT}{dt} = k(T - T_m).$$
- $$L\frac{d\mathbf{i}}{dt} + R\mathbf{i} = E(t).$$
- $$m\frac{dv}{dt} = mg - kv.$$
- $$m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = F(t)$$