Bernoulli Equation

Larry Caretto
Mechanical Engineering 390
Fluid Mechanics

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California State University Northridge

Outline

- Review forces on submerged surfaces
- Streamlines
- Bernoulli equation derivation
- Constant density flows
- Bernoulli equation for ideal gas flows
- Continuity equation
- Cavitation
- Flow measurement

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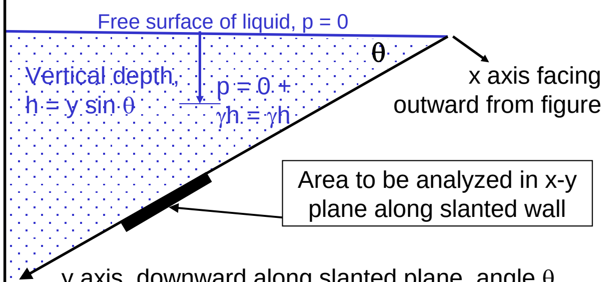
Review Slanted Surface



y axis, downward along slanted plane, angle θ with liquid surface, y = 0 at liquid surface

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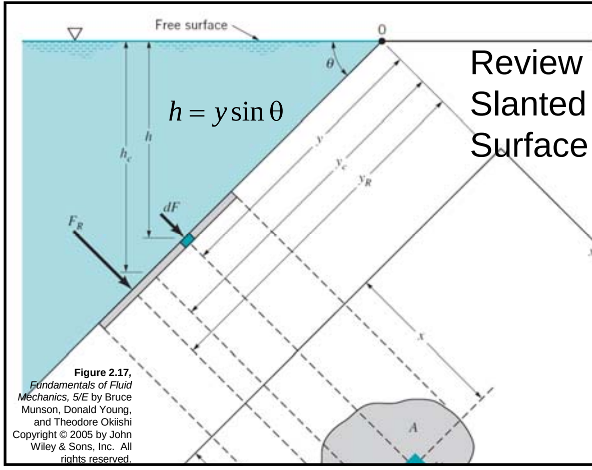
Review Slanted Surface



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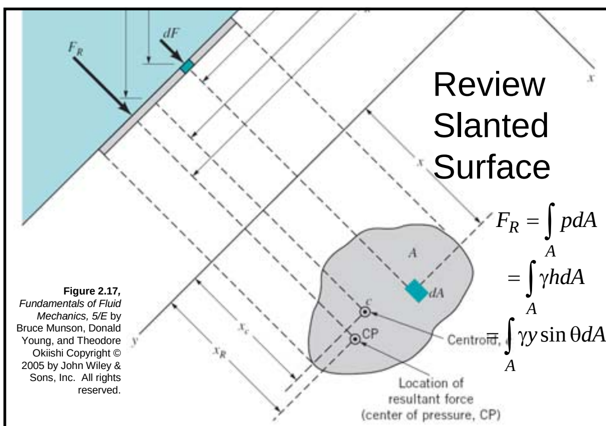
Review Slanted Surface



$$F_R = \int_A p\,dA = \int_A \gamma h\,dA = \int_A \gamma y \sin\theta\,dA$$

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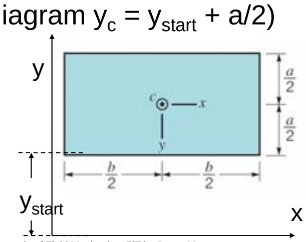
Review Resultant Force

$$F_R = \gamma \sin\theta \int_A y\,dA = \gamma y_c A \sin\theta$$

- Center of pressure, **not y_c**, is ***location*** of resultant force (in diagram $y_c = y_{start} + a/2$)

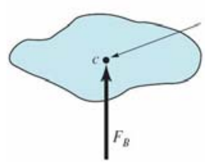
$$y_{CP} = \frac{I_x}{Ay_c} = \frac{I_{xc}}{Ay_c} + y_c$$

$I_x = I_{xc} + Ay_C^2$ (in diagram $I_{xc} = ba^3/12$ and $A = ab$

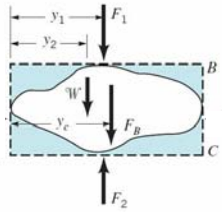


California State University Northridge **Figure 2.18(a),** *Fundamentals of Fluid Mechanics, 5/E* by Bruce Munson, Donald Young, and Theodore Okiishi Copyright © 2005 by John Wiley & Sons, Inc. All rights reserved. 6

Review Buoyancy



- Buoyant force, F_B, due to difference in pressure between top and bottom



- $F_B = \gamma_{fluid} V_b$
- Passes through the centroid of the submerged body

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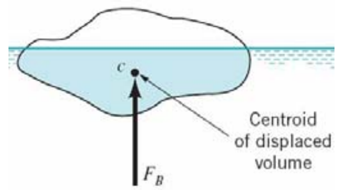
Review Buoyancy II



- If the object is light enough, it will float on the top of the liquid
- This analysis ignores the weight of the top layer of fluid assumed to be air
  - Can consider buoyancy of an object be-tween two fluid layers with different densities

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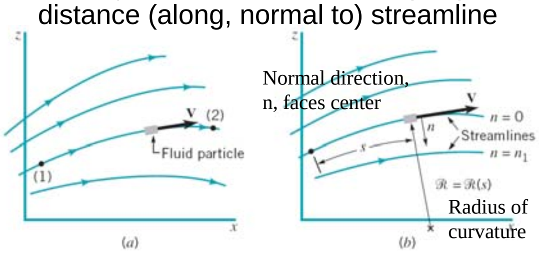
Streamlines

- A line everywhere tangent to a the velocity vector is a streamline (s, n) = distance (along, normal to) streamline



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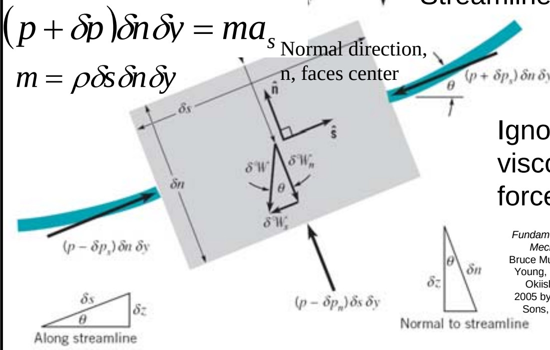
$$-\gamma \delta s \delta n \delta y \sin\theta + (p - \delta p)\delta n \delta y - (p + \delta p)\delta n \delta y = m a_s$$

$$m = \rho \delta s \delta n \delta y$$

F= ma Balance along Streamline



Ignore viscous forces

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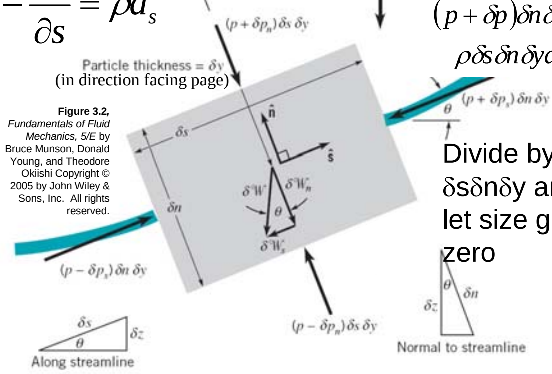
$$-\gamma \sin\theta - \frac{\partial p}{\partial s} = \rho a_s$$

$$-\gamma \delta s \delta n \delta y \sin\theta + (p - \delta p)\delta n \delta y - (p + \delta p)\delta n \delta y = \rho \delta s \delta n \delta y a_s$$



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Divide by δsδnδy and let size go to zero

F = ma Along Streamline

- Start with equation from previous slide and define acceleration terms

$$-\gamma \sin\theta - \frac{dp}{ds} = \rho a_s = \rho \frac{dV}{dt} = \rho\left[\frac{\partial V}{\partial t} + \frac{dV}{ds}\frac{ds}{dt}\right]$$

- Assume steady flow so $\partial V/\partial t = 0$ and substitute V for ds/dt

$$-\gamma \sin\theta - \frac{dp}{ds} = \rho \frac{dV}{ds}\frac{ds}{dt} = \rho \frac{dV}{ds} V = \frac{1}{2}\rho \frac{d\left(V^2\right)}{ds}$$

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F = ma Along Streamline II

- From diagram $\sin\theta = dz/ds$; substitute this for $\sin\theta$, multiply by ds and integrate

$$-\gamma\sin\theta - \frac{dp}{ds} = -\gamma\frac{dz}{ds} - \frac{dp}{ds} = \frac{1}{2}\rho\frac{d\left(V^2\right)}{ds}$$

$$\frac{\rho g dz}{\rho} + \frac{dp}{\rho} + \frac{1}{2}d\left(V^2\right) = g dz + \frac{dp}{\rho} + \frac{1}{2}d\left(V^2\right) = 0$$

$$g\left(z_2 - z_1\right) + \int_1^2 \frac{dp}{\rho} + \frac{\left(V_2^2 - V_1^2\right)}{2} = 0$$

Bernoulli Equation

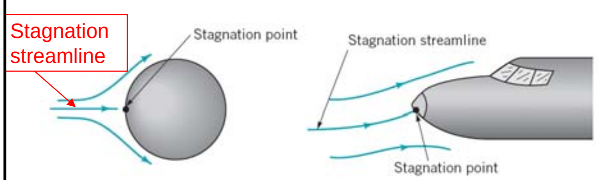
- Limited to steady, inviscid, streamline flow
- Restriction to steady flow comes from assumption that velocity changes in space, but not with time ($\partial V/\partial t = 0$)
- Have to know ρ-p equation to integrate
  - Simplest relation is constant density

For constant density
$$g\left(z_2 - z_1\right) + \frac{p_2 - p_1}{\rho} + \frac{\left(V_2^2 - V_1^2\right)}{2} = 0$$

Bernoulli Equation

Stagnation Point Flow

- External flow over a body
- At one point, called stagnation point, velocity is zero



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Stagnation Streamline

- What is the pressure at a stagnation point if, at another point along the streamline, the pressure, elevation and velocity are p_1, z_1 and V_1?
- Assume steady, inviscid, incompressible flow along the streamline
- For stagnation, $V_2 = 0$

$$g\left(z_2 - z_1\right) + \frac{p_2 - p_1}{\rho} + \frac{\left(V_2^2 - V_1^2\right)}{2} = 0$$

Stagnation Streamline II

- Solve Bernoulli equation for p_2 with $V_2 = 0$

$$g\left(z_2 - z_1\right) + \frac{p_2 - p_1}{\rho} + \frac{\left(V_2^2 - V_1^2\right)}{2} = 0$$

$$p_2 - p_1 = \frac{\rho V_1^2}{2} - \rho g\left(z_2 - z_1\right) = \frac{\rho V_1^2}{2} - \gamma\left(z_2 - z_1\right)$$

- For $z_1 = z_2$ $\quad p_2 - p_1 = \frac{\rho V_1^2}{2}$
- Find $p_2 - p_1$ for $V_1 = 30$ m/s and ρ = 1.2 kg/m³ (air) and ρ =1000 kg/m³ (water)

Stagnation Streamline III

- Find $p_2 - p_1$ for $V_1 = 30$ m/s and ρ = 1.2 kg/m³ (air) and ρ =1000 kg/m³ (water)

$$p_2 - p_1 = \frac{\rho V_1^2}{2} = \frac{1}{2}\frac{1.2\,kg}{m^3}\left(\frac{30\,m}{s}\right)^2\frac{kPa\cdot m\cdot s^2}{1000\,kg} = 0.54\,kPa$$

$$p_2 - p_1 = \frac{\rho V_1^2}{2} = \frac{1}{2}\frac{1000\,kg}{m^3}\left(\frac{30\,m}{s}\right)^2\frac{kPa\cdot m\cdot s^2}{1000\,kg} = 450\,kPa$$

Head Terms

- Bernoulli equation between two points along a streamline (ρ = constant)

$$g(z_2 - z_1) + \frac{p_2 - p_1}{\rho} + \frac{(V_2^2 - V_1^2)}{2} = 0$$

- This is equivalent to saying that p + ρV²/2 + γz is constant along a streamline – divide by g (recall γ = ρg)

$$z + \frac{p}{\gamma} + \frac{V^2}{2g} = \text{Constant along a streamline}$$

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Head Terms II

- Terms in equation just written have dimensions of length called head

$$z + \frac{p}{\gamma} + \frac{V^2}{2g} = \text{Constant along a streamline}$$

  – z is elevation head
  – p/γ is pressure head
  – $V^2/2g$ is velocity head

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Energy Terms

- Bernoulli equation terms $\rho V^2/2 + \gamma z + p$ can also be interpreted as energy terms per unit volume
  – Kinetic energy $mV^2/2$ divided by volume so that m/volume = density, ρ
  – Potential energy mgz divided by volume so that mg/volume = γ
  – Pressure energy is product of p times volume – dividing (p times volume) by volume leaves p

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Forces Normal to Streamline

- Similar force balance involving pressure and weight as forces
- Result is $\gamma \frac{dz}{dn} + \frac{\partial p}{\partial n} + \frac{\rho V^2}{\Re} = 0$ $\Re$ is radius of curvature
- For negligible density $\frac{\partial p}{\partial n} = -\frac{\rho V^2}{\Re}$
- For swirling flows, pressure **decreases** towards the center

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Dynamic and Total Pressure

- Bernoulli equation analysis of stagnation point flow shows stagnation pressure, $p_2 = p_1 + \rho V_1^2/2$
- Call $\rho V^2/2$ the dynamic pressure
- Call $p + \rho V^2/2$ the stagnation pressure
- Call pressure, p, static pressure to distinguish this from other pressures
- Total pressure is $p + \gamma z + \rho V^2/2$

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Static and Stagnation Pressure

- Parallel streamlines
- h is static pressure at (1)
- H is stagnation pressure at (2)

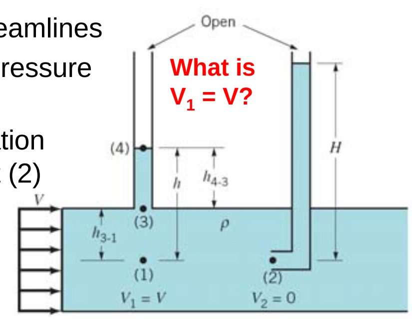

Figure 3.4, *Fundamentals of Fluid Mechanics, 5/E* by Bruce Munson, Donald Young, and Theodore Okiishi

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Finding $V_1 = V$

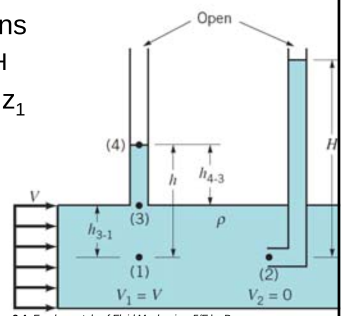
- Piezometer relations
  - $p_1 = \gamma h$ and $p_2 = \gamma H$
- Bernoulli with $z_2 - z_1 = V_2 = 0$

  $p_1 + \rho V_1^2 = p_2$

  $\gamma h + (\gamma/g)V_1^2 = \gamma H$

  $V_1^2 = g(H - h)$

  $V_1 = g(H - h)^{1/2}$



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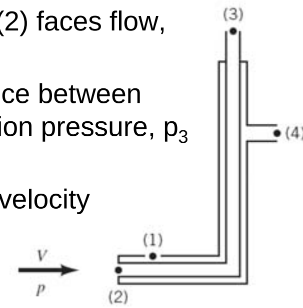
Pitot-static Tube

- Concentric tubes (2) faces flow, (1) normal to flow
- Measures difference between static and stagnation pressure, $p_3 - p_4 = \rho V^2/2$
- Used to measure velocity
- $V = [(p_3 - p_4)/\rho]^{1/2}$



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Free Jets

- Look at streamline from (1) to (2) to (5)

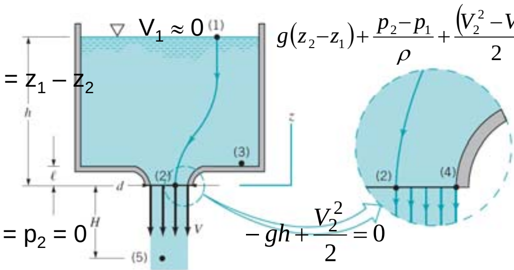
$$g(z_2 - z_1) + \frac{p_2 - p_1}{\rho} + \frac{(V_2^2 - V_1^2)}{2} = 0$$

$V_1 \approx 0$

$h = z_1 - z_2$

$p_1 = p_2 = 0$

$$-gh + \frac{V_2^2}{2} = 0$$



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Ideal Gas Isentropic Flow

- For ideal gas $P = \rho RT$
- For ideal gas with frictionless flow and no heat transfer (constant entropy or isentropic flow) $P/\rho^k = C$ ($k = c_p/c_v$)
  - $k = 1.4$ for air at room temperature
  - For $P/\rho^k = C$, $P = C\rho^k$ and $dp = Ck\rho^{k-1}d\rho$

$$\int_1^2 \frac{dp}{\rho} = \int_1^2 \frac{Ck\rho^{k-1}d\rho}{\rho} = Ck\int_1^2 \rho^{k-2}d\rho = \left.\frac{Ck\rho^{k-1}}{k-1}\right|_1^2 =$$

$$\frac{k}{k-1}\left[\frac{C\rho_2^k}{\rho_2} - \frac{C\rho_1^k}{\rho_1}\right] = \frac{k}{k-1}\left[\frac{p_2}{\rho_2} - \frac{p_1}{\rho_1}\right]$$

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Ideal Gas Isentropic Flow II

- Start with integral form of Bernoulli $g(z_2 - z_1) + \int_1^2 \frac{dp}{\rho} + \frac{(V_2^2 - V_1^2)}{2} = 0$
- Substitute ideal gas isentropic flow result $\int_1^2 \frac{dp}{\rho} = \frac{k}{k-1}\left(\frac{p_2}{\rho_2} - \frac{p_1}{\rho_1}\right)$

$$g(z_2 - z_1) + \frac{k}{k-1}\left(\frac{p_2}{\rho_2} - \frac{p_1}{\rho_1}\right) + \frac{(V_2^2 - V_1^2)}{2} = 0$$

- Equation applies to steady, frictionless ideal gas flow along streamline with no heat transfer or viscous forces

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Incompressible Error

- What is the error for assuming incompressible flow in a gas
- Compute pressure ratio p_2/p_1, where p_2 is stagnation pressure, as a function of initial Mach number, $Ma_1 = V_1/a_1$
- a = sound speed; $a^2 = kRT$ where k = heat capacity ratio c_p/c_v (T in kelvins)
  - Use of Mach number based on accurate analysis of compressible flow

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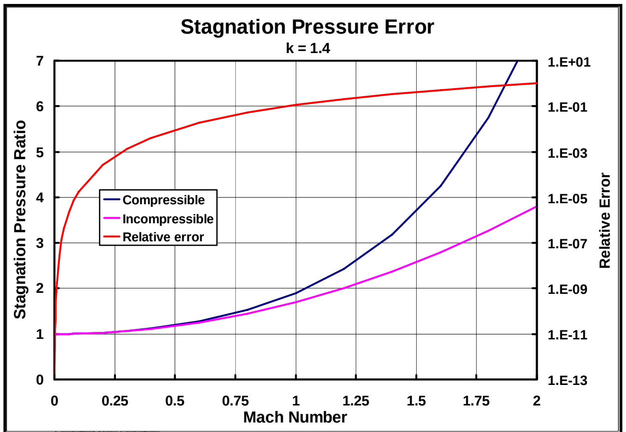

Usual Rule of Thumb

- Accurate results for gas flows can be found from incompressible flow equations provided that the Mach number is less than 0.3
  – Ma = 0.3 gives relative error of 0.13% for computation of stagnation pressure
- Results less accurate for Ma > 0.3
- Catastrophic errors for Ma > 1
  – 1145% relative error for Ma = 4
  – Shock wave for Ma > 1 changes equation

Continuity Equation

- Consider internal (confined) flow
- Look at simple example shown in figure
- Conservation of mass for steady flow
- Mass flow in = mass flow out

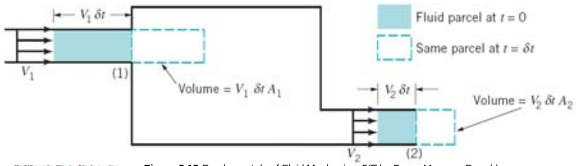

Figure 3.15 *Fundamentals of Fluid Mechanics, 5/E* by Bruce Munson, Donald Young, and Theodore Okiishi

Continuity Equation II

- Volume flow rate, Q = δ∀/δt, = VAδt/δt = VA
- Mass flow rate, $\dot m$ = ρQ = ρVA
- Mass in = mass out gives $\dot m_1 = \dot m_2$
- Continuity equation: $\rho_1 V_1 A_1 = \rho_2 V_2 A_2$

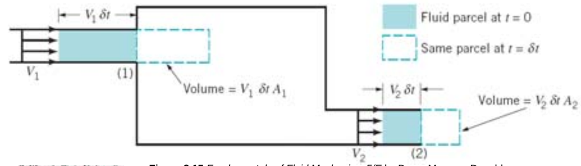

Figure 3.15 *Fundamentals of Fluid Mechanics, 5/E* by Bruce Munson, Donald Young, and Theodore Okiishi

Continuity Equation III

- For more complex confined flows there may be several inflows and outflows
- Also, flow may be unsteady
- Still conserve mass

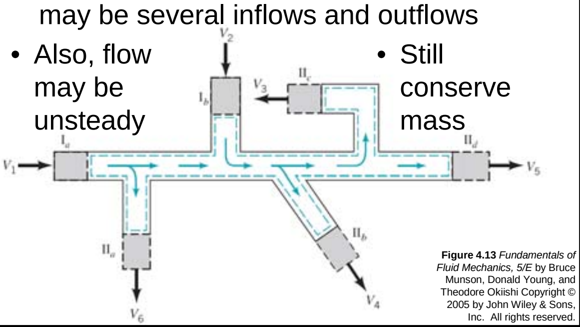

Figure 4.13 *Fundamentals of Fluid Mechanics, 5/E* by Bruce Munson, Donald Young, and Theodore Okiishi

Continuity Equation III

$$\left.\frac{dm}{dt}\right|_{\substack{control\\volume}} = \sum \dot m_{in} - \sum \dot m_{out} = \sum \rho_{in} A_{in} V_{in} - \sum \rho_{out} A_{out} V_{out}$$

- For steady flow dm/dt = 0

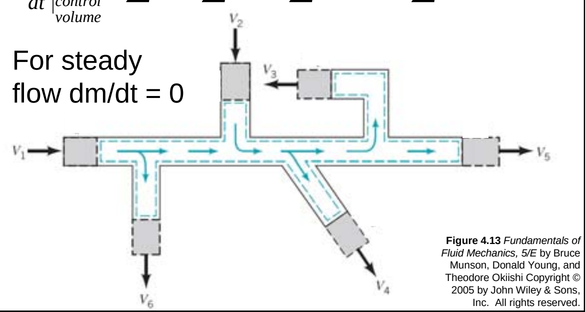

Figure 4.13 *Fundamentals of Fluid Mechanics, 5/E* by Bruce Munson, Donald Young, and Theodore Okiishi

Cavitation

- Change from small to large velocity ($V_1 << V_2$ will cause large pressure decrease: $p_2 = p_1 + \rho(V_1^2 – V_2^2)/2$
- Could reduce pressure to vapor pressure of liquid
- This causes vapor bubbles to form and burst exerting pressure on surfaces
- Cavitation can, over time, damage surfaces and must be avoided

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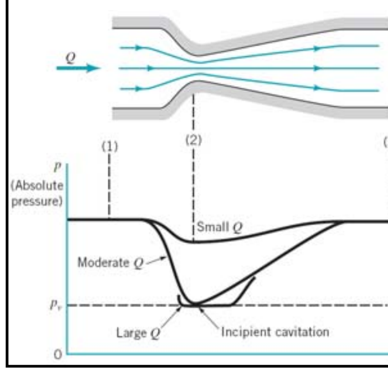
Cavitation II



- Large flow needs need very high velocity for smaller area
- Will decrease p, but cannot decrease p below p_v

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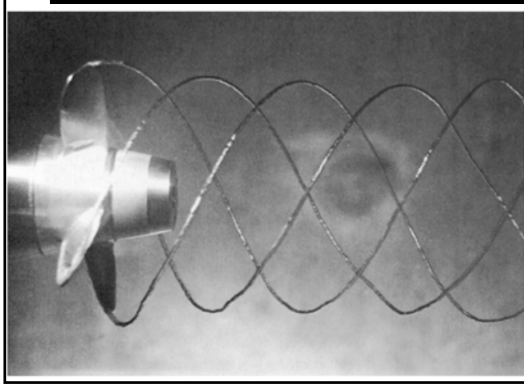
Bubbles from Propeller



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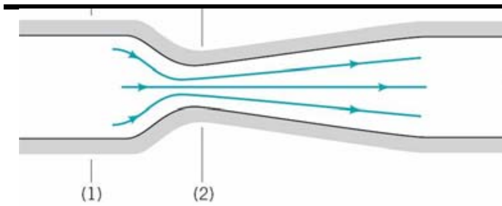
Flowrate Measurement



- Venturi meter shown here is example of flow meter
  - Measure velocity (and flow rate) by measuring pressure difference $p_1 – p_2$

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Flowrate Measurement II

- Flow meters use basic idea of Bernoulli equation and continuity
  - Continuity: area differences give related velocity differences: $V_2 = V_1A_1/A_2$
  - Bernoulli: velocity differences give V^2 difference in terms of pressure difference: $V_2^2 – V_1^2 = 2(p_1 – p_2)/\rho$
    - Assumes $z_2 = z_1$
  - Substitute continuity result, $V_2 = V_1A_1/A_2$

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Flowrate Measurement III

- Result of substitution
  - Continuity: $V_2 = V_1A_1/A_2$
  - Bernoulli: $V_2^2 – V_1^2 = 2(p_1 – p_2)/\rho$
  - Combination: $V_2^2 – V_1^2 = (V_1A_1/A_2)^2 – V_1^2 = V_1^2[1 – (A_1/A_2)^2] = 2(p_1 – p_2)/\rho$

$$V_1 = \sqrt{\frac{2(p_1 - p_2)}{\rho\left[1-(A_2/A_1)^2\right]}}$$

- For circular pipes $(A_1/A_2)^2 = (D_1/D_2)^4$
- Bernoulli-continuity combination important in problem solving

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Flowrate Measurement V

- Is measuring flow rates really this easy?
  - No. The approach used here ignores viscous forces that are ignored in the Bernoulli equation
  - Consideration of these forces is usually done by empirical modifications to basic equation on previous chart
  - Can also apply these ideas to open channel flows where flow rates are measured by elevation differences

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Flowrate Measurement VI

- Basic relation: velocity (or flow rate) proportional to square root of Δp

$$Q = V_1 A_1 = A_1 \sqrt{\frac{2(p_1 - p_2)}{\rho\left[1 - (A_2/A_1)^2\right]}}$$

- Problem: Find V_1 for D_1 = 0.1 m, D_2 = 0.05 m, $p_1 - p_2$ = 100 Pa, water (ρ = 1000 kg/m^3)

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Flowrate Measurement VI

$$V_1 = \sqrt{\frac{2(p_1 - p_2)}{\rho\left[1 - (A_2/A_1)^2\right]}} = \sqrt{\frac{2(100\ Pa)\dfrac{N}{Pa \cdot m^2}\dfrac{kg \cdot m}{N \cdot s^2}}{\dfrac{1000\ kg}{m^3}\left[1 - \left(\dfrac{\pi}{4}(0.05\ m)^2 \Big/ \dfrac{\pi}{4}(0.1\ m)^2\right)^2\right]}}$$

$$V_1 = \sqrt{\frac{0.2}{1 - \left(\dfrac{0.05\ m}{0.1\ m}\right)^4}\frac{m}{s}}$$

- V_1 = 0.462 m/s

- Problem: Find V_1 for D_1 = 4 in, D_2 = 2 in, $p_1 - p_2$ = 10 psf, air (ρ = 0.00238 slugs/ft^3)

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Flowrate Measurement VII

$$V_1 = \sqrt{\frac{2(p_1 - p_2)}{\rho\left[1 - (A_2/A_1)^2\right]}} = \sqrt{\frac{2\left(10\ \dfrac{lb_f}{ft^2}\right)\dfrac{slug \cdot ft}{lb_f \cdot s^2}}{\dfrac{0.00238\ slug}{ft^3}\left[1 - \left(\dfrac{\pi}{4}(4\ in)^2 \Big/ \dfrac{\pi}{4}(2\ in)^2\right)^2\right]}}$$

$$V_1 = \sqrt{\frac{20}{0.00238}\frac{ft^2}{s^2}\frac{1}{1 - \left(\dfrac{4\ in}{2\ in}\right)^4}}$$

- V_1 = 94.7 ft/s

- A given Δp means a larger V with air as compared to water

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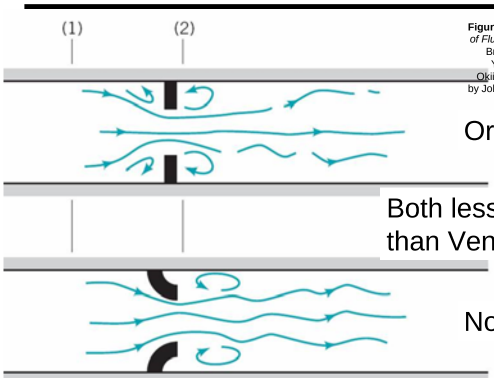
Other Flow Meters



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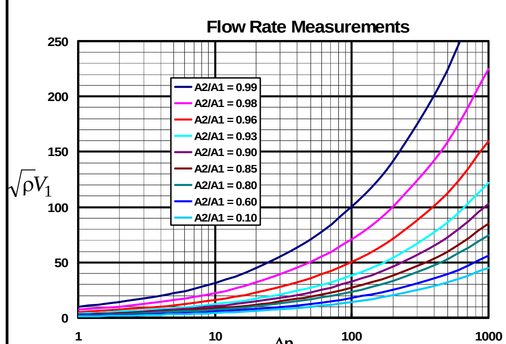
Orifice

Both less ideal than Venturi

Nozzle



For Δp in Pa and ρ in kg/m^3, V is in m/s

For Δp in psf and ρ in slugs/ft^3, V is in ft/s

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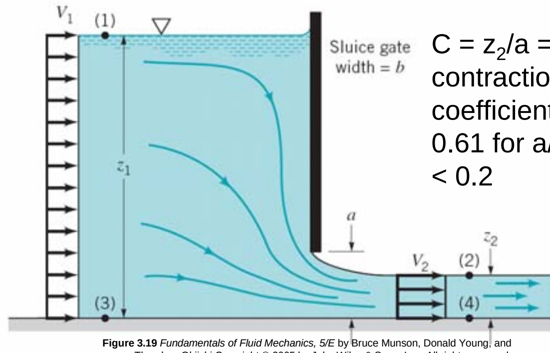
Open Channel Flows

- Measurements by sluice gates and weirs
- Measure height difference, $z_1 - z_2 = H$, at between two points with p = 0
- Rectangular flow areas with width b has areas bz_2 and bz_1 so that $V_2bz_2 = V_1bz_1$ and $z_1 + V_1^2/2g = z_2 + V_2^2/2g = z_1 + (V_2z_2/z_1)^2/2g$

$$V_2 = \sqrt{\frac{2g(z_1 - z_2)}{1 - (z_2/z_1)^2}}$$

Sluice Gate



C = z_2/a = contraction coefficient ≈ 0.61 for $a/z_1 < 0.2$

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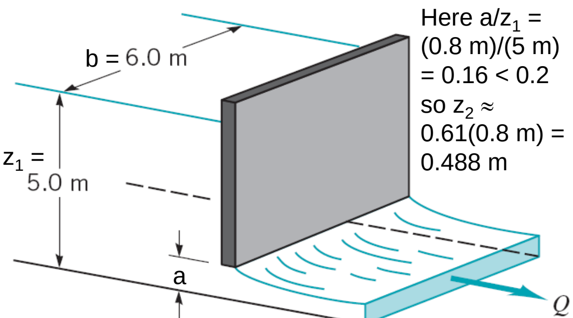
Sluice Gate II



Here a/z_1 = (0.8 m)/(5 m) = 0.16 < 0.2 so $z_2 \approx$ 0.61(0.8 m) = 0.488 m

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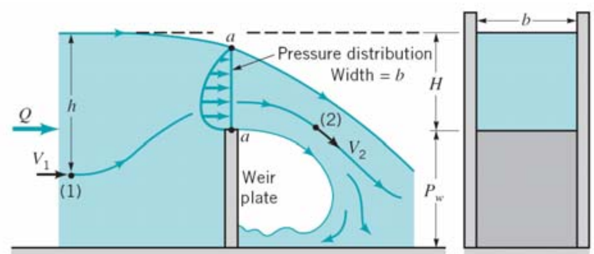
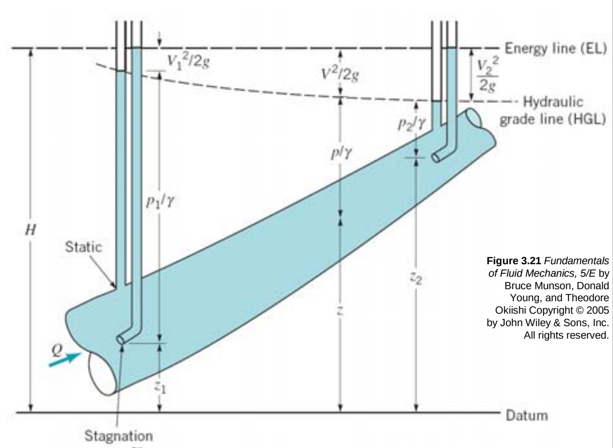
Weir



- Height H is measure of flow rate: Q ~ $Hb(2gH)^{1/2}$

Energy/Hydraulic Grade

- Natural definition of lines in a flow from Bernoulli equations
  - Energy line = z + p/ρ + $V^2/2g$ = H = total head, a constant for an inviscid steady flow
    - A series of pitot tube measurements will always produce the energy line
  - Hydraulic Grade Line = z + p/ρ = energy line – $V^2/2g$
    - A series of piezometer tubes will always produce the hydraulic grade line



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