Generalized Harmonic Evolutions of Binary Black Hole Spacetimes

Lee Lindblom

California Institute of Technology

AMS Meeting :: New Orleans :: 7 January 2007
Frans Pretorius performed first numerical BBH inspiral, merger and ringdown calculations in 2005. Pretorius Inspiral Movies
Frans Pretorius performed first numerical BBH inspiral, merger and ringdown calculations in 2005. Pretorius Inspiral Movies

Caltech/Cornell collaboration and the AEI/Pittsburgh collaboration perform successful BBH simulations in 2006 using GH methods.
Frans Pretorius performed first numerical BBH inspiral, merger and ringdown calculations in 2005.  
Caltech/Cornell collaboration and the AEI/Pittsburgh collaboration perform successful BBH simulations in 2006 using GH methods.

Outline of this talk:
- Review Generalized Harmonic (GH) form of the Einstein system.
- Constraint damping.
- Boundary conditions.
Methods of Specifying Spacetime Coordinates

We often decompose the 4-metric into its 3+1 parts:

\[ ds^2 = \psi_{ab} dx^a dx^b = -N^2 dt^2 + g_{ij} (dx^i + N^i dt)(dx^j + N^j dt). \]

The lapse \( N \) and shift \( N^i \) specify how coordinates are laid out on a spacetime manifold:

\[ \vec{n} = \partial_\tau = (\partial_t - N^k \partial_k)/N. \]
Methods of Specifying Spacetime Coordinates

- We often decompose the 4-metric into its 3+1 parts:
  \[ ds^2 = \psi_{ab}dx^adx^b = -N^2dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt). \]
  The lapse \( N \) and shift \( N^i \) specify how coordinates are laid out on a spacetime manifold: \( \vec{n} = \partial_\tau = (\partial_t - N^k \partial_k)/N. \)

- An alternate way to specify the coordinates is through the gauge source function \( H^a \):

- Let \( H^a \) denote the function obtained by the action of the covariant scalar wave operator on the coordinates \( x^a \):
  \[ H^a \equiv \nabla^c \nabla_c x^a = -\Gamma^a, \]
  where \( \Gamma^a = \psi^{bc} \Gamma^a_{bc} \) and \( \psi_{ab} \) is the 4-metric.
Methods of Specifying Spacetime Coordinates

- We often decompose the 4-metric into its 3+1 parts:
  \[ ds^2 = \psi_{ab} dx^a dx^b = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt). \]
  The lapse \( N \) and shift \( N^i \) specify how coordinates are laid out on a spacetime manifold: \( \vec{n} = \partial_\tau = (\partial_t - N^k \partial_k)/N. \)

- An alternate way to specify the coordinates is through the gauge source function \( H^a \):

- Let \( H^a \) denote the function obtained by the action of the covariant scalar wave operator on the coordinates \( x^a \):
  \[
  H^a \equiv \nabla^c \nabla_c x^a = -\Gamma^a,
  \]
  where \( \Gamma^a = \psi^{bc} \Gamma^a_{bc} \) and \( \psi_{ab} \) is the 4-metric.

- Specifying coordinates by the generalized harmonic (GH) method can be accomplished by choosing a gauge-source function \( H_a(x, \psi) = \psi_{ab} H^b \), and requiring that \( H_a(x, \psi) = -\Gamma_a = -\psi_{ab} \psi^{cd} \Gamma^b_{cd}. \)
The Einstein equations are manifestly hyperbolic when coordinates are specified using a GH gauge function:

\[
R_{ab} = -\frac{1}{2} \psi^{cd} \partial_c \partial_d \psi_{ab} + \nabla (a \Gamma_b) + F_{ab}(\psi, \partial \psi),
\]

where \( \psi_{ab} \) is the 4-metric, and \( \Gamma_a = \psi^{bc} \Gamma_{abc} \). The vacuum Einstein equation, \( R_{ab} = 0 \), has the same principal part as the scalar wave equation when \( H_a(x, \psi) = -\Gamma_a \) is imposed.
Important Properties of the GH Method

- The Einstein equations are manifestly hyperbolic when coordinates are specified using a GH gauge function:

\[ R_{ab} = -\frac{1}{2} \psi^{cd} \partial_c \partial_d \psi_{ab} + \nabla (a \Gamma_b) + F_{ab}(\psi, \partial \psi), \]

where \( \psi_{ab} \) is the 4-metric, and \( \Gamma_a = \psi^{bc} \Gamma_{abc} \). The vacuum Einstein equation, \( R_{ab} = 0 \), has the same principal part as the scalar wave equation when \( H_a(x, \psi) = -\Gamma_a \) is imposed.

- Imposing coordinates using a GH gauge function profoundly changes the constraints. The GH constraint, \( C_a = 0 \), where

\[ C_a = H_a + \Gamma_a, \]

depends only on first derivatives of the metric. The standard Hamiltonian and momentum constraints, \( M_a = 0 \), are determined by the derivatives of the gauge constraint \( C_a \):

\[ M_a \equiv \left[ R_{ab} - \frac{1}{2} \psi_{ab} R \right] n^b = \left[ \nabla (a C_b) - \frac{1}{2} \psi_{ab} \nabla^c C_c \right] n^b. \]
Pretorius (based on a suggestion from Gundlach, et al.) modified the GH system by adding terms proportional to the gauge constraints:

\[ 0 = R_{ab} - \nabla_\perp (a \xi_b) + \gamma_0 \left[ n_\perp (a \xi_b) - \frac{1}{2} \psi_{ab} n^c \xi_c \right], \]

where \( n^a \) is a unit timelike vector field. Since \( \xi_a = H_a + \Gamma_a \) depends only on first derivatives of the metric, these additional terms do not change the hyperbolic structure of the system.
Pretorius (based on a suggestion from Gundlach, et al.) modified the GH system by adding terms proportional to the gauge constraints:

\[
0 = R_{ab} - \nabla(aC_b) + \gamma_0 \left[ n(aC_b) - \frac{1}{2} \psi_{ab} n^c C_c \right],
\]

where \( n^a \) is a unit timelike vector field. Since \( C_a = H_a + \Gamma_a \) depends only on first derivatives of the metric, these additional terms do not change the hyperbolic structure of the system.

Evolution of the constraints \( C_a \) follow from the Bianchi identities:

\[
0 = \nabla^c \nabla_c C_a - 2\gamma_0 \nabla^c \left[ n(cC_a) \right] + C^c \nabla(cC_a) - \frac{1}{2} \gamma_0 \ n_a C^c C_c.
\]

This is a damped wave equation for \( C_a \), that drives all small short-wavelength constraint violations toward zero as the system evolves (for \( \gamma_0 > 0 \)).
Kashif Alvi (2002) derived a nice (symmetric hyperbolic) first-order form for the generalized-harmonic evolution system:

\[ \Phi_{kab} = \partial_k \psi_{ab}, \]

\[ \partial_t \psi_{ab} - N^k \partial_k \psi_{ab} = -N \Pi_{ab}, \]

\[ \partial_t \Pi_{ab} - N^k \partial_k \Pi_{ab} + Ng^{ki} \partial_k \Phi_{iab} \approx 0, \]

\[ \partial_t \Phi_{iab} - N^k \partial_k \Phi_{iab} + N \partial_i \Pi_{ab} \approx 0. \]
First Order Generalized Harmonic Evolution System

- Kashif Alvi (2002) derived a nice (symmetric hyperbolic) first-order form for the generalized-harmonic evolution system:

\[
\begin{align*}
\Phi_{kab} & = \partial_k \psi_{ab}, \\
\partial_t \psi_{ab} - N^k \partial_k \psi_{ab} & = -N \Pi_{ab}, \\
\partial_t \Pi_{ab} - N^k \partial_k \Pi_{ab} + Ng^{ki} \partial_k \Phi_{iab} & \simeq 0, \\
\partial_t \Phi_{iab} - N^k \partial_k \Phi_{iab} + N \partial_i \Pi_{ab} & \simeq 0.
\end{align*}
\]

- This system has two immediate problems:

  - This system has new constraints, \(C_{kab} = \partial_k \psi_{ab} - \Phi_{kab}\), that tend to grow exponentially during numerical evolutions.

  - This system is not linearly degenerate, so it is possible (likely) that shocks will develop (e.g. the shift evolution equation is of the form \(\partial_t N^i - N^k \partial_k N^i \simeq 0\)).
Improved First-Order GH Evolution System

- We can correct these problems by adding additional multiples of the constraints to the evolution system:

\[
\begin{align*}
\partial_t \psi_{ab} - (1 + \gamma_1) N^k \partial_k \psi_{ab} &= -N \Pi_{ab} - \gamma_1 N^k \Phi_{kab}, \\
\partial_t \Pi_{ab} - N^k \partial_k \Pi_{ab} + Ng^{ki} \partial_k \Phi_{iab} - \gamma_1 \gamma_2 N^k \partial_k \psi_{ab} &\approx -\gamma_1 \gamma_2 N^k \Phi_{kab}, \\
\partial_t \Phi_{iab} - N^k \partial_k \Phi_{iab} + N \partial_i \Pi_{ab} - \gamma_2 N \partial_i \psi_{ab} &\approx -\gamma_2 N \Phi_{iab}.
\end{align*}
\]
We can correct these problems by adding additional multiples of the constraints to the evolution system:

\[ \partial_t \psi_{ab} - (1 + \gamma_1) N^k \partial_k \psi_{ab} = -N \Pi_{ab} - \gamma_1 N^k \Phi_{kab}, \]

\[ \partial_t \Pi_{ab} - N^k \partial_k \Pi_{ab} + N g^{ki} \partial_k \Phi_{iab} - \gamma_1 \gamma_2 N^k \partial_k \psi_{ab} \simeq -\gamma_1 \gamma_2 N^k \Phi_{kab}, \]

\[ \partial_t \Phi_{iab} - N^k \partial_k \Phi_{iab} + N \partial_i \Pi_{ab} - \gamma_2 N \partial_i \psi_{ab} \simeq -\gamma_2 N \Phi_{iab}. \]

This improved GH evolution system has several nice properties:

- This system is linearly degenerate for \( \gamma_1 = -1 \) (and so shocks should not form from smooth initial data).
- The \( \Phi_{iab} \) evolution equation can be written in the form,
  \[ \partial_t C_{iab} - N^k \partial_k C_{iab} \simeq -\gamma_2 NC_{iab}, \]
  so the new constraints are damped when \( \gamma_2 > 0 \).
- This system is symmetric hyperbolic for all values of \( \gamma_1 \) and \( \gamma_2 \).
Numerical Tests of the First-Order GH System

- 3D numerical evolutions of static black-hole spacetimes illustrate the constraint damping properties of the GH evolution system.
- These evolutions are stable and convergent when $\gamma_0 = \gamma_2 = 1$.

The boundary conditions used for this simple test problem freeze the incoming characteristic fields to their initial values.
Boundary Conditions

Boundary conditions are straightforward to formulate for first-order hyperbolic evolutions systems,

\[ \partial_t u^\alpha + A^k{}_{\alpha\beta}(u) \partial_k u^\beta = F^\alpha(u). \]

For the GH system \( u^\alpha = \{\psi_{ab}, \Pi_{ab}, \Phi_{kab}\} \).

Find the eigenvectors of the characteristic matrix \( s_k A^k{}_{\alpha\beta} \) at each boundary point:

\( \hat{e}_\alpha \sim s_k A^k{}_{\alpha\beta} = v(\hat{e}_\alpha) \hat{e}_\alpha \),

where \( s_k \) is the outward directed unit normal to the boundary.

For hyperbolic evolution systems the eigenvectors \( \hat{e}_\alpha \) are complete:

\[ \det \hat{e}_\alpha \sim 0 \]

So we define the characteristic fields:

\( u^\alpha = \{\psi_{ab}, \Pi_{ab}, \Phi_{kab}\} \).
Boundary Conditions

- Boundary conditions are straightforward to formulate for first-order hyperbolic evolutions systems,
  \[
  \partial_t u^\alpha + A^k{}_{\alpha \beta}(u) \partial_k u^\beta = F^\alpha(u).
  \]

  For the GH system \( u^\alpha = \{\psi_{ab}, \Pi_{ab}, \Phi_{kab}\} \).

- Find the eigenvectors of the characteristic matrix \( s_k A^k{}_{\alpha \beta} \) at each boundary point:
  \[
  e^{\widehat{\alpha}}_\alpha s_k A^k{}_{\alpha \beta} = v(\widehat{\alpha}) e^{\widehat{\alpha}}_\beta,
  \]
  where \( s_k \) is the outward directed unit normal to the boundary.
Boundary Conditions

- Boundary conditions are straightforward to formulate for first-order hyperbolic evolutions systems,

\[ \partial_t u^\alpha + A^k{}^\alpha{}_\beta (u) \partial_k u^\beta = F^\alpha (u). \]

For the GH system \( u^\alpha = \{ \psi_{ab}, \Pi_{ab}, \Phi_{kab} \} \).

- Find the eigenvectors of the characteristic matrix \( s_k A^k{}^\alpha{}_\beta \) at each boundary point:

\[ e^{\hat{\alpha}}{}^\alpha s_k A^k{}^\alpha{}_\beta = v(\hat{\alpha}) e^{\hat{\alpha}}{}^\beta, \]

where \( s_k \) is the outward directed unit normal to the boundary.

- For hyperbolic evolution systems the eigenvectors \( e^{\hat{\alpha}}{}^\alpha \) are complete: \( \det e^{\hat{\alpha}}{}^\alpha \neq 0 \). So we define the characteristic fields:

\[ u^{\hat{\alpha}} = e^{\hat{\alpha}}{}^\alpha u^\alpha. \]
Boundary Conditions

- Boundary conditions are straightforward to formulate for first-order hyperbolic evolutions systems,
  \[ \partial_t u^\alpha + A^{k\alpha \beta}(u) \partial_k u^\beta = F^\alpha(u). \]

  For the GH system \( u^\alpha = \{ \psi_{ab}, \Pi_{ab}, \Phi_{kab} \}. \)

- Find the eigenvectors of the characteristic matrix \( s_k A^{k\alpha \beta} \) at each boundary point:
  \[ e^{\hat{\alpha} \alpha} s_k A^{k\alpha \beta} = v(\hat{\alpha}) e^{\hat{\alpha} \beta}, \]

  where \( s_k \) is the outward directed unit normal to the boundary.

- For hyperbolic evolution systems the eigenvectors \( e^{\hat{\alpha} \alpha} \) are complete: \( \det e^{\hat{\alpha} \alpha} \neq 0 \). So we define the characteristic fields:
  \[ u^{\hat{\alpha}} = e^{\hat{\alpha} \alpha} u^\alpha. \]

- A boundary condition must be imposed on each incoming characteristic field (i.e. every field with \( v(\hat{\alpha}) < 0 \)), and must not be imposed on any outgoing field (i.e. any field with \( v(\hat{\alpha}) > 0 \)).
Evolutions of a Perturbed Schwarzschild Black Hole

- A black-hole spacetime is perturbed by an incoming gravitational wave that excites quasi-normal oscillations.
- Use boundary conditions that Freeze the remaining incoming characteristic fields.
- The resulting outgoing waves interact with the boundary of the computational domain and produce constraint violations.

Lapse Movie  Constraint Movie
The evolution of the constraints,
\[ c^A = \{ C_a, C_{kab}, M_a \approx n^c \partial_c C_a, C_{ka} \approx \partial_k C_a, C_{klab} = \partial_{[k} \Phi_{l]ab} \} \]
determined by the evolution of the fields \( u^\alpha = \{ \psi_{ab}, \Pi_{ab}, \Phi_{kab} \} \):
\[ \partial_t c^A + A^k A_B(u) \partial_k c^B = F^A_B(u, \partial u) c^B. \]
The evolution of the constraints,
\[ c^A = \{ C_a, C_{kab}, M_a \approx n^c \partial_c C_a, C_{ka} \approx \partial_k C_a, C_{klab} = \partial[l \Phi]_{ab} \} \]
determined by the evolution of the fields \( u^\alpha = \{ \psi_{ab}, \Pi_{ab}, \Phi_{kab} \} \):
\[
\partial_t c^A + A^k A_B(u) \partial_k c^B = F^A_B(u, \partial u) c^B.
\]
This constraint evolution system is symmetric hyperbolic with principal part:
\[
\partial_t C_a \approx 0,
\partial_t M_a - N^k \partial_k M_a - N g^{ij} \partial_i C_{ja} \approx 0,
\partial_t C_{ia} - N^k \partial_k C_{ia} - N \partial_i M_a \approx 0,
\partial_t C_{iab} - (1 + \gamma_1) N^k \partial_k C_{iab} \approx 0,
\partial_t C_{ijab} - N^k \partial_k C_{ijab} \approx 0.
\]
Constraint Evolution for the First-Order GH System

- The evolution of the constraints,
  \[ c^A = \{ C_a, C_{kab}, M_a \approx n^c \partial_c C_a, C_{ka} \approx \partial_k C_a, C_{klab} = \partial_{[k} \Phi_{l]ab} \} \]
  are determined by the evolution of the fields \( u^\alpha = \{ \psi_{ab}, \Pi_{ab}, \Phi_{kab} \} \):
  \[ \partial_t c^A + A^k A_B(u) \partial_k c^B = F^A_B(u, \partial u) c^B. \]

- This constraint evolution system is symmetric hyperbolic with principal part:
  \[ \partial_t C_a \approx 0, \]
  \[ \partial_t M_a - N^k \partial_k M_a - N g^{ij} \partial_i C_{ja} \approx 0, \]
  \[ \partial_t C_{ia} - N^k \partial_k C_{ia} - N \partial_i M_a \approx 0, \]
  \[ \partial_t C_{iab} - (1 + \gamma_1) N^k \partial_k C_{iab} \approx 0, \]
  \[ \partial_t C_{ijab} - N^k \partial_k C_{ijab} \approx 0. \]

- An analysis of this system shows that all of the constraints are damped in the WKB limit when \( \gamma_0 > 0 \) and \( \gamma_2 > 0 \). So, this system has constraint suppression properties that are similar to those of the Pretorius (and Gundlach, et al.) system.
Construct the characteristic fields, $\hat{c}^A = e^A_B c^B$, associated with the constraint evolution system,

$$\partial_t c^A + A^k_B \partial_k c^B = F^A_B c^B.$$
Constraint Preserving Boundary Conditions

- Construct the characteristic fields, $\hat{c}^\hat{A} = \hat{e}^\hat{A}_A c^A$, associated with the constraint evolution system, $\partial_t c^A + A^k A_B \partial_k c^B = F^A_B c^B$.

- Split the constraints into incoming and outgoing characteristics: $\hat{c} = \{\hat{c}^-, \hat{c}^+\}$. 
Construct the characteristic fields, \( \hat{\mathbf{c}}^A = e^A \mathbf{c}^A \), associated with the constraint evolution system, \( \partial_t \mathbf{c}^A + A^k \mathbf{A}_B \partial_k \mathbf{c}^B = F^A \mathbf{c}^B \).

Split the constraints into incoming and outgoing characteristics: \( \hat{\mathbf{c}} = \{ \hat{\mathbf{c}}^-, \hat{\mathbf{c}}^+ \} \).

The incoming characteristic fields must vanish on the boundaries, \( \hat{\mathbf{c}}^- = 0 \), if the influx of constraint violations is to be prevented.
Construct the characteristic fields, $\hat{c}^\hat{A} = e^{\hat{A}}_A c^A$, associated with the constraint evolution system, $\partial_t c^A + A^k_A B \partial_k c^B = F^A_B c^B$.

Split the constraints into incoming and outgoing characteristics: $\hat{c} = \{\hat{c}^-, \hat{c}^+\}$.

The incoming characteristic fields must vanish on the boundaries, $\hat{c}^- = 0$, if the influx of constraint violations is to be prevented.

The constraints depend on the primary evolution fields (and their derivatives). We find that $\hat{c}^-$ for the GH system can be expressed:

$$\hat{c}^- = d_\perp \hat{u}^- + \hat{F}(u, d_\parallel u).$$
Constraint Preserving Boundary Conditions

- Construct the characteristic fields, \( \hat{c}^A = e^A_A c^A \), associated with the constraint evolution system, \( \partial_t c^A + A^A_B \partial_k c^B = F^A_B c^B \).

- Split the constraints into incoming and outgoing characteristics: \( \hat{c} = \{ \hat{c}^-, \hat{c}^+ \} \).

- The incoming characteristic fields must vanish on the boundaries, \( \hat{c}^- = 0 \), if the influx of constraint violations is to be prevented.

- The constraints depend on the primary evolution fields (and their derivatives). We find that \( \hat{c}^- \) for the GH system can be expressed:
  \[
  \hat{c}^- = d_\perp \hat{u}^- + \hat{F}(u, d_\parallel u).
  \]

- Set boundary conditions on the fields \( \hat{u}^- \) by requiring
  \[
  d_\perp \hat{u}^- = -\hat{F}(u, d_\parallel u).
  \]
More Boundary Condition Issues

- Constraints can not provide BCs for all incoming fields.
- Physical gravitational-wave degrees-of-freedom must have BCs determined by the physics of the situation:

\[ \partial_t \Psi_0 = 0 \]

This condition is translated into a BC by expressing \( \Psi_0 \) in terms of the incoming characteristic fields:

\[ \Psi_0 = d_{\perp} \hat{u} - d_{\parallel} \hat{F}(u, d_{\parallel}) \]

Initial-boundary problem for first-order GH evolution system is well-posed for algebraic boundary conditions on \( \hat{u}_\alpha \).

Constraint preserving and physical boundary conditions involve derivatives of \( \hat{u}_\alpha \), and standard well-posedness proofs fail.

Oliver Rinne (2006) used Fourier-Laplace analysis to show that these BC satisfy the Kreiss (1970) condition which is necessary for well-posedness (but not sufficient for this type of BC).

Help Wanted!

New analysis methods are needed to prove (or disprove) complete well-posedness for this type of BC.
More Boundary Condition Issues

- Constraints can not provide BCs for all incoming fields.
- Physical gravitational-wave degrees-of-freedom must have BCs determined by the physics of the situation:
  - Isolated systems (no incoming gravitational waves) are modeled by imposing a BC that sets the time-dependent part of the incoming components of the Weyl tensor to zero: \( \partial_t \Psi_0 = 0 \).
  - This condition is translated into a BC by expressing \( \Psi_0 \) in terms of the incoming characteristic fields: \( \Psi_0 = d_\perp \hat{u}^- + \hat{F}(u, d_\parallel u) \).
More Boundary Condition Issues

- Constraints can not provide BCs for all incoming fields.
- Physical gravitational-wave degrees-of-freedom must have BCs determined by the physics of the situation:
  - Isolated systems (no incoming gravitational waves) are modeled by imposing a BC that sets the time-dependent part of the incoming components of the Weyl tensor to zero: \( \partial_t \Psi_0 = 0 \).
  - This condition is translated into a BC by expressing \( \Psi_0 \) in terms of the incoming characteristic fields: \( \Psi_0 = d_{\perp} \hat{u}^- + \hat{F}(u, d_{\parallel} u) \).

- Initial-boundary problem for first-order GH evolution system is well-posed for algebraic boundary conditions on \( \hat{u}^\alpha \).
- Constraint preserving and physical boundary conditions involve derivatives of \( \hat{u}^\alpha \), and standard well-posedness proofs fail.

Oliver Rinne (2006) used Fourier-Laplace analysis to show that these BC satisfy the Kreiss (1970) condition which is necessary for well-posedness (but not sufficient for this type of BC).

Help Wanted!
New analysis methods are needed to prove (or disprove) complete well-posedness for this type of BC.

Lee Lindblom (Caltech)
More Boundary Condition Issues

- Constraints can not provide BCs for all incoming fields.
- Physical gravitational-wave degrees-of-freedom must have BCs determined by the physics of the situation:
  - Isolated systems (no incoming gravitational waves) are modeled by imposing a BC that sets the time-dependent part of the incoming components of the Weyl tensor to zero: \( \partial_t \psi_0 = 0 \).
  - This condition is translated into a BC by expressing \( \psi_0 \) in terms of the incoming characteristic fields: \( \psi_0 = d_\perp \hat{u}^{-} + \hat{F}(u, d_\parallel u) \).
- Initial-boundary problem for first-order GH evolution system is well-posed for algebraic boundary conditions on \( \hat{u}^\alpha \).
- Constraint preserving and physical boundary conditions involve derivatives of \( \hat{u}^\alpha \), and standard well-posedness proofs fail.
- Oliver Rinne (2006) used Fourier-Laplace analysis to show that these BC satisfy the Kreiss (1970) condition which is necessary for well-posedness (but not sufficient for this type of BC).
More Boundary Condition Issues

- Constraints can not provide BCs for all incoming fields.
- Physical gravitational-wave degrees-of-freedom must have BCs determined by the physics of the situation:
  - Isolated systems (no incoming gravitational waves) are modeled by imposing a BC that sets the time-dependent part of the incoming components of the Weyl tensor to zero: \( \partial_t \psi_0 = 0 \).
  - This condition is translated into a BC by expressing \( \psi_0 \) in terms of the incoming characteristic fields: \( \psi_0 = d_\perp \hat{u}^- + \hat{F}(u, d_\parallel u) \).

Initial-boundary problem for first-order GH evolution system is well-posed for algebraic boundary conditions on \( \hat{u}^\alpha \).

- Constraint preserving and physical boundary conditions involve derivatives of \( \hat{u}^\alpha \), and standard well-posedness proofs fail.
- Oliver Rinne (2006) used Fourier-Laplace analysis to show that these BC satisfy the Kreiss (1970) condition which is necessary for well-posedness (but not sufficient for this type of BC).
- Help Wanted! New analysis methods are needed to prove (or disprove) complete well-posedness for this type of BC.
Numerical Tests of Boundary Conditions

- Compare the solution obtained on a “small” computational domain with a reference solution obtained on a “large” domain where the boundary is not in causal contact with the comparison region.

Solution Differences

- Solutions using “Freezing” BC (dashed curves) have differences and constraints that do not converge to zero.

- Solutions using constraint preserving and physical BC (solid curves) have much smaller differences and constraints that converge to zero.
Summary

- Generalized Harmonic method produces manifestly hyperbolic representations of the Einstein equations for any choice of coordinates.
- Constraint damping makes the modified GH equations stable for numerical simulations.
- Constraint preserving boundary conditions have been implemented and tested for the GH system.
- Binary black hole simulations have been successfully performed using GH methods by several groups using very different numerical methods.