An appendix to ”Some harmonic analysis questions suggested by Anderson-Bernoulli models” (SVW): A general contraction property of $PSL(2,R)$ (T.Wolff)

Notation and Definitions:

$PSL(2,R) = SL(2,R)/\{I,-I\}$ determines a matrix within a non-zero multiple

$R^* = R \cup \{\infty\}$

$H = \{z = x + iy \in \mathbb{C} : y > 0\}$

Regard $PSL(2,R)$ acting on $H \cup R^*$ via Mobius transformations:

$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $gz = \frac{az+b}{cz+d}$
Define elementary subgroups or stabilizers of $G = PSL(2, R)$. $g$ is

elliptic, if $z \in H$ then $F_z = \{ g \in G : gz = z \}$

parabolic, if $x \in R^*$, then
$F_x = \{ g \in G : gx = x \}$

hyperbolic, if $x, y \in R^*, x \neq y$, then
$F_{x,y} = \{ g \in G : g$ maps the set $\{x, y\}$ into itself $\}$

Conjugacy classes
$g = hm_k h^{-1}$, we write $g \sim m_k$:

elliptic, $g \sim m_k = k z$, for some $k$ satisfying $|k| = 1, k \neq 1$

parabolic, $g \sim m_1(z) = z + 1$ with fixed point at $\infty$

hyperbolic, $g \sim m_k(z) = k z$ $|k| \neq 1$ with fixed points 0 and $\infty$
A representation $\rho$ of $G$ and a $\mathbb{C}$ H.S. $V$:

$\rho : G \rightarrow GL(V)$

$\rho(1) = 1$

$\rho(g_1g_2) = \rho(g_1)\rho(g_2) \quad g_1, g_2 \in G$

$\rho \rightarrow \rho(g)v$ is continuous for every $v \in V$

An irreducible representation has no nontrivial closed $G$-invariant spaces. [ie. there is no invariant subspace $U$ of $V$ such that $\rho(G)U \subset U$ except zero and the $V$. ]

Irreducible unitary representations of $SL(2, R)$ are classified into the principle, complementary, and discrete series, limit of discrete series, and the trivial one.

For example, for $0 < \alpha < 1$

$C^\alpha \left[ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right] f(x) = (-bx + d)^{-1-\alpha} f\left(\frac{ax-c}{-bx+d}\right)$

acting on

$W^{-\alpha/2} = \{ f : \|f\|^2 = \int_{\mathbb{R}} |\hat{f}(\xi)|^2 |\xi|^{-\alpha} d\xi < \infty \}$
The Iwasawa decomposition: $G = KAN$

$$K = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

$$A = \begin{pmatrix} k^{1/2} & 0 \\ 0 & k^{-1/2} \end{pmatrix}$$

$$N = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$$

THIS REALLY SIMPLIFIES THINGS!

Notice that $\rho(A)f = k^{\frac{1+\alpha}{2}} f(kx)$

and $\rho(N^t)f = f(x - k)$
Define a probability measure $\mu$ on $PSL(2, R)$.

Assumption: $\text{supp}\mu$ is not contained in a left coset of the elementary subgroups (Furstenberg, 1963).

Recall a left coset is $gF = \{gF : g \in G\}$.

Theorem 1: Let $\rho$ be a unitary representation such that for some $\epsilon$, the direct integral decomposition of $\rho$ does not contain the trivial nor $C^\alpha$, $\alpha > 1 - \epsilon$.

Define

$$\rho(\mu) = \int_G \rho(g) d\mu(g)$$

then $\|\rho(\mu)\| < 1$. 
Let $g, g_1, g_2 \in PSL(2, R) = G$ and $f \in L^2$.

What does $||\rho(\mu)|| < 1$ mean?

$$||\int_G \rho(g) d\mu(g)f||^2$$

$$= \int < \rho(g_1)f, \rho(g_2)f > d\mu(g_1)d\mu(g_2)$$

$$= 1 - 1/2 \int ||\rho(g_1)f - \rho(g_2)f||^2 d\mu(g_1)d\mu(g_2)$$

Need to $||\rho(g_1)f - \rho(g_2)f|| \geq \epsilon$ with positive probability to prove the result.
Definition: Let $||f|| = 1$. $f$ is an $\epsilon$-invariant vector for an operator $T$ if $||Tf - f|| < \epsilon$.

A vector is $\epsilon$-invariant for a collection of operators $\mathcal{T}$ if it is $\epsilon$-invariant for each $T \in \mathcal{T}$.

Main ideas in the proof:

Step 1: Produce an $\epsilon > 0$ such that $\mathcal{T}$ the image under $\rho$ of the set $\Gamma \subset PSL(2, R)$ has no $\epsilon$-invariant vectors where

$$\Gamma = \{p_1, p_2\}, \ p_1 \ and \ p_2 \ are \ parabolic \ with \ different \ fixed \ points$$

$$\Gamma = \{h_1, h_2, h_3\} \ with \ each \ h_i \ hyperbolic \ without \ common \ fixed \ points$$

Step 2: Prove that it suffices to study these cases; ie. there is a positive probability that a triple $g_1, g_2, \ and \ g_3$ has no $\epsilon$-invariant vectors.
Reformulation of the problem using ideas from Goldsheid and Margulis, 1989.

Definition of Zariski closure of a set:

Let $M$ be an algebraic manifold and $A \subset M$.

$S = \{ p \text{ polynomials} : p(x) = 0 \text{ for all } x \in A \}.$

The Zariski closure of $A$ is

$Z(A) = \{ z \in M : p(z) = 0 \text{ for all } p \in S \}.$

Now we can rephrase Theorem 1.

Theorem 2: Let $\rho$ be a representation of $SL(2, R)$ which does not weakly contain the trivial one and assume $E \subset SL(2, R)$ is not contained in any Zariski closed subgroup. Then for suitable $\epsilon > 0$ there are no vectors $\epsilon$-invariant for $\rho(E)$. 

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The condition excluding the weak containment of trivial one is the same as excluding the complementary representation $C^\alpha$, $\alpha > 1 - \epsilon$. The problem stems from the failure of property T for $SL(2, R)$, i.e. there are subgroups with almost invariant vectors but no invariant vectors.
The purpose of a general spectral gap property for measures (Wolff) is to prove a related result on general non-compact semisimple groups instead of $SL(2, \mathbb{R})$.

Problem: There is no simple classification theorem for representations of $SL(n, \mathbb{R})$. We will need to place certain restrictions on the type of representations considered.

Theorem 3: Assume for some finite dimensional representation $\Phi : G \to SL(n, \mathbb{R})$ the Zariski closure of the group generated by $\Phi(\text{supp} \mu)$ is noncompact and acts strongly irreducibly. Let $\rho$ be a representation of $G$ unitarily induced from the stabilizer subgroup $\{g \in G : \Phi(g)l = l\}$ of a one dimensional subspace $l$. Then $\rho(\mu)$ has spectral radius less than 1.

Application:

(Shubin, Vakilian, Wolff) Gives a quantitative bound on the largest Lyapunov exponent for the Anderson model on the strip.
In fact the main theorem is too general to state in this talk, but heuristically it says:

If $\mu$ is a probability measure on $G \subset GL(n, R)$ for which the Zariski closure of $\text{supp}\mu$ is big, then for suitable unitary representations $\rho$ of $G$, $\hat{\mu}(\rho)$ has norm less than 1.

The key point is that $\text{supp}\mu$ is allowed to be small.

Around the same time, Y. Shalom independently obtained related results in *Explicit Kazhdan constants for representations of semisimple and arithmetic groups*, Ann. Inst. Fourier Grenoble 50, 3 (2000), 833-863 Theorem C.

Shalom’s condition is that the topological closure of the $\text{supp}\mu$ is nonamenable.
Frequency concentration and localization lengths for the Anderson model at small disorders (Schlag, Shubin, Wolff)

\[ Hu = \Delta_{\mathbb{Z}^d} u + \lambda \omega u \quad u \in l^2(\mathbb{Z}^d) \quad d = 1, 2 \]

\( \{\omega_n\}_n \in \mathbb{Z}^d \) is iid with \( E\omega_0 = 0 \) and \( E\omega_0^2 = 1 \)

Goal: estimate localization lengths as \( \lambda \to 0 \).

A simple perturbation argument gives a localization length of \( \lambda^{-1} \).

Theorem 4: The Fourier Transform (FT) of "most eigenfunctions" restricted to cubes \([-N, N]^d\) are concentrated on annuli of thickness \( \lambda^2 \). These annuli are neighborhoods of that curve which supports the FT of the corresponding eigenfunctions of \( \Delta \) with the same energy.

The uncertainty principle implies that the localization length is \( \lambda^{-2} \) (up to logarithms) away from the edges of the spectrum.

"Most eigenfunctions" means up to a set of density \( o(1) \) as \( \lambda \to 0 \).
Take the FT of the 2D discrete Laplacian to find the multiplier:

\[ m(\theta_1, \theta_2) = 2 \cos(2\pi \theta_1) + 2 \cos(2\pi \theta_2) \]

Look at the level curves \( m(\theta_1, \theta_2) = E \)
Notation:

Let $0 < \delta < \epsilon < 1/2$

Let $P_\epsilon$ denote the restriction to the annulus of width $\epsilon$ for fixed $E$ around $m(\theta_1, \theta_2)$

$Q \subset Z^2$ is a square of side length $L\epsilon^{-1}$, $L \geq 1$.

$\omega_Q$ the restriction of $\omega$ to $Q$

Goal: control $\|P_\epsilon \omega_Q P_\delta \|

Main Ingredients:

(1) an almost orthogonality argument

For all $m \geq 1$,

$\|P_\epsilon u\|_2^2 \leq C \sum_{Q \in Q_\epsilon,L} \|P_\epsilon \chi_Q u\|_2^2 + C m \epsilon^{-1} L^{-m} \|u\|_2^2$

for all $u \in l^2(Z^2)$ where $Q_{\epsilon,L}$ be a partition of $Z^2$ into squares of side length $L\epsilon^{-1}$ where $L \geq 1$. 

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(2) entropy estimate

The entropy bounds limit the number of functions tested on \( \| P_\varepsilon w Q P_\delta \| \) to
\[
M \leq C_1 \exp(C \exp(C \kappa^{-3} L^2 \alpha / \varepsilon)).
\]
Here \( 0 < \kappa < 1 \) and \( 0 < \alpha < \pi \).

(3) basic probabilistic estimate

Fix some small \( \eta > 0 \). Then
\[
P[\| P_\varepsilon w Q P_\delta \| \geq C_\eta A \sqrt{\varepsilon}] \leq C M^2 \varepsilon^{3/2} \exp(-\frac{A^2}{\sqrt{\varepsilon} |\log \varepsilon|^2})
\]
for all \( A \geq \varepsilon^{-\eta} \)

(4) obtain the estimate
\[
\|u\|_4^4 \leq C \lambda^{2-\eta} \|u\|_2^4
\]
avay from the band edges
Optimality?

The order $\lambda^2$ for the concentration of the FT is optimal.

1D - Suppose it could be improved:

i.e. for some $\eta > 0$

$$||u||_4^4 \leq c\lambda^{2+\eta}||u||_2^4$$

By the theory of Anderson localization there is some $n_0 \in \mathbb{Z}$ such that for any $\epsilon > 0$, $|u(n)| \leq C_\epsilon e^{-(\gamma(E,\lambda)-\epsilon)|n-n_0|}$ with $\gamma(E,\lambda) > 0$.

By Figotin-Pastur formula $\gamma(E,\lambda) \sim C(E)\lambda^2$ as $\lambda \to 0$.

These inequalities are incompatible. This implies $\lambda^2$ is optimal in 1D.

2D- it is suspected that localization lengths are exponential in $\lambda^{-2}$
J. Bourgain extended these results to random potentials

\[ V_\omega(n) = \omega_n \nu_n \]

\{\omega_n\} are Bernoulli or normalized Gaussians

\{\nu_n\} satisfying \( \sup |\nu_n||n|^\rho < C \) with \( \rho > 1/2 \).

- proved \( H \) has only a.c. spectrum away from the band edges and \( \text{E}=0 \)

- established the existence and completeness of the wave operators