Bidder Welfare in an Auction with a Buyout Option

Timothy Mathews*

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Abstract

A buyout option enhances an auction by allowing a bidder to purchase the item at a pre-specified price (instead of attempting to obtain the item by way of auction). A comparison is made between the ex ante welfare of bidders in an auction with a buyout option to a traditional auction with no such option. The impact on bidder welfare is shown to depend upon the distribution from which bidder valuations are drawn. In comparison to a traditional auction with no buyout option, when a buyout option is in place either: all bidders are weakly better off (in which case the option results in an ex ante Pareto improvement) or bidders with “relatively high valuations” are worse off.

Keywords: Auctions, Internet, Buyout Option, Bidder Welfare.


*Department of Economics, California State University-Northridge, 18111 Nordhoff St., Northridge, CA 91330-8374, USA; e-mail: tmathews@csun.edu; telephone: (818) 677-6696.
1 Introduction

One of the many ways in which internet auction sites have changed the rules from those in traditional auctions is by offering a “buyout” or “auction stop” option, like the “buy it now” option offered by industry leader eBay (www.eBay.com). When such an option is available, any potential bidder can choose to stop the auction and purchase the item at the buyout price, a pre-specified price chosen by the seller. Lucking-Reiley (2000, p. 245) noted that such options essentially allow “the bidder to buy an early end to the auction by submitting a sufficiently high bid.”

Options of this nature are now commonplace on the internet. eBay’s “buy it now” feature has become very popular since its introduction in November 2000. Fixed price trading (of which sales by way of “buy it now” are the primary component) accounted for $2.0 billion of eBay’s gross merchandise sales during the third quarter of 2003 and $2.2 billion of eBay’s gross merchandise sales during each of the first two quarters of 2004. These amounts represent 28%, 28%, and 27% of total gross merchandise sales during each respective quarter. Clearly such options are being used extensively.

When discussing options of this nature, it is important to recognize the difference between a “permanent” buyout option in which the option is available for the entire duration of the auction and a “temporary” buyout option in which the option may cease to be available before the auction ends. When available on eBay, the buyout option is only available so long as no traditional bids have been placed. Thus, it is not only possible for this temporary option to be no longer available before the conclusion of the auction, but any potential bidder has a great deal of control over whether or not the option is available to other bidders. Similarly, on the auction sites LabX

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1 These figures are from eBay’s Fourth Quarter 2003, First Quarter 2004, and Second Quarter 2004 Financial Results, available online at http://investor.ebay.com/financial.cfm?ssPageName=CMDV:AB.

2 If the seller has specified a positive secret reserve price, the buyout option is available so long as no bids exceeding this secret reserve price have been submitted.
(www.LabX.com) and Mackley & Company (www.mackley.com) the buyout option (if not exercised) will cease to be available at some point before the conclusion of the auction, a time which cannot be influenced by bidder behavior. In contrast, sites such as Yahoo! (www.auctions.yahoo.com) and Amazon (auctions.amazon.com) offer permanent buyout options which are available for the entire duration of the auction. A bidder cannot alter the availability of a permanent buyout option, short of exercising the option.

Budish and Takeyama (2001) analyze an auction with a permanent buyout option in which two bidders with valuations drawn from a discrete distribution compete for a single object. They illustrate that when bidders are risk averse an optimally set buyout price can increase the expected revenue of the seller. Thus, a risk neutral seller will find such an option beneficial.

Mathews and Katzman (2006) consider an auction with a temporary buyout option in which \( n + 1 \) bidders attempt to acquire a single item. It is shown that a risk averse seller facing risk neutral bidders will offer a buyout option which will be exercised with positive probability.

Lopomo (1998) analyzes a class of selling mechanisms called “simple sequential auctions,” in which the seller is restricted to choosing a sequence of increasing bid prices and non-increasing ask prices. An auction with a permanent buyout option (with a potentially changing, but non-increasing, buyout price) falls within this class of selling mechanisms. Lopomo shows that to maximize expected revenue, a seller will choose a sequence of ask prices such that risk neutral bidders never purchase the item at the ask price.

Reynolds and Wooders (2003) compare permanent and temporary options of this nature. When a seller faces two risk averse bidders with uniformly distributed valuations, options of each format may potentially lead to higher expected revenue than similar auctions without such options. Further, for a common buyout price, the expected revenue of the seller is greater if the buyout option is permanent than if the buyout option is temporary.

Kirkegaard and Overgaard (2003) focus on a sequence of two auctions in which each of two bidders has a positive but decreasing valuation for
each item. Supposing that all auction participants are risk neutral and non-discounting, they show that a seller may wish to use such an option when bidders desire multiple units. Specifically, offering a buyout option in the first of two auctions can increase the expected revenue in the first auction (at the expense of expected revenue in the second auction, as well as total expected revenue across both auctions). Additionally, total expected revenue may be increased by potentially offering a buyout option in later auctions, conditional upon the outcome of earlier auctions.

Mathews (2004) analyzes an auction with a temporary buyout option with rules mirroring those on eBay, in which \( n + 1 \) bidders with uniformly distributed valuations compete for a single item. The focus of the analysis is the impact of discounting on the part of auction participants. Discounting by either the seller or the bidders can lead to the seller choosing a buyout price which results in the option being exercised with positive probability.

Mathews (2003) introduces risk aversion on the part of the seller into the model developed by Mathews (2004). Again, a risk averse seller facing risk neutral bidders will offer a buyout option which will be exercised with positive probability.

In the aforementioned studies, the potential allocative inefficiency of such options has been well noted. When bidders exercise such an option with positive probability, the option may be successfully exercised by and the item awarded to a bidder other than the one with the highest valuation. As a result, the bidder with the highest valuation may clearly be worse off ex post than if such an option is not offered (further, it is clear that bidders other than the high valuation bidder may be better off ex post when such an option is in place).

However, it may be that all bidders are weakly better off ex ante when the seller chooses to offer a buyout option which is exercised with positive probability. In such instances, offering a buyout option results in an ex ante

\[\text{3This possibility was briefly noted by Mathews (2003 and 2004) and Mathews and Katzman (2006).}\]
Pareto improvement in comparison to a traditional auction without such an option. A generalized version of the model developed in Mathews (2004) is analyzed in order to more fully examine the impact of such an option on the ex ante welfare of auction participants.

The model analyzed by Mathews (2004) is described and generalized in Section 2. Equilibrium behavior for risk neutral, non-discounting bidders is characterized in Section 3. The choice of buyout price by the seller is analyzed in Section 4. The resulting impact on the ex ante welfare of auction participants is examined in Section 5. Section 6 concludes.

2 A Model of eBay’s “Buy it Now” Option

As noted, Buyout option rules differ across auction sites. The most common online auction rules are those of eBay. The auctions on eBay are ascending price auctions with proxy bidding. The seller can choose to offer a buyout option, an option referred to as “buy it now” by eBay. The buyout price is chosen by the seller when the item is initially listed for sale, and the option is available only before any bids are placed. If the option is exercised, the auction stops immediately, with the bidder exercising the option receiving the item for certain at the buyout price.

Under the rules of eBay, a bidder exercising the option knows with certainty that he will receive the item. Also, a bidder is able to make an available buyout option no longer available to others by simply submitting a bid. The model developed in Mathews (2004) incorporates both of these important characteristics of the timing of eBay’s “buy it now” option, while the models

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4The rules on eBay are the most common for two reasons. Not only is eBay by far the largest internet auction site, but also numerous auction sites have emerged with rules identical to those of eBay (for example: “eBiddin” (www.eBiddin.com), “Trade Me” (www.trademe.co.nz), and “Auction.com” (www.auction.com)).

5A proxy bid will “compete” at the minimum level necessary to be the leading bid. For a further explanation of the proxy bidding rules on eBay, see pages.ebay.com/help/buyerguide/bidding-prxy.html.

The model outlined below was developed in Mathews (2004) and closely resembles the “buy it now” option available on eBay. Consider a situation in which \( n + 1 \) bidders, each with an independent private valuation \( v_i \in [0, 1] \), attempt to acquire a single object. The model consists of a two stage game. Each potential bidder \( i \) realizes an independent private valuation \( v_i \) before stage one. During stage one the seller chooses a buyout price, \( \bar{B} \). Stage two consists of an ascending price auction with proxy bidding occurring over continuous time \( T = [0, 1] \). During this auction the buyout option is available so long as no bids have been submitted.

Each potential bidder \( i \) realizes an “arrival time” \( t_i \in [0, 1] \), during stage two. Bidder \( i \) may “actively participate” in the auction only after his arrival.\(^6\) If the option is exercised, the transaction occurs at the time when the option is exercised. If the option is not exercised (and the item is sold by way of auction), the item is sold at time \( t = 1 \) to the bidder submitting the highest proxy bid for an amount equal to the second highest proxy bid submitted.

At any time \( t \), a bidder that has arrived can observe the exact time at which any previous proxy bids were submitted. Further, such a bidder can observe the amount of every proxy bid previously submitted, except for the proxy bid that is currently the highest. The exact amount of the proxy bid that is currently highest is observed by only the bidder that submitted the bid. Let \( h(t) \) denote this information at time \( t \). This observable history of play is an element of \( H(t) \), the set of possible histories at time \( t \). Let \( A_i(h(t)) \) denote the set of possible actions for bidder \( i \) to take given the history \( h(t) \). At any time after \( t_i \), bidder \( i \) can: exercise the buyout option if it is available; place or update a proxy bid; or take no action. A strategy of bidder \( i \) is a function specifying an action for bidder \( i \) at time \( t \) from the set \( A_i(h(t)) \) for

\(^6\)By “actively participate” we mean submit (or update) a proxy bid or exercise the buyout option (if it is available). A bidder may increase, but not decrease, his proxy bid at any time before the close of the auction.
every possible \( h(t) \in H(t) \) for every \( t \in [0, 1] \).

The payoff of each participant is defined in the usual manner. If a bidder obtains the item, his payoff is the difference between his valuation and the amount he pays. A bidder not obtaining the item and making no payments realizes a payoff of zero. The payoff of the seller is equal to the total revenue received. If the option is exercised the seller receives \( \bar{B} \); if the option is not exercised, the seller receives an amount equal to the second highest proxy bid submitted.

The model proposed here closely resembles the eBay setup in that: the option is available so long as no traditional proxy bids have been placed; if a bidder exercises an available option he will receive the item for certain; and a bidder can make the option no longer available to others by placing a bid.

This model was analyzed by Mathews (2003 and 2004) assuming: the \( v_i \)'s are independently distributed \( U[0,1] \); and the \( t_i \)'s are independent, each with the cumulative distribution function \( J(t) \) such that \( J(0) = 0, J(1) = 1 \), and \( J'(t) > 0 \) for \( 0 < t < 1 \). Mathews (2004) shows that discounting on either side of the transaction (by either bidders or the seller) can motivate the seller to choose a buyout price low enough so that bidders exercise the option with positive probability. Mathews (2003) establishes that a risk averse seller facing risk neutral, non-discounting bidders will find such an option beneficial. In each case, it was noted that risk neutral, non-discounting bidders are weakly better off ex ante when the seller chooses a buyout price low enough so that the option is exercised with positive probability.

In order to analyze ex ante bidder welfare more generally, the assumption of uniformly distributed bidder valuations is relaxed. It is instead supposed that bidders have independent private valuations drawn from \( F(v) \) such that \( F(0) = 0, F(1) = 1, \) and \( f(v) = F'(v) > 0 \) for all \( 0 < v < 1 \).

The game is solved recursively. First, equilibrium behavior is identified for non-discounting, risk neutral bidders in the second stage of the game as a function of \( \bar{B} \). The choice of \( \bar{B} \) by the seller is subsequently examined, assuming that all potential bidders follow the strategies identified as an equi-
librium in stage two. It is shown that risk aversion or time impatience on
the part of the seller will make the offering of such an option beneficial.

The primary contribution of the current study is a detailed examination
of the ex ante welfare of bidders in this more general setting. When the
seller enhances a traditional auction by offering a buyout option, the sub-
sequent impact on ex ante bidder welfare is such that (in comparison to a
similar auction with no buyout option) either: all bidders are better off (in
which case the option results in an ex ante Pareto improvement); or bidders
with “relatively high valuations” are worse off. Which of these two possible
outcomes results depends upon the functional form of $F(v)$.

3 Equilibrium Bidder Behavior

Attention is restricted to risk neutral, non-discounting bidders. Specifically,
it is assumed throughout that each bidder has utility for receiving an object
valued at $v$ for a payment of $y$ at time $t$ of $u(x) = x$, with $x = v - y$.

The first goal is to determine if a symmetric equilibrium exists such that
in stage two every bidder $i$ follows a strategy of the form: if the buyout
option is available at time $t_i$, exercise the option immediately if $\bar{B} < B(v_i)$
and submit a proxy bid of $b_i = v_i$ immediately if $\bar{B} \geq B(v_i)$, for some
prespecified function $B(v)$; if the option is not available at time $t_i$, submit
a proxy bid of $b_i = v_i$ immediately. An equilibrium of this form specifies
strategies which are essentially independent of the history of play $h(t)$. This
results from the fact that by the time bidder $i$ observes any action by a rival
bidder, either there is no available action for $i$ to take or $i$ has a weakly
dominant strategy. If bidder $i$ observes that a rival has exercised the option,
then it is too late for $i$ to take any action. If bidder $i$ observes that a rival has
submitted a proxy bid, then $i$ has a weakly dominant strategy of submitting
a proxy bid of $b_i = v_i$.

If $B(v)$ is strictly increasing and continuous in $v$ for $v \in [0, 1)$, define $\bar{v}$
to be the unique value of $v \in [0, 1]$ such that $B(\bar{v}) = \bar{B}$. If $B(v) < \bar{B}$ for
all \( v \in [0, 1] \), define \( \bar{v} = 1 \). In equilibrium, a bidder will exercise the option immediately upon arrival if the option is available and \( v_i > \bar{v} \).

The following theorem specifies equilibrium bidder behavior during stage two. All results are proved in Appendix A.\(^7\)

**Theorem 1** A symmetric equilibrium exists in which each bidder \( i \): exercises the buyout option at time \( t_i \) if the option is available and \( \bar{B} < v_i - \int_0^{v_i} (v_i - x)nF(x)^{n-1}f(x)dx \); and otherwise submits a proxy bid of \( b_i = v_i \) at time \( t_i \).

Let \( \sigma^* \) denote the strategies described in Theorem 1.\(^8\) There is no claim that \( \sigma^* \) is the unique equilibrium for bidders in the second stage.\(^9\) In order to proceed, it is assumed that bidders follow \( \sigma^* \) during stage two.\(^10\)

A bidder \( i \) playing as dictated by Theorem 1 will exercise an available option immediately upon arrival if \( \bar{B} \) is “low enough,” with \( B(v) \) specifying precisely what “low enough” means as a function of \( v_i \). Further, a bidder choosing not to exercise an available option will immediately make the option no longer available to others by submitting a bid.

Note that if the first bidder to arrive submits a bid, the resulting outcome (in terms of the allocation of the item, as well as the payoff of every bidder) is identical to that of a traditional sealed bid second price auction. The payoff of a bidder with valuation \( v_i \) from competing in such an auction against \( n \) rivals with valuations drawn from \( F(v) \) is \( \pi_{SPA}(v_i) = \int_0^{v_i} (v_i - x)nF(x)^{n-1}f(x)dx \).

\(^7\)To simplify the presentation, all results are proved for the case in which \( n + 1 > 2 \). All results also hold if \( n + 1 = 2 \).

\(^8\)It should be noted that for \( F(v) = v \), the strategies described in Theorem 1 do correspond to those identified by Mathews (2004) for non-discounting, risk neutral bidders.

\(^9\)In fact, it is known that \( \sigma^* \) is not the unique equilibrium. For example, similar equilibria characterized by \( B(v) = v - \int_0^v (v - x)nF(x)^{n-1}f(x)dx \) exist, for which a bidder not exercising an available option is free to submit any bid \( b_i = b \in (0, v_i] \) at time \( t_i \) so long as he updates his proxy bid to \( b_i = v_i \) at time \( t = 1 \).

\(^10\)Note that for the other equilibria characterized by \( B(v) \) the payoff of each player is the same as for \( \sigma^* \), for any specific \( \bar{B} \).
From here, when the auction is enhanced with a buyout option the first bidder to arrive will choose to exercise the option if and only if $\bar{B} < v_i - \pi_{SPA}(v_i)$, or equivalently $v_i - \bar{B} > \pi_{SPA}(v_i)$. That is, he will choose to exercise the option only if the certain payoff from doing so exceeds his expected payoff from attempting to obtain the item by way of auction.

Since $B(v)$ can also be expressed as $B(v) = v - \int_0^v F(x)dx$, we observe that $B'(v) = 1 - F(v)^n$ (which is positive for all $v \in [0, 1]$). Further, $B(0) = 0$ and $B(1) = 1 - \int_0^1 F(x)^n dx = \int_0^1 x^n F(x)^{n-1} f(x)dx$ imply: if $\bar{B} = 0$ the first bidder to arrive will exercise the option, whereas if $\bar{B} \geq 1 - \int_0^1 F(x)^n dx$ the first bidder to arrive will immediately submit a bid (making the option no longer available to others). Finally, whenever $\bar{B} < 1 - \int_0^1 F(x)^n dx$ the option is exercised with positive probability.

If the seller chooses $\bar{B} < 1 - \int_0^1 F(x)^n dx$ (so that $\bar{v} < 1$), the option will be exercised if and only if the first bidder to arrive has a valuation $v_i > \bar{v}$. When the option is exercised, the item may be awarded to a bidder other than the one with the highest valuation. The resulting allocation is ex post inefficient if the first bidder to arrive exercises the option, but does not have the highest valuation.

4 Choice of Buyout Price by the Seller

Continuing to solve the game recursively, the choice of $\bar{B}$ by the seller is examined, assuming that bidders follow $\sigma^*$. Since $B'(v) > 0$ for $v \in [0, 1)$, this choice of $\bar{B}$ by the seller can be analyzed as a choice of $\bar{v}$. Let $\bar{v}^*$ denote the optimal value of $\bar{v}$ and $\bar{B}^*$ denote the optimal value of $\bar{B}$. It will be shown that the expected revenue of the seller is increasing in $\bar{v}$ for $\bar{v} \in [0, 1]$, implying that a risk neutral, non-discounting seller will choose a buyout price high enough so that the option is never exercised ($\bar{v}^* = 1$). However, a seller that is either risk averse or time impatient will choose a buyout price low enough so that the option is exercised with positive probability ($\bar{v}^* < 1$).
Thus, the results of this section collectively illustrate that the motivations for a seller to offer a buyout option which were identified by Mathews (2003 and 2004) extend to a more general environment in which bidder valuations need not be uniformly distributed.

4.1 Expected Revenue of Seller

When bidders follow $\sigma^*$, the expected revenue of the seller is

$$\Pi_s(\bar{v}) = \int_{\bar{v}}^1 B(\bar{v}) f(y) dy$$

$$+ \int_0^{\bar{v}} \left\{ \int_0^y x g(x) dx + \int_0^y \left[ \int_0^x y k(z) dz + \int_y^x z k(z) dz \right] g(x) dx \right\} f(y) dy$$

where: $y$ is the valuation of the first bidder to arrive and $f(y)$ is the probability density function of $y$; $x$ is the largest valuation of all bidders other than the first bidder to arrive and $g(x) = nF(x)^{n-1} f(x)$ is the probability density function of $x$; and $z$ is the second largest valuation of all bidders other than the first bidder to arrive and $k(z) = \frac{(n-1)F(z)^{n-2} f(z)}{F(x)^{n-1}}$ is the probability density function of $z$ conditional upon a realized value of $x$.

This function can be expressed as:

$$\Pi_s(\bar{v}) = B(\bar{v})[1 - F(\bar{v})] + \int_{\bar{v}}^0 \left\{ \int_0^y nx F(x)^{n-1} f(x) dx \right\} f(y) dy$$

$$+ \int_0^{\bar{v}} \left\{ \int_0^y y F(y)^{n-1} + \int_y^x (n-1) z F(z)^{n-2} f(z) dz \right\} n f(x) dx \right\} f(y) dy.$$ 

A risk neutral, non-discounting seller is concerned only with expected revenue. Proposition 1 states the optimal choice of $\bar{v}$ for such a seller.

**Proposition 1** For a risk neutral, non-discounting seller: $\bar{v}^* = 1$.

Thus, such a seller chooses a buyout price high enough so that the option is never exercised.
4.2 Risk Averse Seller

Now consider a risk averse expected utility maximizing seller, with utility for a realization of revenue \( x \) equal to \( u_r(x) \), such that \( u'_r(x) > 0 \) and \( u''_r(x) < 0 \). The expected utility of such a seller as a function of \( \bar{v} \) is

\[
U_{s,r}(\bar{v}) = u_r(B(\bar{v})) [1 - F(\bar{v})] + \int_0^{\bar{v}} \left\{ \int_0^y n u_r(x) F(x)^{n-1} f(x) dx \right\} f(y) dy
+ \int_0^{\bar{v}} \left\{ \int_y^1 \left[ u_r(y) F(y)^{n-1} + \int_y^x (n-1) u_r(z) F(z)^{n-2} f(z) dz \right] n f(x) dx \right\} f(y) dy.
\]

Proposition 2 characterizes the choice of \( \bar{v} \) for a risk averse seller.

**Proposition 2** For a risk averse seller with \( u_r(x) \) such that \( u'_r(x) > 0 \) and \( u''_r(x) < 0 \): \( \bar{v}^* < 1 \).

Thus, a risk averse seller will choose a buyout price low enough so that the option is exercised with positive probability. Even though the option reduces expected revenue, a risk averse seller finds such an option beneficial since it also reduces the risk inherent with selling an item by way of auction.

4.3 Time Impatient Seller

Finally consider a seller that discounts future transactions. Specifically, consider a seller with utility for revenue of \( x \) received at time \( t \) equal to \( u_d(x,t) = x \delta(t) \), where \( \delta(t) \) is such that \( \delta(t) > 0 \) and \( \delta'(t) < 0 \) for all \( t \in [0,1] \). For this seller, \( \frac{\partial u_d(x,t)}{\partial t} < 0 \), indicating that the seller prefers a given transaction to occur “sooner” as opposed to “later.”\(^{11}\) In this case, the

\(^{11}\)Mathews (2004) considered discounting auction participants with preferences given by \( u(x,t) = x \delta^t \) with \( \delta \in (0,1) \). The specific functional form considered by Mathews (2004) falls within the broader class of functions considered here.
expected utility of the seller can be expressed as a function of \( \bar{v} \) as

\[
U_{s,d}(\bar{v}) = [1 - F(\bar{v})] \int_0^1 u_d(B(\bar{v}), w) j(1)(w)dw + \int_0^{\bar{v}} \left\{ \int_0^y u_d(x, 1) g(x)dx \right\} f(y)dy
\]

\[
+ \int_0^{\bar{v}} \left\{ \int_0^y u_d(y, 1) k(z)dz + \int_0^\bar{v} u_d(z, 1) k(z)dz \right\} g(x)dx \right\} f(y)dy,
\]

where \( w \) is the arrival time of the first bidder to arrive and \( j(1)(w) = (n + 1) \int_1 w [1 - J(w)]^n \) is the probability density function of \( w \).

This function can be expressed as:

\[
U_{s,d}(\bar{v}) = [1 - F(\bar{v})] \int_0^1 u_d(B(\bar{v}), w)(n + 1) j(w)[1 - J(w)]^n dw
\]

\[
+ \int_0^{\bar{v}} \left\{ \int_0^y u_d(x, 1)nF(x)^{n-1}f(x)dx \right\} f(y)dy
\]

\[
+ \int_0^{\bar{v}} \left\{ \int_0^y u_d(y, 1)(n - 1)F(z)^{n-2}f(z)dz \right\} nf(x)dx \right\} f(y)dy
\]

\[
+ \int_0^{\bar{v}} \left\{ \int_0^y u_d(z, 1)(n - 1)F(z)^{n-2}f(z)dz \right\} nf(x)dx \right\} f(y)dy.
\]

Proposition 3 characterizes the choice of \( \bar{v} \) for such a time impatient seller.

**Proposition 3** For a seller with utility for revenue of \( x \) received at time \( t \) equal to \( u_d(x, t) = x\delta(t) \), where \( \delta(t) \) is such that \( \delta(t) > 0 \) and \( \delta'(t) < 0 \) for all \( t \in [0, 1] \): \( \bar{v}^* < 1 \).

That is, a seller that discounts future revenue will choose a buyout price low enough so that the option is exercised with positive probability. While the option does decrease expected revenue, it potentially allows the transaction to occur sooner than it otherwise would. A seller that discounts future transactions is willing to accept this reduction in expected payoff, in order to obtain the potential benefit of having the transaction occur earlier.

In summary, while a risk neutral non-discounting seller will not find such an option beneficial, a seller that is either risk averse or time impatient will
choose $B^* < 1 - \int_0^1 F(x)dx$, leading to $\bar{v}^* < 1$. Any bidder with $v_i \in [\bar{v}^*, 1]$ will exercise the option immediately upon arrival.

5 Ex Ante Auction Participant Welfare

The seller will choose $\bar{v} < 1$ if and only if his ex ante payoff from doing so exceeds his ex ante payoff from $\bar{v} = 1$. Suppose this is the case. The remainder of the analysis is concerned with the ex ante welfare of bidders when $\bar{v}^* < 1$.

Mathews (2003 and 2004) analyzed this model assuming $F(v) = v$. It was shown that for any $\bar{v} < 1$, all bidders are weakly better off than they would be if the item were instead sold by way of auction with no buyout option. Specifically, any bidder with $v_i \in [0, \bar{v}]$ or $v_i = 1$ has the same ex ante expected payoff under the two selling mechanisms, while bidders with $v_i \in (\bar{v}, 1)$ have a strictly higher ex ante payoff when the buyout option is in place. Since all agents are better off ex ante than if the option is not offered, enhancing an auction with a buyout option leads to an ex ante Pareto improvement.

Bidder welfare is analyzed assuming valuations are drawn from a general distribution $F(v)$ for which $F(0) = 0$, $F(1) = 1$, and $F'(v) = f(v) > 0$ for all $0 < v < 1$. For a general function $F(v)$, it may or may not be the case that all bidders are better off ex ante when such an option is in place.

It is straightforward to verify that in an ascending price auction with proxy bidding and no buyout option it is a weakly dominant strategy for each bidder $i$ to ultimately submit a bid of $b_i = v_i$. When all bidders follow this strategy, the outcome is simply that the bidder with the highest valuation receives the item and pays an amount equal to the second highest valuation of all bidders. This is the same outcome that results in a standard sealed bid second price auction when each bidder follows the weakly dominant strategy of submitting a bid equal to his true valuation. The expected payoff of a
Consider the case in which a buyout option can be offered. Suppose bidders follow $\sigma^*$ and the seller chooses $\bar{v}^* < 1$. Clearly the seller is better off ex ante than he would be if he were not able to offer such an option. Also, any bidder with $v_i \in [0, \bar{v}]$ is just as well off with or without the option in place (since the outcome for such a bidder is the same in either case for every possible realization of valuations and arrival times of the $n$ other bidders). It remains to compare $\pi_i A$ to the expected payoff from competing in an auction with a buyout option for bidders with $v_i \in (\bar{v}, 1]$.

A bidder with $v_i \in (\bar{v}, 1]$ will exercise an available option immediately upon arrival. Thus, his ex ante expected payoff is

$$\pi_i B(v_i, \bar{v}) = \frac{1}{n+1} [v_i - B(\bar{v})] + \frac{n}{n+1} \int_0^{\bar{v}} \int_0^{v_i} (v_i - y)(n-1)F(y)^{n-2}f(y)dyf(x)dx$$

$$+ \frac{n}{n+1} \int_0^{\bar{v}} \int_0^{v_i} (v_i - y)(n-1)F(y)^{n-2}f(y)dyf(x)dx$$

$$= \frac{1}{n+1} (v_i - \bar{v} + \bar{v}F(\bar{v})^n) - n \int_0^{\bar{v}} xF(x)^{n-1}f(x)dx$$

$$+ \frac{n}{n+1} \left( v_i F(v_i)^{n-1}F(\bar{v}) - \int_0^{\bar{v}} x(n-1)F(x)^{n-2}f(x)F(\bar{v})dx \right).$$

Define the gain in the ex ante expected payoff of such a bidder by

$$\pi_i G(v_i, \bar{v}) = \pi_i B(v_i, \bar{v}) - \pi_i A(v_i).$$

A bidder $i$ with $v_i \in (\bar{v}, 1]$ is strictly better off when the buyout option is offered if $\pi_i G(v_i, \bar{v}) > 0$ (and weakly better off if $\pi_i G(v_i, \bar{v}) \geq 0$).

With a buyout option in place either all bidders are weakly better off or bidders with “relatively high valuations” are worse off. As stated by Theorem 2, which outcome arises depends upon whether a bidder with the highest
possible valuation (that is, a bidder with \( v_i = 1 \)) is made better off or worse off by the option.

**Theorem 2** If \( \pi_{iG}(1, \bar{v}) \geq 0 \), then \( \pi_{iG}(v_i, \bar{v}) \geq 0 \) for all \( v_i \in (\bar{v}, 1] \). If \( \pi_{iG}(1, \bar{v}) < 0 \), then there exists a unique \( v_C \in (\bar{v}, 1) \) such that: \( \pi_{iG}(v_i, \bar{v}) \geq 0 \) for all \( v_i \in [\bar{v}, v_C] \) and \( \pi_{iG}(v_i, \bar{v}) < 0 \) for \( v_i \in (v_C, 1) \).

From Theorem 2 it follows that if \( \pi_{iG}(1, \bar{v}) \geq 0 \), then all bidders are better off when the seller offers a buyout option. As a result, such an option results in an ex ante Pareto improvement, compared to a similar auction with no such option. If instead \( \pi_{iG}(1, \bar{v}) < 0 \), then bidders with \( v_i \leq v_C \) are better off ex ante with the option in place, but those with \( v_i > v_C \) are worse off. As such, the option does not result in an ex ante Pareto improvement.

Thus, enhancing a traditional auction with a buyout option affects ex ante bidder welfare in one of two possible ways: either all bidders are weakly better off when such an option is in place or bidders with “relatively high valuations” are worse off with such an option in place.

Corollary 1 provides a sufficient condition to ensure that all bidders are better off with the option in place.

**Corollary 1** If \( f'(v) \geq 0 \) for all \( v \in [0, 1] \), then \( \pi_{iG}(1, \bar{v}) \geq 0 \) for any \( \bar{v} < 1 \).

By Corollary 1, all bidders are weakly better off with the option in place whenever \( f(v) \) is non-decreasing in \( v \) for all \( v \in [0, 1] \). Again, if this is the case, then an auction with a buyout option results in an ex ante Pareto improvement, compared to a similar auction with no such option.

When bidder valuations are uniformly distributed, \( f'(v) = 0 \). Thus, the observation of Mathews (2003 and 2004) that such an option results in an ex ante Pareto improvement in this case follows from Corollary 1.

Corollary 2 specifies a sufficient condition under which bidders with “rel-
Corollary 2 If $f'(v) < 0$ for all $v \in [0, 1]$, then $\pi_iG(1, \bar{v}) < 0$ for any $\bar{v} < 1$.

By Corollary 2, $\pi_iG(1, \bar{v}) < 0$. In this case there exists a unique $v_C \in (\bar{v}, 1)$ such that $\pi_iG(v_i, \bar{v}) \geq 0$ for $v_i \leq v_C$, while $\pi_iG(v_i, \bar{v}) < 0$ for $v_i > v_C$. Those bidders with $v_i > v_C$ are strictly worse off ex ante with the option in place. As a result, such an option does not lead to an ex ante Pareto improvement.

Note that while the sufficient conditions identified in Corollaries 1 and 2 are restrictive, if applicable these results allow for a very straightforward characterization of ex ante bidder welfare. This is because each result specifies the sign of $\pi_iG(1, \bar{v})$ for any possible value of $\bar{v} < 1$. Thus, if $F(v)$ is such that one of these two Corollaries can be applied, the impact of a buyout option on ex ante bidder welfare can be addressed without having to determine the specific value of $\bar{v} < 1$ which the seller would optimally choose.\footnote{Alternatively, it could be shown that if $f'(v) < 0$ for $v \approx 1$, then $\pi_iG(1, \bar{v}) < 0$ for $\bar{v} \approx 1$. Thus, so long as $f(v)$ is decreasing for $v \approx 1$ there are some buyout prices which would decrease the ex ante expected payoff of those bidders with “relatively high valuations.”}

Finally, as an application of Corollaries 1 and 2, suppose bidder valuations are drawn from $F(v) = v^\alpha$, with $\alpha \in (0, \infty)$. In this case $f(v) = \alpha v^{\alpha-1}$ and $f'(v) = \alpha(\alpha - 1)v^{\alpha-2}$.

For $\alpha \geq 1$, $f'(v) \geq 0$ for all $v \in [0, 1]$. By Corollary 1, all bidders are weakly better off whenever $\bar{v} < 1$, implying that the option leads to an ex ante Pareto improvement compared to a similar auction with no buyout option. If $\alpha = 1$, bidder valuations are uniformly distributed, which was the case analyzed by Mathews (2003 and 2004).

For $\alpha < 1$, $f'(v) < 0$ for all $v \in [0, 1]$. By Corollary 2, $\pi_iG(1, \bar{v}) < 0$ for any $\bar{v} < 1$. Thus, there exists a unique $v_C \in (\bar{v}, 1)$ such that: $\pi_iG(v_i, \bar{v}) \geq 0$ for $v_i \in [\bar{v}, v_C]$, while $\pi_iG(v_i, \bar{v}) < 0$ for $v_i \in (v_C, 1]$. Since bidders with $v_i > v_C$ are worse off, the option does not result in an ex ante Pareto improvement.
So, while the conditions identified in Corollaries 1 and 2 are restrictive, a functional form of $F(v)$ has been identified for which these conditions allow insights into ex ante bidder welfare to be made with ease.

6 Conclusion

A generalized version of the model of an auction with a buyout option developed by Mathews (2004) has been analyzed, relaxing the assumption of uniformly distributed bidder valuations. Equilibrium behavior was characterized for risk neutral, non-discounting bidders. When facing such bidders, a seller that is either risk averse or time impatient will choose a buyout price low enough so that the option is exercised with positive probability. It is noted, as in previous studies, that such an option leads to allocative inefficiency with strictly positive probability.

Mathews (2003 and 2004) and Mathews and Katzman (2006) recognized the potential for all bidders to be better off ex ante when such an option is offered. In such cases, allowing the seller to offer a buyout option leads to an ex ante Pareto improvement (compared to a similar auction with no such option). In light of this observation, the impact of a buyout option on ex ante auction participant welfare was examined in greater detail. Ex ante bidder welfare is shown to critically depend upon the distribution from which bidder valuations are drawn.

In comparison to a similar auction with no buyout option, when the seller chooses a buyout price low enough so that the option is exercised with positive probability, either: all bidders are weakly better off ex ante or bidders with “relatively high valuations” are strictly worse off ex ante. Thus, while it is possible for a buyout option to result in an ex ante Pareto improvement, it could instead be the case that some bidders are strictly worse off ex ante when the seller is able to offer such an option.
Appendix A

Proof of Theorem 1. This result can be proven by verifying that no bidder wishes to unilaterally deviate from the proposed strategy.

Consider a bidder $i$ choosing to submit a bid. Once a bid is submitted, the buyout option is no longer available. In terms of the final outcome, all that matters is the proxy bids submitted at $t = 1$. Further, for bidders with independent private valuations, the choice of proxy bid at $t = 1$ is equivalent to the choice of a bid in a sealed bid second price auction. Thus, a potential bidder $i$ choosing to bid has a dominant strategy of submitting (and updating as often as desired) a proxy bid of $b_i = b \in (0, v_i]$ at time $t_i$, as long as a proxy bid of $b_i = v_i$ is submitted before $t = 1$. The expected payoff of bidder $i$ from submitting a proxy bid of $b_i = v_i$ is

$$
\int_0^v (v_i - x) nF(x)^{n-1} f(x) dx.
$$

It is straightforward to characterize the optimal choice of a bidder $i$ arriving at a time $t_i$ when the option is not available. If the option is not available, either the option has been exercised or a proxy bid has been submitted. If the option has been exercised, the auction has ended and it is too late for $i$ to take any action; if a proxy bid has been submitted, $i$ has a dominant strategy of submitting a proxy bid of $b_i = v_i$ at any time before $t = 1$.

Now consider a bidder $i$ with valuation $v_i$ arriving at a time $t_i$ when the option is available. Such a bidder can take one of three possible courses of action at time $t_i$: exercise the option immediately, submit a bid immediately, or wait until time $t_i + \epsilon$ to take any action. If this bidder decides to submit a bid immediately upon arrival, his expected payoff is $\pi_{BA} = \int_0^v (v_i - x) nF(x)^{n-1} f(x) dx$. If this bidder decides to exercise the option immediately upon arrival, he realizes a payoff of $\pi_{OA} = v_i - \bar{B}$. Thus, such a bidder is better off exercising the option immediately (as opposed to submitting a bid immediately) if and only if $\bar{B} < B(v_i)$, with

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13In equilibrium, each bidder $i$ will act immediately if the option is available at $t_i$. Thus, only the first bidder to arrive is faced with this decision of what to do when the option is available.
\[ B(v) = v - \int_0^v (v - x) nF(x)^{n-1} f(x) dx. \]

Note that \( B'(v) = 1 - F(v)^n > 0 \) for \( v \in [0, 1) \). Let \( \bar{v} \) denote the unique value of \( v \) such that \( B(\bar{v}) = \bar{B} \).

It remains to show that it is best for bidder \( i \) to act immediately at time \( t_i \), as opposed to waiting until a later time \( t_i + \epsilon \). Let \( \Pr(R) \) denote the probability with which a rival bidder would arrive during the interval \( (t_i, t_i + \epsilon) \). Let \( \hat{\pi} \) denote the payoff of bidder \( i \) from waiting until time \( t_i + \epsilon \) if a rival bidder arrives before \( t_i + \epsilon \). If a rival bidder arrives at \( t_j \in (t_i, t_i + \epsilon) \), this rival will either exercise the option or bid immediately upon arrival. If this rival exercises the option, the payoff of \( i \) is zero; if this rival submits a bid, the expected payoff of \( i \) is \( \pi_B \). Letting \( \Pr(B) \) denote the probability with which a rival bidder arriving during \( (t_i, t_i + \epsilon) \) will submit a bid, \( \pi_{BL} = \Pr(B)\pi_B \leq \pi_B \).

Thus, if \( i \) waits until \( t_i + \epsilon \) to bid, his expected payoff is

\[ \pi_{BL} = [1 - \Pr(R)] \pi_B + [\Pr(R)] \hat{\pi} \leq \pi_B. \]

If \( i \) waits until \( t_i + \epsilon \) to exercise the option, his expected payoff is

\[ \pi_{OL} = [1 - \Pr(R)] (v_i - \bar{B}) + [\Pr(R)] \hat{\pi} \leq [1 - \Pr(R)] \pi_{OA} + [\Pr(R)] \pi_B. \]

It must be shown that: for a bidder with \( v_i \geq \bar{v} \), \( \pi_{OA} \geq \max \{ \pi_{BL}, \pi_{OL} \} \); and for a bidder with \( v_i < \bar{v} \), \( \pi_B \geq \max \{ \pi_{BL}, \pi_{OL} \} \). First consider a bidder with \( v_i \geq \bar{v} \). For this bidder \( \pi_{OA} \geq \pi_B \), implying

\[ \max \{ \pi_{BL}, \pi_{OL} \} \leq [1 - \Pr(R)] \pi_O + [\Pr(R)] \pi_B \leq \pi_{OA}. \]

Now consider a bidder with \( v_i < \bar{v} \). For this bidder \( \pi_{OA} < \pi_B \), implying

\[ \max \{ \pi_{BL}, \pi_{OL} \} \leq [1 - \Pr(R)] \pi_B + [\Pr(R)] \pi_B \leq \pi_B. \]

In either case \( i \) realizes a lower expected payoff by waiting until time \( t_i + \epsilon \). Thus, the conjectured strategies specify equilibrium bidder behavior during stage two. Q.E.D.
Proof of Proposition 1. For this seller,

\[ \Pi_s'(\bar{v}) = -f(\bar{v})B(\bar{v}) + B'(\bar{v})(1 - F(\bar{v})) + f(\bar{v})\int_0^{\bar{v}} nxF(x)^{n-1}f(x)dx \]

\[ + f(\bar{v})\int_{\bar{v}}^1 \left[ \bar{v}F(\bar{v})^{n-1} + \int_{\bar{v}}^x (n-1)zF(z)^{n-2}f(z)dz \right] nf(x)dx, \]

which can be expressed as \( \Pi_s'(\bar{v}) = f(\bar{v})\gamma(\bar{v}) \) with

\[ \gamma(\bar{v}) = \frac{1 - F(\bar{v})}{f(\bar{v})} (1 - F(\bar{v})^n) - \bar{v} + \bar{v}F(\bar{v})^n + \bar{v}nF(\bar{v})^{n-1} - \bar{v}nF(\bar{v})^n \]

\[ + \int_{\bar{v}}^1 \left\{ \int_{\bar{v}}^{\bar{v}} (n-1)zF(z)^{n-2}f(z)dz \right\} nf(x)dx. \]

The desired result will follow if \( \Pi_s'(\bar{v}) \geq 0 \) for all \( \bar{v} \in (0, 1) \). Since \( f(\bar{v}) > 0 \) for all \( \bar{v} \in (0, 1) \), this is the case so long as \( \gamma(\bar{v}) > 0 \) for all \( \bar{v} \in (0, 1) \). Since

\[ \int_{\bar{v}}^{\bar{v}} (n-1)zF(z)^{n-2}f(z)dz > \int_{\bar{v}}^{\bar{v}} (n-1)\bar{v}F(z)^{n-2}f(z)dz \]

it follows that

\[ \gamma(\bar{v}) > \frac{1 - F(\bar{v})}{f(\bar{v})} (1 - F(\bar{v})^n) - \bar{v} + \bar{v}F(\bar{v})^n + \bar{v}nF(\bar{v})^{n-1} - \bar{v}nF(\bar{v})^n \]

\[ + \int_{\bar{v}}^1 \left\{ \int_{\bar{v}}^{\bar{v}} (n-1)\bar{v}F(z)^{n-2}f(z)dz \right\} nf(x)dx \]

\[ = \frac{1 - F(\bar{v})}{f(\bar{v})} (1 - F(\bar{v})^n) > 0. \]

As a result, \( \Pi_s'(\bar{v}) \geq 0 \) for all \( \bar{v} \in (0, 1) \), implying that for a risk neutral, non-discounting seller: \( \bar{v}^* = 1 \). Q.E.D.

Proof of Proposition 2. For a risk averse seller

\[ U_{s,r}'(\bar{v}) = u_r'(B(\bar{v})) B'(\bar{v})[1 - F(\bar{v})] - u(B(\bar{v}))f(\bar{v}) + f(\bar{v})\int_0^{\bar{v}} nu_r(x)F(x)^{n-1}f(x)dx \]

\[ + f(\bar{v})\int_{\bar{v}}^1 u_r(\bar{v})F(\bar{v})^{n-1} + \int_{\bar{v}}^{\bar{v}} (n-1)u_r(z)F(z)^{n-2}f(z)dz \right\} nf(x)dx. \]
Since $B(1) = \int_0^1 xnF(x)^{n-1}f(x)dx$, we have:

$$U'_{s,r}(1) = -f(1) \left[ u_r(B(1)) - \int_0^1 u_r(x)nF(x)^{n-1}f(x) \right]$$

$$= -f(1) \left[ u_r \left( \int_0^1 xnF(x)^{n-1}f(x)dx \right) - \int_0^1 u_r(x)nF(x)^{n-1}f(x) \right].$$

For any strictly concave function $u_r(x)$ it must be that

$$u_r \left( \int_0^1 xnF(x)^{n-1}f(x)dx \right) > \int_0^1 u_r(x)nF(x)^{n-1}f(x),$$

implying $U'_{s,r}(1) < 0$. As a result, for a risk averse seller $\bar{v}^* < 1$. Q.E.D.

**Proof of Proposition 3.** For such a time impatient seller

$$U'_{s,d}(\bar{v}) = \left[ 1 - F(\bar{v}) \right] \frac{1}{\partial} \frac{\partial u_d(B(\bar{v}), w)}{\partial B(\bar{v})} B'(\bar{v})(n + 1)j(w)[1 - J(w)]^n dw$$

$$-f(\bar{v}) \int_0^1 u_d(B(\bar{v}), w)(n + 1)j(w)[1 - J(w)]^n dw$$

$$+ f(\bar{v}) \int_0^\bar{v} u_d(x, 1)nF(x)^{n-1}f(x)dx$$

$$+ f(\bar{v}) \int_0^\bar{v} \left[ \int_0^x u_d(\bar{v}, 1)(n - 1)F(z)^{n-2}f(z)dz \right] n_f(x)dx$$

$$+ f(\bar{v}) \int_0^\bar{v} \left[ \int_0^x u_d(z, 1)(n - 1)F(z)^{n-2}f(z)dz \right] n_f(x)dx.$$
\[
\begin{align*}
&\int_0^1 u_d(B(1, 1)(n + 1)j(w)[1 - J(w)]^n dw - \int_0^1 u_d(x, 1)nF(x)^{n-1}f(x)dx. \\
&\text{Since } B(1) = \int_0^1 xnf(x)^{n-1}f(x)dx, \text{ for a seller with } u_d(x, t) = x\delta(t) \text{ we have}
\end{align*}
\]

\[
\begin{align*}
&\int_0^1 u_d(B(1, 1)(n + 1)j(w)[1 - J(w)]^n dw - \int_0^1 u_d(x, 1)nF(x)^{n-1}f(x)dx \\
&= u_d \left( \int_0^1 xnf(x)^{n-1}f(x)dx, 1 \right) - \int_0^1 u_d(x, 1)nF(x)^{n-1}f(x)dx \\
&= \int_0^1 xnf(x)^{n-1}f(x)dx - \int_0^1 x\delta(1)nF(x)^{n-1}f(x)dx = 0.
\end{align*}
\]

As a result,

\[
\int_0^1 u_d(B(1, w)(n + 1)j(w)[1 - J(w)]^n dw - \int_0^1 u_d(x, 1)nF(x)^{n-1}f(x)dx > 0,
\]

implying \(U'_s, \bar{v} < 0\). Thus, such a seller will choose \(\bar{v}^* < 1\). \(Q.E.D.\)

**Proof of Theorem 2.** Begin by noting that \(\pi_{iG}(\bar{v}, \bar{v}) = 0\). Further,

\[
\frac{\partial \pi_{iG}(v_i, \bar{v})}{\partial v_i} = \frac{1}{n + 1} + \frac{n}{n + 1}F(v_i)^{n-1}F(\bar{v}) - F(v_i)^n,
\]

which is strictly positive at \(v_i = \bar{v}\) for any \(\bar{v} < 1\). From here,

\[
\frac{\partial^2 \pi_{iG}(v_i, \bar{v})}{\partial v_i^2} = nf(v_i)F(v_i)^{n-2}\left\{\frac{n - 1}{n + 1}F(\bar{v}) - F(v_i)\right\},
\]

which is clearly negative for all \(v_i \in [\bar{v}, 1]\).

It follows that \(\pi_{iG}(v_i, \bar{v}) \geq 0\) for all \(v_i \in [\bar{v}, 1]\) if and only if \(\pi_{iG}(1, \bar{v}) \geq 0\). If instead \(\pi_{iG}(1, \bar{v}) < 0\), then there exists a unique \(v_C \in (\bar{v}, 1)\) such that: \(\pi_{iG}(v_i, \bar{v}) > 0\) for \(v_i \in (\bar{v}, v_C)\); \(\pi_{iG}(v_C, \bar{v}) = 0\); and \(\pi_{iG}(v_i, \bar{v}) < 0\) for \(v_i \in (v_C, 1]\). \(Q.E.D.\)
Proof of Corollary 1. Let
\[
\psi(\bar{v}) = \pi_{iG}(1, \bar{v})
\]
\[
= n \int_{\bar{v}} \frac{1}{n+1} xF(x)^{n-1}f(x)dx - \frac{1}{n+1} \bar{v}(1 - F(\bar{v})^n)
- \left( \frac{n}{n+1} \right) \left( 1 - F(\bar{v}) + \int_{\bar{v}} x(n-1)F(x)^{n-2}f(x)F(\bar{v})dx \right).
\]
Begin by noting that $\psi(1) = 0$. Further,
\[
\psi'(\bar{v}) = \frac{1}{n+1} nf(\bar{v}) \int_{\bar{v}} F(x)^{n-1}dx - \frac{1}{n+1}(1 - F(\bar{v})^n).
\]
If $f'(v) \geq 0$ for all $v \in [0,1]$, then
\[
\int_{\bar{v}} nF(x)^{n-1}f(\bar{v})dx \leq \int_{\bar{v}} nF(x)^{n-1}f(x)dx = 1 - F(\bar{v})^n.
\]
From here it follows that $\psi'(\bar{v}) \leq 0$ for all $\bar{v} \in [0,1]$, implying $\psi(\bar{v}) = \pi_{iG}(1, \bar{v}) \geq 0$ for all $\bar{v} \in [0,1]$. Q.E.D.

Proof of Corollary 2. In the proof of Corollary 1 we defined
\[
\psi(\bar{v}) = \pi_{iG}(1, \bar{v})
\]
\[
= n \int_{\bar{v}} xF(x)^{n-1}f(x)dx - \frac{1}{n+1} \bar{v}(1 - F(\bar{v})^n)
- \left( \frac{n}{n+1} \right) \left( 1 - F(\bar{v}) + \int_{\bar{v}} x(n-1)F(x)^{n-2}f(x)F(\bar{v})dx \right)
\]
and noted that $\psi(1) = 0$. In order to obtain the desired result, it must simply be shown that if $f'(v) < 0$ for all $v \in [0,1]$, then $\psi'(\bar{v}) > 0$ for all $\bar{v} \in [0,1]$. Recall that
\[
\psi'(\bar{v}) = \frac{1}{n+1} nf(\bar{v}) \int_{\bar{v}} F(x)^{n-1}dx - \frac{1}{n+1}(1 - F(\bar{v})^n).
\]
If \( f'(v) < 0 \) for all \( v \in [0, 1] \), then
\[
\int_{\bar{v}}^{1} nF(x)^{n-1} f(\bar{v}) \, dx > \int_{\bar{v}}^{1} nF(x)^{n-1} f(x) \, dx = 1 - F(\bar{v})^n.
\]
As a result, \( \psi'(\bar{v}) > 0 \) for all \( \bar{v} \in [0, 1) \), implying \( \psi(\bar{v}) = \pi_{iG}(1, \bar{v}) < 0 \) for all \( \bar{v} \in [0, 1) \). \( Q.E.D. \)

References


