Wave Equation and Introduction to Classification of PDEs

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Seminar in Engineering Analysis
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Overview

- Review last class
  - Wave equation solutions by separation of variables and D’Alembert approach
- Wave equation solution with boundaries
- Characteristics and classification of partial differential equations
  - General analysis
  - Parabolic equations
  - Elliptic equations
  - Hyperbolic equations

Course Items

- Notes on wave equation on web site
- Midterm – Wednesday, March 11
  - Covers material on diffusion and Laplace equations
  - Includes material up to and including lecture and homework for March 2
  - Open textbook and notes, including homework solutions
  - Use existing solutions to answer questions

Review Gradients

- Gradients of Laplace equation solutions often proportional to flux terms
  - Heat flux and temperature gradient
  - Diffusion flux and mass fraction gradient
  - Velocity and velocity potential in ideal flow
  - In constant potential plot, lines perpendicular to the potential are flux lines

\[
\text{grad } f = \nabla f = \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial z}
\]

\[
\text{V} = \frac{\partial \phi}{\partial x} + i \frac{\partial \phi}{\partial y} + k \frac{\partial \phi}{\partial z}
\]

Review Interpretation of \( \nabla^2 u = 0 \)

- When \( \textbf{v} = -k \text{ grad } u \) is a flux that is the gradient of a scalar, Laplace’s equation for \( u \) says that the net inflow of \( \textbf{v} \) is zero

\[
\iiint_{\text{Enclosed Volume}} \nabla^2 u dV = -\frac{1}{k} \iint_{\text{Surface}} \textbf{v} \cdot \text{n} dA = 0
\]

- Example of this result shown last week
- Result applies to any problem in any geometry with Laplace’s equation
Review Complex Variable

- Cauchy-Riemann conditions
  \[ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \]
  then \[ \frac{df}{dz} = \frac{du}{dz} + i\frac{dv}{dz} = \frac{du}{dx} - i\frac{dv}{dy} \]
- Equivalent to Laplace equation
  - Function \( u(x,y) \) that satisfies Laplace equation in two dimensions, has associated function \( v(x,y) \) that satisfies Laplace
  - Lines of \( u(x,y) \) and \( v(x,y) \) are perpendicular
  - Typically if \( u \) is a potential (e.g., temperature, \( v \) is a corresponding flux)

Review Additional Results

- Cauchy theorem for complex integration shows Laplace equation solutions
  - Have maximum and minimum on boundary
  - If boundary is a constant at all points then solution is the same constant in region
  - Dirichlet problem has unique solution
  - Neumann problem does not
- Kreyszig section 18.6 has proofs

Review Wave Equation

- Wave phenomena: \( u(x,t) \) is wave amplitude varying with space, \( x \), and time, \( t \)
  \[ \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \]
- \( c \) is wave speed
- Can solve by usual separation of variables technique
- Also have D'Alambert solution with arbitrary functions \( F \) and \( G \) with coordinates \( \xi = x + ct \) and \( \eta = x - ct \)

Review Separation of Variables

- Usual assumption \( u(x,t) = X(x)T(t) \)
  \[ \frac{1}{c^2} \frac{d^2 T(t)}{dt^2} = \frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} = -\lambda^2 \]
  Result is function of \( t \) equal to function of \( x \)
  \[ u(x,t) = T(t)X(x) = \left[ A\sin(\lambda ct) + B\cos(\lambda ct) \right] \left[ C\sin(\lambda x) + D\cos(\lambda x) \right] \]
- Use above solution as starting point
  - Boundary conditions at \( x = 0 \) and \( x = L \)
  - Initial conditions on \( u \) and \( \frac{\partial u}{\partial t} \) at \( x = 0 \)

Review Separation of Variables

- Solution for \( u(x,t) \) with initial and boundary conditions
  - \( u(x,0) = f(x) \); \( \partial u/\partial x \big|_{x=0} = g(x) \)
  - \( u(0,t) = u(L,t) = 0 \)
  \[ c \] is wave speed
  \[ u(x,t) = \sum_{n=1}^{\infty} \left( A_n \sin \left( \frac{n\pi ct}{L} \right) + B_n \cos \left( \frac{n\pi ct}{L} \right) \right) \sin \left( \frac{n\pi x}{L} \right) \]
  \[ A_n = \frac{2}{L} \int_{0}^{L} f(x) \sin \left( \frac{n\pi x}{L} \right) dx \quad B_n = \frac{2}{L} \int_{0}^{L} g(x) \sin \left( \frac{n\pi x}{L} \right) dx \]

General Solution

- Substitute \( A_n \) and \( B_n \) from equations just found and substitute into previous solution
  \[ u(x,t) = \sum_{n=1}^{\infty} \left[ A_n \sin \left( \frac{n\pi ct}{L} \right) + B_n \cos \left( \frac{n\pi ct}{L} \right) \right] \sin \left( \frac{n\pi x}{L} \right) \]
  - Examine case where \( g(x) = 0 \) so \( A_n = 0 \)
  \[ u(x,t) = \sum_{n=1}^{\infty} B_n \cos \left( \frac{n\pi ct}{L} \right) \sin \left( \frac{n\pi x}{L} \right) \]
Wave equation and classification of PDEs

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General Solution for \( g(x) = 0 \)

\[
 u(x,t) = \sum_{n=1}^{\infty} B_n \cos \left( \frac{n \pi x}{L} \right) \sin \left( \frac{n \pi t}{L} \right)
\]

- From trig identities for \( \sin(x \pm y) \)
  - \( \sin(x + y) = \sin x \cos y + \sin y \cos x \)
  - \( \sin(x - y) = \sin x \cos y - \sin y \cos x \)
  - \( \sin(x + y) + \sin(x - y) = 2 \sin x \cos y \)

\[
 \sum_{n=1}^{\infty} \left[ \cos \left( \frac{n \pi x}{L} \right) + \sin \left( \frac{n \pi x}{L} \right) \right] = \frac{1}{x(\pi x)}
\]

Similar Solution for \( f(x) = 0 \)

- From trig identities for \( \cos(x \pm y) \)
  - \( \cos(x + y) = \cos x \cos y - \sin y \sin x \)
  - \( \cos(x - y) = \cos x \cos y + \sin y \sin x \)
  - \( \cos(x - y) - \cos(x + y) = 2 \sin x \sin y \)

\[
 u(x,t) = \frac{1}{2} \sum_{n=1}^{\infty} A_n \left[ \cos \left( \frac{n \pi (x - ct)}{L} \right) - \cos \left( \frac{n \pi (x + ct)}{L} \right) \right]
\]

D'Alambert Solution

- Wave phenomena: \( u(x,t) \) is wave amplitude varying with space, \( x \), and time, \( t \)
- \( c \) is wave speed

- D'Alambert solution, shown below, uses arbitrary functions \( F \) and \( G \) with coordinates \( \xi = x + ct \) and \( \eta = x - ct \)

\[
 u = F(\xi) + G(\eta) = F(x + ct) - G(x - ct)
\]

- Proof of solution based on transforming derivatives

Derive D'Alambert Solution

- Transform equation from \( (x,t) \) to \( (\xi,\eta) \)
  - \( \xi = x + ct \) and \( \eta = x - ct \)

\[
 \frac{\partial u}{\partial \xi} = \frac{\partial}{\partial \xi} [F(\xi) + G(\eta)] = F'(\xi) \\
 \frac{\partial u}{\partial \eta} = \frac{\partial}{\partial \eta} [F(\xi) + G(\eta)] = G'(\eta)
\]

- Apply transforms to \( \partial/\partial t \) and \( \partial/\partial x \)

\[
 \frac{\partial^2 u}{\partial \xi \partial \eta} = \frac{\partial}{\partial \xi} \left[ \frac{\partial}{\partial \eta} [F(\xi) + G(\eta)] \right] = \frac{\partial}{\partial \xi} [F'(\xi)] = c [F'(\xi) - G'(\eta)]
\]

Derive D'Alambert Solution II

- Second derivatives satisfy wave equation

\[
 \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left[ \frac{\partial}{\partial \eta} [F(\xi) + G(\eta)] \right] = \frac{\partial}{\partial x} [F'(\xi) - G'(\eta)]
\]

\[
 \frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta \partial \xi} = \frac{\partial}{\partial \xi} \left[ \frac{\partial}{\partial \eta} [F(\xi) + G(\eta)] \right] = \frac{\partial}{\partial \xi} [F'(\xi)] = c [F'(\xi) - G'(\eta)]
\]

- Proof of solution based on transforming derivatives
Solution with Initial Conditions

- Define \( u(x,0) = f(x) \) and \( \frac{\partial u}{\partial t}|_{t=0} = g(x) \)
- Solution, \( u(x,t) \) uses \( f(x \pm ct) \)

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(v) d\nu = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(v) d\nu = \frac{1}{2} \int_{-\infty}^{\infty} g(v) d\nu
\]

- Terms \( f(x + ct) \) and \( f(x - ct) \) satisfy wave equation since any function of these arguments satisfies the equation
- Solution gives \( u(0,x) = f(x) \) as required

Solution with Initial Conditions II

- Integral term satisfies wave equation
- Details of derivation in wave equation notes

\[
\frac{\partial^2}{\partial x^2} \int_{-\infty}^{\infty} g(v) d\nu = c^2 \left[ g'(x + ct) - g'(x - ct) \right]
\]

Solution with Initial Conditions III

- Verify initial condition \( \frac{\partial u}{\partial t}|_{t=0} = g(x) \)
- Proposed solution satisfies wave equation and initial conditions

Compare Solution Approaches

- Separation of variables solutions for \( g(x) = 0 \)

\[
u(x,t) = \sum_{n=1}^{\infty} B_n \sin \left( \frac{n\pi x}{L} \right) \sin \left( \frac{n\pi t}{L} \right)
\]

- Set \( t = 0 \) to get \( f(x) \) initial condition

\[
f(x) = \sum_{n=1}^{\infty} B_n \sin \left( \frac{n\pi x}{L} \right)
\]

- Compare to D’Alambert solution

Compare Solutions II

- \( f(x) \) from last chart gives \( f(x \pm ct) \) below

\[
 f(x + ct) = \sum_{n=1}^{\infty} B_n \sin \left( \frac{n\pi (x + ct)}{L} \right) \quad f(x - ct) = \sum_{n=1}^{\infty} B_n \sin \left( \frac{n\pi (x - ct)}{L} \right)
\]

- D’Alambert solution for \( g(x) = 0 \) is \( u(x,t) = \frac{1}{2} [f(x + ct) + f(x - ct)]/2 \)

\[
u(t,x) = \frac{1}{2} [f(x + ct) + f(x - ct)] + \sum_{n=1}^{\infty} B_n \sin \left( \frac{n\pi x - nc t}{L} \right)
\]

Compare Solutions III

- D’Alambert solution: \( [f(x + ct) + f(x - ct)]/2 \)

\[
u(x,t) = \frac{1}{2} \sum_{n=1}^{\infty} B_n \left[ \sin \left( \frac{n\pi x + nc t}{L} \right) + \sin \left( \frac{n\pi x - nc t}{L} \right) \right]
\]

- This matches separation-of-variables solution modified by trigonometric identities
Wave equation and classification of PDEs

Why D’Alambert

• We see that the solution obtained by separation of variables agrees with the D’Alambert solution for one case
• The D’Alambert solution is more general
• It also provides a basis for propagation of wave shapes without damping
  – Look at meaning of \( f(x + ct) \) and \( f(x - ct) \)
    • If \( f(x) = a \) when \( x = b \) at \( t = 0 \) then at any point where \( x \pm ct = b \), \( f(x \pm ct) = a \)

Meaning of D’Alambert Solution

• Consider case with \( g(x) = 0 \)
• \( u(x,t) = \frac{f(x + ct) + f(x - ct)}{2} \)
• Initial condition is propagated into different spatial regions over time without change in shape
• Boundaries can affect solution
• Examine infinite region with simple \( f(x) \)
  – Triangular: \( f(x) = 1 + x \) \((-1 \leq x < 0); f(x) = 1 - x \) \((0 \leq x \leq 1)\) and \( f(x) = 0 \) otherwise

Meaning of Solution II

• Here is definition of \( f(z) \) for triangular wave from last chart \((z = x \pm ct)\)

<table>
<thead>
<tr>
<th>( z )</th>
<th>( z &lt; -1 )</th>
<th>(-1 \leq z &lt; 0 )</th>
<th>( 0 \leq z &lt; 1 )</th>
<th>( z \geq 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(z) )</td>
<td>0</td>
<td>1 + ( z )</td>
<td>1 - ( z )</td>
<td>0</td>
</tr>
</tbody>
</table>

• For any value of \( ct \) and \( x \), we can find \( z = x \pm ct \) and get the correct value of \( f(z) \) from this initial condition chart

Initial Wave Propagates

• For solution with \( g(x) = 0 \), \( u(x,t) = \frac{f(x + ct) + f(x - ct)}{2} \)
• Given \( x \) and \( ct \) we can compute \( f(x + ct) \) and \( f(x - ct) \) from table on previous chart
• Adding them together and dividing by 2 gives the solution for any \( u(x,t) \)
  – This is specific application of general idea to triangular initial condition

Triangular Initial Conditions

• Region \( 0 \leq x \leq L = 10 \) with \( g(x) = 0 \)
• Triangular initial condition at center
  – \( f(x) = 0 \) for \( x \leq 4 \) and \( x \geq 6 \)
  – \( f(x) = x - 4 \) for \( 4 \leq x \leq 5 \)
  – \( f(x) = 6 - x \) for \( 5 \leq x \leq 6 \)

\[
B_n = \frac{2}{L} \int_{0}^{L} f(x) \sin \left( \frac{m \pi x}{L} \right) dx = \frac{2}{10} \int_{0}^{5} (0) \sin \left( \frac{m \pi x}{10} \right) dx + \frac{5}{10} \int_{5}^{10} (6 - x) \sin \left( \frac{m \pi x}{10} \right) dx
\]

Wave propagation
Triangular Initial Conditions II

\[
B_n = \frac{2}{10} \left[ \int_4^5 (x-4) \sin \left( \frac{m \pi x}{10} \right) dx + \int_5^6 (6-x) \sin \left( \frac{m \pi x}{10} \right) dx \right]
\]

- Details of integration follow last chart of lecture presented in class

\[
B_n = \frac{20}{m^2 \pi^2} \left[ 2 \sin \left( \frac{m \pi}{2} \right) - \sin \left( \frac{2m \pi}{5} \right) - \sin \left( \frac{3m \pi}{5} \right) \right]
\]

- Can also use MATLAB to get \( B_n \) as shown on next chart

What Happens at Boundaries

- Still have basic solution

\[
u(x,t) = \frac{1}{2} \sum_{n=1}^{\infty} B_n \left[ \sin \left( \frac{n \pi (x+ct)}{L} \right) + \sin \left( \frac{n \pi (x-ct)}{L} \right) \right]
\]

- What if \( x \pm ct \) is outside of range \( 0 \leq x \leq L \)?
- Solution in \( 0 \leq x \leq L \) will have components from periodic repetition of sine function just as in Fourier series

Solution is Fourier Series

- Wave equation solution is Fourier sine series which is periodic, odd function

\[
u(x,0) = f(x) = \sum_{n=1}^{\infty} B_n \sin \left( \frac{n \pi x}{L} \right)
\]

- For larger values of \( x, ct \), periodic extensions move into \( 0 \leq x \leq L = 1 \)

B\(_m\) from MATLAB

- EDU>> syms m x I
- EDU>> \((\int((x - 4) \sin(m \pi x/10),4,5) + \int((6-x) \sin(m \pi x/10),5,6))/5\)
- EDU>> I = simplify(ans)
- EDU>> pretty(I)

\[2 \sin \left( \frac{1}{2} m \pi \right) - \sin \left( \frac{2}{5} m \pi \right) - \sin \left( \frac{3}{5} m \pi \right)\]

Solution at Boundaries \( g(x) = 0 \)

- Sine solution defined for limited region, but sine and cosine have periodic repetition for all values of their arguments

\[
u(x,t) = \frac{f(x+ct) + f(x-ct)}{2} = \frac{1}{2} \sum_{n=1}^{\infty} B_n \left[ \sin \left( \frac{n \pi (x+ct)}{L} \right) + \sin \left( \frac{n \pi (x-ct)}{L} \right) \right]
\]

- \( f(x) = 0 \) for \( 0 \leq x \leq 0.4 \) and \( 0.6 \leq x \leq 1 \)
- \( f(x) = 10x - 4 \) for \( 0.4 \leq x \leq 0.5 \)
- \( f(x) = 6 - 10x \) for \( 0.5 \leq x \leq 0.6 \)

Time Evolution

- Look at evolution when \( ct = 0.4 \)

\[
u(x,t) = \frac{f(x+ct) + f(x-ct)}{2}
\]

- For larger values of \( x, ct \), periodic extensions move into \( 0 \leq x \leq L = 1 \)
Wave equation and classification of PDEs

Wave Equation Summary

- View spreadsheet showing wave travel for initial profiles
- Have separation-of-variables solution and D’Alambert solution

\[ u(x,t) = \frac{1}{2} \left[ f(x + ct) + f(x - ct) \right] + \frac{1}{2c} \int_{-ct}^{ct} g(v) dv \]
- D’Alambert solution shows how wave solution in x and t is composed of the initial profiles as traveling waves

Overview of Characteristics

- Lines along which solution with discontinuities can propagate
- Main applications are for wave equation in which characteristics are real
- Has implications for understanding solutions and for numerical analysis
- Determine slope of characteristics by finding directions in which solution for equation is not unique

Classification of PDEs

- The general second-order PDE in two variables is classified as follows
  - If \( B^2 - 4AC < 0 \) the PDE is called elliptic and has no real characteristic directions
  - If \( B^2 - 4AC = 0 \) the PDE is parabolic and has one repeated characteristic direction
  - If \( B^2 - 4AC > 0 \) the PDE is hyperbolic and has two real characteristic directions

Important Equations

- Laplace/Poisson/Helmholtz equations (\( B^2 - 4AC < 0 \)) are elliptic (no real characteristics)
- Diffusion equation (\( B^2 - 4AC = 0 \)) is parabolic (one characteristic)
- Wave equation (\( B^2 - 4AC > 0 \)) is hyperbolic (two real characteristics)
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Wave Equation Characteristics
• Compute characteristic directions

\[
\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0
\]

Wave equation: \( A = 1; \ B = 0; \ C = -c^2; \ B^2 - 4AC = c^2 > 0 \)

\[
A \frac{\partial^2 u}{\partial t^2} + B \frac{\partial^2 u}{\partial x^2} + C \frac{\partial^2 u}{\partial x \partial t} + D \left( t, x, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t} \right) = 0
\]

\[
\frac{dx}{dt} = -B \pm \sqrt{B^2 - 4AC} \quad \frac{dt}{2A} = \frac{-0 \pm \sqrt{0^2 - 4(1)(-c^2)}}{2(1)} = \pm c
\]

Wave Equation Characteristics II
• Compute characteristic directions with order of variables reversed

\[
\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0
\]

Wave equation: \( A = c^2; \ B = 0; \ C = -1; \ B^2 - 4AC = c^2 > 0 \)

\[
A \frac{\partial^2 u}{\partial t^2} + B \frac{\partial^2 u}{\partial x^2} + C \frac{\partial^2 u}{\partial x \partial t} + D \left( t, x, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t} \right) = 0
\]

\[
\frac{dx}{dt} = -B \pm \sqrt{B^2 - 4AC} \quad \frac{dt}{2A} = \frac{-0 \pm \sqrt{0^2 - 4(1)(-c^2)}}{2(-c^2)} = \frac{1}{c}
\]

Wave Propagation
• Characteristic slope \( \frac{dt}{dx} = -1/c \)

• Characteristic slope \( \frac{dx}{dt} = 1/c \)

Behavior of Equation Types
• Domain of dependence for \( u(x_1, y_1) \)
  – The area (in \( x-y \) space) whose \( u \) values affect the value of \( u(x_1, y_1) \)

• Region of influence of \( u(x_1, y_1) \)
  – The area (in \( x-y \) space) whose \( u \) values are affected by the value of \( u(x_1, y_1) \)

• Importance for specifying boundary conditions and for numerical solutions

Hyperbolic Equations
• Domain of dependence shown on previous chart
• Region of influence is region of characteristics leaving \( x_1, y_1 \)
• Conditions outside domain of dependence should not affect solution
• Important point for numerical algorithms which can violate this principle for inappropriate choices of step sizes

Elliptic PDEs
• Imaginary characteristics for elliptic equations like Laplace and Poisson’s
• Entire solution region is both domain of dependence and region of influence
• This means that any change in any boundary condition can affect the solution at any point in the region
  – Effects may be small far from boundary, but will be present
Parabolic PDEs

- Parabolic equations typically involve time and space as coordinates
- Consider region \(0 \leq x \leq L\) and \(t > 0\)
  - Domain of dependence at any point \(x_1, t_1\) is entire domain at previous times: \(0 \leq x \leq L\) and \(0 \leq t < t_1\)
  - Any change in initial conditions or boundary conditions for \(t < t_1\) will change solution here
- Region of influence at \(x_1, t_1\) is entire region for future times \(0 \leq x \leq L\) and \(t > t_1\)

Example Question

- You are solving the diffusion equation in the region \(0 \leq x \leq L\) and \(t > 0\) with an initial condition \(u(x,0) = f(x)\) and the following boundary conditions
  - \(t < 12\) s: \(u(0,t) = u(L,t) = 0\)
  - \(t \geq 12\) s: \(u(0,t) = u(L,t) = a = 1\)
- If you have a solution to this problem for \(a = 1\), how does the solution change for \(t < 12\) s, if you set \(a = 2\)?
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Conclusions

• Wave equation can be solved by conventional separation of variables
• D’Alambert solution is special approach for wave equation
  – Consistent with separation of variables solution, but more general
  – Shows propagation in terms like \( f(x + ct) \) and \( f(x - ct) \)

Notes

• The remaining charts show the integration of the expansions coefficients, \( B_m \), for the triangular initial condition
• The first midterm is scheduled for Thursday, March 4 will cover material up to and including the March 2 homework

Get Integral Terms

• Constant terms in \( B_m \) integral

\[
\frac{2}{10} \left( \int \left( -4 \sin \left( \frac{m \pi x}{10} \right) + \frac{6}{5} \sin \left( \frac{m \pi x}{10} \right) \right) dx \right) =
\]

\[
\frac{2}{10} \left( 4 \cos \left( \frac{m \pi x}{10} \right) - \frac{6}{5} \cos \left( \frac{m \pi x}{10} \right) \right) =
\]

\[
\frac{2}{m \pi} \left( \frac{m \pi 5}{10} - 4 \cos \left( \frac{m \pi 4}{10} \right) - 6 \cos \left( \frac{m \pi 6}{10} \right) + 6 \cos \left( \frac{m \pi 5}{10} \right) \right) =
\]

\[
\frac{20}{m \pi} \left( \frac{m \pi 2}{2} \right) \cos \left( \frac{2 \pi 5}{5} \right) \frac{12}{m \pi} \cos \left( \frac{3 \pi 5}{5} \right)
\]

Use \( \int x \sin ax dx = \frac{1}{a^2} \left( \sin ax - ax \cos ax \right) \)

• \( x \) terms in \( B_m \) integral

\[
I_1 = \frac{2}{10} \left( \int \left( \sin \left( \frac{m \pi x}{10} \right) + \frac{10}{\pi} \sin \left( \frac{m \pi x}{10} \right) \right) \right) =
\]

\[
\frac{2}{10} \left( \sin \left( \frac{m \pi 5}{10} \right) - \sin \left( \frac{m \pi 4}{10} \right) - \sin \left( \frac{m \pi 5}{10} \right) + \sin \left( \frac{m \pi 4}{10} \right) \right) =
\]

\[
\frac{2}{10} \left( \sin \left( \frac{m \pi 6}{10} \right) - \sin \left( \frac{m \pi 5}{10} \right) - \sin \left( \frac{m \pi 6}{10} \right) + \sin \left( \frac{m \pi 5}{10} \right) \right) =
\]

\[
\frac{20}{m \pi} \left( \frac{m \pi 2}{2} \right) \cos \left( \frac{2 \pi 5}{5} \right) \frac{12}{m \pi} \cos \left( \frac{3 \pi 5}{5} \right)
\]

Result for \( B_m \)

\[
B_m = \frac{2}{m \pi} \left( \int \left( 0 \sin \left( \frac{m \pi x}{10} \right) + \frac{10}{\pi} \sin \left( \frac{m \pi x}{10} \right) \right) \right) =
\]

\[
\frac{20}{m \pi} \left( \frac{m \pi 2}{2} \right) \cos \left( \frac{2 \pi 5}{5} \right) \frac{12}{m \pi} \cos \left( \frac{3 \pi 5}{5} \right) + \frac{20}{m \pi} \left( \frac{m \pi 2}{2} \right) \cos \left( \frac{2 \pi 5}{5} \right) \frac{12}{m \pi} \cos \left( \frac{3 \pi 5}{5} \right)
\]

These terms all cancel (note factor of \( 20/m^2\pi^2 \) multiplying last row)