Take your time to read the entire paper before you begin to write, and read each question carefully. Remember that certain questions are worth more points than others. Make a note of the questions that you feel confident you can do, and then do those first: you do not have to proceed through the paper in the order given.

• You have 75 minutes to complete this exam.

• This is a closed book exam, and no notes of any kind are allowed. The use of cell phones, pagers or any text storage or communication device is not permitted.

• Only the Faculty approved TI-30 calculator is allowed.

• The correct answer requires justification written legibly and logically: you must convince me that you know why your solution is correct. Answer these questions in the space provided. Use the backs of pages if necessary.

• Where it is possible to check your work, do so.

• Good Luck!

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Question 1. [18 points] Determine if the following statements are TRUE or FALSE. Justify each answer.

(a) [3 points] The equation \( y'''(1 + y) = 2t \) is a third order linear ODE.

Solution: False. The equation is not linear. It cannot be written in the form \( a_3(t)y''' + a_2(t)y'' + a_1(t)y' + a_0(t)y = g(t) \).

(b) [3 points] The function \( y(x) = e^{-2x} \) is a solution for \( x^2 y'' - 2xy' + 5y = 0 \).

Solution: False. 
\[
x^2 y'' - 2xy' + 5y = x^2(4e^{-2x}) - 2x(-2e^{-2x}) + 5e^{-2x} = 4xe^{-2x} + 4e^{-2x} + 5e^{-2x} = (4x^2 + 4x + 5)e^{-2x}
\]
which is always nonzero.

(c) [3 points] The equation \( xy' = (x^2 + 3)(y^2 + 1) \) is a separable ODE.

Solution: True. \( xy' = (x^2 + 3)(y^2 + 1) \) \( \Rightarrow x \frac{dy}{dx} = (x^2 + 3)(y^2 + 1) \). We can separate variables to have 
\[
\frac{dy}{y^2+1} = \frac{x^2+3}{x} \, dx.
\]

(d) [3 points] \( \frac{dy}{dx} + p(x)y = q(x) \) cannot be solved if \( p(x) \) or \( q(x) \) are discontinuous.

Solution: False. Since there exist many examples where solutions exist.

(e) [3 points] The isoclines of an autonomous ODE, \( \frac{dy}{dx} = f(y) \), are always horizontal.

Solution: True. Since autonomous ODE \( \frac{dy}{dx} = f(y) \). The isoclines of it: \( f(y) = C \) does not depend on \( x \). The slope does not depend on \( x \) in an autonomous ODE.

(f) [3 points] The linear differential equation, \( y' + k_1y = k_2 \), where \( k_1 \) and \( k_2 \) are nonzero constant, always possesses a constant solution.

Solution: True. Rewrite the equation of \( y' + k_1y = k_2 \) to \( y' = k_2 - k_1y \). By setting \( y' = 0 \), i.e., 
\[
k_2 - k_1y = 0 \quad \Rightarrow \quad y = \frac{k_2}{k_1}
\]
since \( (k_1 & k_2) \) are nonzero constant). Then \( y = \frac{k_2}{k_1} \) is the constant solution.
**Question 2.** [10 points] Consider the following differential equation

$$\frac{dy}{dx} = \sqrt{xy}$$

determine a region of the xy-plane for which the given differential equation would have a unique solution whose graph passes through a point \((x_0, y_0)\) in the region.

**Solution:** Let \(f(x, y) = \sqrt{xy}\), then

- \(f(x, y) = \sqrt{xy}\) is continuous when \(x \geq 0, y \geq 0\) or \(x \leq 0, y \leq 0\);
- \(\frac{\partial f}{\partial y} = \frac{1}{2}(xy)^{\frac{1}{2}-1} \cdot x = \frac{1}{2} \frac{x}{\sqrt{xy}}\)

Therefore, by the Fundamental Theorem of Existence & Uniqueness, in the region of \(R = \{(x, y)| x > 0, y > 0, \text{ or } x < 0, y < 0\}\) would have a unique solution whose graph passes through a point \((x_0, y_0)\) in the region.
Question 3. [30 points] Consider the first order autonomous differential equation

\[ \frac{dy}{dt} = y^2(9-y^2) \]

Answer the following questions without solving the equation.

(a) [6 points] Denote \( f(y) = y^2(9-y^2) \), sketch the graph of \( f(y) \) versus \( y \);

**Solution:** The graph of \( f(y) \) versus \( y \) is given as follows.

![Graph of f(y) versus y](image1.png)

Figure 1: The graph of \( f(y) \) versus \( y \)

(b) [4 points] Determine the equilibrium solutions of the given differential equation;

**Solution:** Solving \( f(y) = 0 \), i.e., \( y^2(9-y^2) = 0 \), we get the equilibrium solutions: \( y = 0, 3, -3 \).

(c) [7 points] Draw the phase line for the given differential equation;

**Solution:** The phase line diagram is

![Phase line diagram](image2.png)

Figure 2: The phase line diagram
(d) [3 points] Classify the stability of each equilibrium found in Part (b);

**Solution:**

- $y = 3$ is asymptotically stable;
- $y = 0$ is semi-stable;
- $y = -3$ is unstable.

(e) [10 points] Sketch several graphs of solutions in the $ty$-plane, making sure you have at least one graph representing each type;

**Solution:**

![Figure 3: Typical solutions](image)

Note that the concavity (inflection points) of the solution curves can be determined by checking the sign of $\frac{\partial^2 y}{\partial t^2} = f(y) \cdot f'(y)$. Therefore solving $f'(y) = 0$, i.e., $2y(9 - y^2) + y^2(-2y) = 0$, we get the inflection points for the solution curves are $y = \frac{3}{2} \sqrt{2}$ and $y = -\frac{3}{2} \sqrt{2}$.

(f) [2 points] If $y(10) = -3$, what is $y(0)$?

**Solution:** $y(0) = -3$.

(g) [2 points] If $y(1) = -\pi$, what is $\lim_{t \to \infty} y(t)$?

**Solution:** $\lim_{t \to \infty} y(t) = -\infty$.

(h) [2 points] If $y(-\pi) = 1$, what is $\lim_{t \to \infty} y(t)$?

**Solution:** $\lim_{t \to \infty} y(t) = 3$. 
Question 4. [12 points] Solve the initial value problem

\[ x \frac{dy}{dx} + (x + 2)y = \cos(x)e^{-x}, \quad \text{with} \quad y(\pi) = 0. \]

On what interval around \( x = \pi \) is the solution defined?

Solution: Divide the above equation by \( x \), we have

\[ \frac{dy}{dx} + \frac{x + 2}{x}y = \frac{\cos(x)e^{-x}}{x}. \]

This is a 1st-order linear equation with \( p(x) = 1 + \frac{2}{x} \) and \( q(x) = \frac{\cos(x)e^{-x}}{x} \). The integrating factor is

\[ \mu(x) = e^{\int p(x)dx} = e^{(1+\frac{2}{x})dx} = e^{x+2\ln x} = x^2 e^x. \]

Multiplying through by this function, we obtain

\[ \frac{d}{dx}(x^2 e^x y) = \frac{\cos(x)e^{-x}}{x} \cdot x^2 e^x = x \cos(x). \]

To integrate the right side, we need to use integration by parts,

\[ \int x \cos x \, dx = \int x d(\sin x) = x \sin x - \int \sin x \, dx = x \sin x - (-\cos x) + c. \]

Therefore, \( x^2 e^x y = -x \cos x + \sin x + c \) \Rightarrow \( y = e^{-x}(\frac{\sin x}{x^2} - \frac{\cos x}{x} + \frac{c}{x^2}). \)

This gives the general solution of the DE. To determine the constant \( c \), we impose the initial condition \( y(\pi) = 0 \) \Rightarrow \( 0 = e^{-\pi}(\frac{\sin \pi}{\pi^2} - \frac{\cos \pi}{\pi} + \frac{c}{\pi^2}) \Rightarrow 0 = e^{-\pi}(0 + \frac{1}{\pi} + \frac{c}{\pi^2}) \Rightarrow c = -\pi. \)

Thus, the solution of this IVP is \( y(x) = e^{-x}(\frac{\sin x}{x^2} - \frac{\cos x}{x} - \frac{\pi}{x^2}). \) Note that the solution grows unbounded as \( x \) approaches 0. So, the interval around \( x = \pi \) over which the solution is well-defined (or valid) is \( x \in (0, \infty) \).
**Question 5.**  [10 points] Solve the following differential equation by separation of variables

\[ e^x y \frac{dy}{dx} = e^{-y} + e^{-2x-y} \]

**Solution:**

\[
e^x y \frac{dy}{dx} = e^{-y}(1 + e^{-2x})
\]

\[ ye^{y} dy = \frac{1+e^{-2x}}{e^x} dx \]

\[ = (e^{-x} + e^{-3x}) dx \]

\[ \int ye^{y} dy = \int (e^{-x} + e^{-3x}) dx \]

Then

\[ \int ye^{y} dy = ye^{y} - \int e^{y} dy = ye^{y} - e^{y} \]

and

\[ \int (e^{-x} + e^{-3x}) dx = -e^{-x} - \frac{1}{3} e^{-3x} + c \]

\[ \Rightarrow \]

\[ ye^{y} - e^{y} = -e^{-x} - \frac{1}{3} e^{-3x} + c \]

i.e.,

\[ (y - 1)e^{y} = -e^{-x} - \frac{1}{3} e^{-3x} + c \]
Question 6. [16 points] Consider the following differential equation
\[ \frac{y}{x} + 2e^x + (\ln x - 2) \frac{dy}{dx} = 0 \]

(a) [6 points] Verify the above equation is an exact equation;
Solution:
\[ \frac{y}{x} + 2e^x + (\ln x - 2) \frac{dy}{dx} = 0 \implies \left( \frac{y}{x} + 2e^x \right) dx + (\ln x - 2) \, dy = 0 \]
\[ \implies M(x, y) = \frac{y}{x} + 2e^x, \quad N(x, y) = \ln x - 2 \]
\[ \implies \frac{\partial M}{\partial y} = \frac{1}{x}, \quad \frac{\partial N}{\partial x} = \frac{1}{x} \]
i.e.,
\[ \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \]
So this DE is exact.

(b) [10 points] Find the solution for \( x > 0 \) with the initial condition: \( y(1) = e \), and determine its interval of definition.
Solution: Since the equation is exact, there exists a function \( f(x, y) \) such that \( \frac{\partial f}{\partial x} = \frac{y}{x} + 2e^x \) and \( \frac{\partial f}{\partial y} = \ln x - 2 \). Using the first of these equations, we have \( \int \frac{\partial f}{\partial x} \, dx = \int \left( \frac{y}{x} + 2e^x \right) \, dx \). \( f(x, y) = y \ln x + 2e^x + g(y) \). Now, \( \frac{\partial f}{\partial y} = \ln x + g'(y) = \ln x - 2 \implies g'(y) = -2 \implies g(y) = -2y \). Therefore, \( f(x, y) = y \ln x + 2e^x - 2y = c \) is the general solution of this DE. Applying the I.C., \( y(1) = e \), we have \( e \ln 1 + 2e^1 - 2e = c \implies c = 0 \). We have the solution of the IVP is \( y \ln x + 2e^x - 2y = 0 \implies y = \frac{2e^x}{\ln x} \), where \( 2 - \ln x \neq 0 \implies x \neq e^2 \). The interval of definition is \((0, e^2)\).
Question 7. [14 points] Given the following differential equation

\[(x + ye^{y/x})dx - xe^{y/x}dy = 0,\]

(a) [4 points] Verify the above equation is homogeneous;
Solution: \(M(x, y) = (x + ye^{y/x}), N(x, y) = -xe^{y/x}.\) So, \(N(tx, ty) = -txe^{ty/x} = tN(x, y)\) and \(M(tx, ty) = (tx + yte^{ty/x}) = t(x + ye^{y/x}) = tM(x, y).\) Therefore, it is homogeneous.

(b) [10 points] Solve the above equation with the initial condition: \(y(1) = 0.\)
Solution: Let \(y = ux, i.e., u = \frac{y}{x}.\) \(dy = udx + xdu.\) So \((x + ye^{y/x})dx - xe^{y/x}dy = 0 \Rightarrow (x + uxe^u)dx - xe^u(udx + xdu) = 0 \Rightarrow xdx + uxe^u dx - xue^u dx - x^2e^u du = 0.\) Therefore, \(xdx = x^2e^u du \Rightarrow e^udu = \frac{1}{x}dx.\) So, \(\int e^udu = \int \frac{1}{x}dx\) and therefore \(e^u = \ln x + c.\) So \(e^{y/x} = \ln x + c \Rightarrow y(1) = 0 \Rightarrow e^{0} = \ln 1 + c \Rightarrow c = 1.\) So, \(e^{y/x} = \ln x + 1.\)
Question 8.  [16 points] Solve the following differential equations by using an appropriate substitution:

(a) [10 points] \[ t \frac{dy}{dt} - (1 + t)y = ty^2 \]

Solution: \[ \begin{align*} \frac{dy}{dt} - \frac{1 + t}{t}y &= y^2 \Rightarrow \frac{dy}{dt} - (1 + \frac{1}{t})y = y^2 \Rightarrow \frac{dy}{dt} &= (1 + \frac{1}{t})y + y^2. \end{align*} \]

\[ n = 2, \ u = y^{1-2} = y^{-1}. \]

\[ \frac{du}{dx} = -\frac{1}{t} \]

\[ \Rightarrow \frac{du}{dx} = -(1 + \frac{1}{t})y^{-1} - 1 \]

\[ = -(1 + \frac{1}{t})u - 1 \]

so that \[ \frac{du}{dt} + (1 + \frac{1}{t})u = -1. \] So \( p(t) = 1 + \frac{1}{t} \) and \( \mu(t) = e^{\int 1 + \frac{1}{t} \, dt} = e^{\frac{t}{t} \ln t} = te^t. \)

So \[ \frac{d}{dt}(te^t u) = -te^t. \] Also, \( te^t u = -\int te^t \, dt = -te^t + e^t + c. \) \( u = -1 + \frac{1}{t} + \frac{c}{t} e^{-t} \) and \( y^{-1} = -1 + \frac{1}{t} + \frac{c}{t} e^{-t}. \)

(b) [6 points] \[ \frac{dy}{dx} = \frac{1-x-y}{x+y} \]

Solution: \( u = x + y, \ u \frac{du}{dx} = 1 + \frac{dy}{dx} \Rightarrow \frac{du}{dx} = \frac{1}{u} \Rightarrow u - 1 \Rightarrow \frac{du}{dx} = \frac{1}{u}. \) So, \( u \frac{du}{dx} = \frac{1}{2} u^2 = x + c. \) So, \( u^2 = 2x + c \) and \( (x + y)^2 = 2x + c. \)
Question 9. [BONUS: 12 points] Find a function whose square plus the square of its derivative is 1.

Solution: Let \( y(x) \) be the function we are looking for. Therefore, \( y(x) \) should be the solution of the following equation:

\[
y^2 + \left( \frac{dy}{dx} \right)^2 = 1
\]

then

\[
\Rightarrow \left( \frac{dy}{dx} \right)^2 = 1 - y^2 \quad \Rightarrow \quad \frac{dy}{dx} = \pm \sqrt{1 - y^2}
\]

(A) \[\frac{dy}{dx} = \sqrt{1 - y^2} \quad \Rightarrow \quad \frac{1}{\sqrt{1 - y^2}} \quad dy = dx \quad \Rightarrow \quad \sin^{-1} y = x + c \quad \Rightarrow \quad y = \sin(x + c).\]

(B) \[\frac{dy}{dx} = -\sqrt{1 - y^2} \quad \Rightarrow \quad \frac{1}{\sqrt{1 - y^2}} \quad dy = -dx \quad \Rightarrow \quad \sin^{-1} y = -x + c \quad \Rightarrow \quad y = \sin(c - x).\]

Therefore, the following functions can be the solution to this problem:

\[
y = \sin x, \quad y = -\sin x, \quad y = \cos x, \quad y = -\cos x, \quad y = 1, \quad y = -1, \text{ etc.}
\]