Review Sheet for Midterm Exam

Problem 1. Determine whether the following statements are True or False. Justify your answer.

(a) For $A \in \mathbb{R}^{m \times n}$, $\text{range}(A) = \text{range}(A^\top A)$

(b) If the vectors $a_1$, $a_2$ and $b$ are orthogonal, then the projection of $b$ onto the plane containing $a_1$ and $a_2$ equals $\|b\|_2$

(c) $(AB)^+ = B^+A^+$

(d) The system $Ax = b$ with

$$ A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 1 & -1 \\ 0 & 2 & 2 \\ 1 & -3 & -1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 1 \end{pmatrix} $$

has unique least squares solution.

(e) If $q_1, q_2, \ldots, q_k$ are orthogonal vectors in $\mathbb{R}^n$, and $v \in \mathbb{R}^n$ then

$$ r = v - \sum_{i=1}^{k} (v^\top q_i) q_i, $$

is orthogonal to $q_i$, $i = 1, 2, \ldots, k$.

(f) If $AA^\top = A^\top A$, then $\|Ax\|_2 = \|A^\top x\|_2$

(g) Given a vector $u \in \mathbb{R}^n$, the trace of the matrix $P = \frac{uu^\top}{u^\top u}$ always equals 1

Problem 2. Suppose the singular value decomposition of $A \in \mathbb{R}^{m \times n}$, $m \geq n$, is $A = U\Sigma V^\top$, and denote by $A^+$ the pseudoinverse of $A$

(a) Show that the SVD of $A^+$ is $A^+ = V\Sigma^+ U^\top$

(b) Describe $\Sigma^+$ (i.e., size and entries $\sigma_{i,j}^+$)

(c) What is the pseudoinverse of $A^+$?

(d) Show that $AA^+$ is a projector. What subspace does $AA^+$ projects onto?

Problem 3. Find the projection matrix $P$ onto the space spanned by $u = (1, 0, 1)^\top$ and $v = (1, 1, -1)^\top$

Problem 4. What multiple of $a = (1, 1, 1)$ is closest to the point $b = (2, 4, 4)$?

Problem 5. In $n$ dimensions, what angle does the vector $(1, 1, 1, \ldots, 1)^\top$ make with the coordinate axis? What is the projection matrix $P$ onto that vector?
Problem 6. Suppose $P$ is the projection matrix onto the subspace $S$ and $Q$ the projection matrix onto the orthogonal complement $S^\perp$.

(a) What is $P + Q$?

(b) What is $PQ$?

(c) Show that $P - Q$ is its own inverse.

Problem 7. Find the matrix that projects every point in the plane onto the line $x + 3y = 0$.

Problem 8. If the vectors $a_1$, $a_2$, and $b$ are orthogonal, what are $A^\top A$ and $A^\top b$?

Problem 9. If $V$ is the subspace spanned by $(1, 1, 0, 1)^\top$ and $(0, 0, 1, 0)^\top$, find

(a) a basis for the orthogonal complement $V^\perp$

(b) the projection matrix $P$ onto $V$

(c) the vector in $V$ closest to the vector $b = (0, 1, 0, -1)^\top$ in $V^\perp$.

Problem 10. Let $a_1 = \left(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right)^\top$, $a_2 = \left(-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)^\top$, $a_3 = \left(\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right)^\top$, and $b = (0, 3, 0)^\top$.

(a) Project $b$ onto $a_1$ and $a_2$ and find its projection onto the plane containing $a_1$ and $a_2$.

(b) Find the projection of $b$ onto $a_3$, then add up the three one-dimensional projections. Why is $P = a_1a_1^\top + a_2a_2^\top + a_3a_3^\top = I$?

Problem 11. Let

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

(a) Compute $\|A^{-1}\|_2 = \frac{1}{\lambda_1}$, $\|A\|_2 = \lambda_2$, $\kappa(A) = \frac{\lambda_2}{\lambda_1}$

(b) Find a right side $b$ and a perturbation $\delta b$ so that the error, $\frac{\|\delta x\|}{\|x\|} = \kappa \frac{\|\delta b\|}{\|b\|}$, is largest.

Problem 12. Give an asymptotic estimate of the number of flops (+, −, ×, and ÷) in algorithms 10.1, 10.2, and 10.3 in the textbook.