High-order Central Schemes for Hyperbolic Systems of Conservation Laws

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Outline

- Central schemes for hyperbolic conservation laws: overview and implementation
- Central schemes and MHD equations: the $\nabla \cdot \mathbf{B} = 0$ constraint
- Some examples: Euler equations of Gas Dynamics and Ideal MHD equations
We consider hyperbolic conservation laws in general

In one space dimension:

\[ u_t + f(u)_x = 0, \]

and two space dimensions:

\[ u_t + f(u)_x + g(u)_y = 0, \]

with some initial data

\[ u(x, y, 0) = u_0(x, y), \]

where the Jacobian matrices \( \frac{\partial f}{\partial u} \) and \( \frac{\partial g}{\partial u} \) are diagonizable with real eigen values.
Challenges

- discontinuous solutions: even when the initial conditions are smooth, they evolve into steep gradients
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- we seek efficient numerical schemes capable of handling these challenges
What do central schemes offer?

simplicity: no Riemann solvers

Upwind Scheme

Requires a Riemann solver to distinguish from right- and left-going waves

Central Scheme

Evolves solution over staggered grid, no Riemann solver is needed, but staggering requires smaller time step
More Advantages

- ease of implementation: no dimensional splitting in multidimensional models
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- straightforward extension to higher space dimensions
- highly adaptable implementation: minor changes required to solve different problems
- easy to parallelize: sequential function calls $\rightarrow$ concurrent function calls
We begin by integrating the conservation law

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\[
\frac{1}{\Delta x} \int_{t^n}^{t^{n+1}} \int_{x - \frac{\Delta x}{2}}^{x + \frac{\Delta x}{2}} u_t \, dt \, dx = -\frac{1}{\Delta x} \int_{t^n}^{t^{n+1}} \int_{x - \frac{\Delta x}{2}}^{x + \frac{\Delta x}{2}} f(u)_x \, dt \, dx
\]

over the control volume \([x_j, x_{j+\frac{1}{2}}] \times [t^n, t^{n+1}],\)
Fully-discrete Central Schemes – One Dimension

We begin by integrating the conservation law

\[ u_t + f(u)_x = 0 \]

over the control volume \([x_j, x_{j+1}] \times [t^n, t^{n+1}]\), this leads the equivalent cell average formulation

\[ \bar{u}_{j+1/2}^{n+1} = \bar{u}_{j+1/2}^n - \frac{1}{\Delta x} \int_{t^n}^{t^{n+1}} \left[ f(u(x_{j+1}, t)) - f(u(x_j, t)) \right] dt \]

We now proceed in two steps:
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$$\bar{u}_{j+\frac{1}{2}}^{n+1} = \bar{u}_{j+\frac{1}{2}}^n - \frac{1}{\Delta x} \int_{t^n}^{t^{n+1}} \left[ f(u(x_{j+1}, t)) - f(u(x_j, t)) \right] dt$$

We now proceed in two steps:

1. From the cell averages $\{\bar{u}_j^n\}$, a non-oscillatory polynomial reconstruction,

$$\tilde{u}(x, t^n) = \sum_j p_j(x, t^n) \cdot 1_{l_j},$$

is formed to recover $\{\bar{u}_{j+\frac{1}{2}}^n\}$; where $l_j = [x_j - \Delta x/2, x_j + \Delta x/2]$. 
2. Time evolution
Fully-discrete Central Schemes – One Dimension

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The fully discrete approximation reads:

- predictor:
  \[ u_j^{n+\frac{1}{2}} := \bar{u}_j^n - \frac{\lambda}{2} f_j', \quad \lambda = \frac{\Delta t}{\Delta x}, \]
2. Time evolution

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- approximate flux integrals with simple quadrature formulae (e.g., midpoint or Simpson’s).

The fully discrete approximation reads:

- predictor:
  \[
  u_j^{n+\frac{1}{2}} := \bar{u}_j^n - \frac{\lambda}{2} f'_j, \quad \lambda = \frac{\Delta t}{\Delta x},
  \]

- corrector:
  \[
  \bar{u}_{j+\frac{1}{2}}^{n+1} = \frac{1}{2}[\bar{u}_j^n + \bar{u}_{j+1}^n] + \frac{1}{8}[u'_j - u'_{j+1}] - \lambda [f(u_{j+\frac{1}{2}}^{n+\frac{1}{2}}) - f(u_j^{n+\frac{1}{2}})].
  \]
The staggered scheme can be extended to two space dimensions
Fully-discrete Central Schemes – Two Dimensions

The staggered scheme can be extended to two space dimensions

- predictor

\[ u^{n+\frac{1}{2}}_{j,k} := \bar{u}^n_{j,k} - \frac{\lambda}{2} f_{j,k} - \frac{\mu}{2} g_{j,k}, \]

where \( \lambda = \frac{\Delta t}{\Delta x} \) and \( \mu = \frac{\Delta t}{\Delta z} \)
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- **predictor**

\[ u_{j,k}^{n+\frac{1}{2}} := \bar{u}_{j,k}^n - \frac{\lambda}{2} f_{j,k} - \frac{\mu}{2} g_{j,k}, \]

where \( \lambda = \frac{\Delta t}{\Delta x} \) and \( \mu = \frac{\Delta t}{\Delta z} \)

- **corrector**

\[
\begin{align*}
\bar{u}_{j+\frac{1}{2},k+\frac{1}{2}}^{n+1} &= \frac{1}{4} (\bar{u}_{j,k}^n + \bar{u}_{j+1,k}^n + \bar{u}_{j,k+1}^n + \bar{u}_{j+1,k+1}^n) + \frac{1}{16} (u'_{j,k} - u'_{j+1,k}) \\
&- \frac{\lambda}{2} \left[ f(u_{j+1,k}^{n+\frac{1}{2}}) - f(u_{j,k}^{n+\frac{1}{2}}) \right] + \frac{1}{16} (u'_{j,k+1} - u'_{j+1,k+1}) - \frac{\lambda}{2} \left[ f(u_{j+1,k+1}^{n+\frac{1}{2}}) - f(u_{j,k+1}^{n+\frac{1}{2}}) \right] \\
&+ \frac{1}{16} (u'_{j,k} - u'_{j,k+1}) - \frac{\mu}{2} \left[ g(u_{j,k+1}^{n+\frac{1}{2}}) - g(u_{j,k}^{n+\frac{1}{2}}) \right] \\
&+ \frac{1}{16} (u'_{j+1,k} - u'_{j+1,k+1}) - \frac{\mu}{2} \left[ g(u_{j+1,k+1}^{n+\frac{1}{2}}) - g(u_{j+1,k}^{n+\frac{1}{2}}) \right]
\end{align*}
\]
Modified central differencing (Kurganov and Tadmor, 2000)

Using the information provided by the local speed of propagation,

$$a_{j + \frac{1}{2}}^n = \max_{u \in C(u_{j + \frac{1}{2}}^- : u_{j + \frac{1}{2}}^+)} \rho \left( \frac{\partial f}{\partial u}(u) \right),$$

where

$$u_{j + \frac{1}{2}}^+ := p_{j+1}(x_{j + \frac{1}{2}}) \quad \text{and} \quad u_{j + \frac{1}{2}}^- := p_j(x_{j + \frac{1}{2}}),$$
Semi-discrete Central Schemes – One Dimension

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- we distinguish between the regions where the solution remains smooth – no Riemann fans, and regions where discontinuities propagate.
Semi-discrete Central Schemes – One Dimension

- two sets of evolved values are calculated:
  - staggered values over non-smooth regions \( \{ \bar{w}_{j+\frac{1}{2}}^{n+1} \} \)
  - non-staggered evolution over smooth regions \( \{ \tilde{w}_j^{n+1} \} \)
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- the values in these two sets can be interpolated and reprojected as cell averages \( \{ \bar{u}_j^{n+1} \} \) onto the original non-staggered grid (Jiang et. al., 1998)
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- Or one can take the limit as \( \Delta t \to 0 \) to arrive at the semi-discrete formulation:

\[
\frac{d}{dt} \bar{u}_j(t) = \lim_{\Delta t \to 0} \frac{\bar{u}_j^{n+1} - \bar{u}_j^n}{\Delta t} = -\frac{H_{j+\frac{1}{2}}(t) - H_{j-\frac{1}{2}}(t)}{\Delta x},
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\]

where \( H_{j+\frac{1}{2}}(t) := \frac{f(u_{j+\frac{1}{2}}^{+}(t)) + f(u_{j+\frac{1}{2}}^{-}(t))}{2} - \frac{a_{j+\frac{1}{2}}(t)}{2} \left[ u_{j+\frac{1}{2}}^{+}(t) - u_{j+\frac{1}{2}}^{-}(t) \right] \)
Semi-discrete Central Schemes – Two Dimensions

Similarly, in two space dimensions, we apply:

\[
\begin{align*}
D_{jk} &= \frac{1}{2}(x_{j+1} - x_j) \left( z_{k+1} - z_k - z_k + z_{k-1} \right) \\
\end{align*}
\]
Semi-discrete Central Schemes – Two Dimensions

Similarly, in two space dimensions, we apply:

- staggered evolution over red cells
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- staggered evolution over red cells
- staggered evolution in one direction over green strips
- non-staggered evolution over $D_{j,k}$, and

- reprojecting over original cells and taking the limit as $\Delta t \to 0$, we arrive at:

$$
\frac{d}{dt} \bar{u}_j(t) = - \frac{H^x_{j + \frac{1}{2}, k}(t) - H^x_{j - \frac{1}{2}, k}(t)}{\Delta x} - \frac{H^z_{j, k + \frac{1}{2}}(t) - H^z_{j, k - \frac{1}{2}}(t)}{\Delta z}
$$
Central Schemes – Reconstruction

Example of a non-oscillatory reconstruction:
second order minmod reconstruction (Van Leer, 1979)

\[ p_{j,k}(x, z) = \bar{u}_{j,k}^n + u'_j, k \left( \frac{x - x_j}{\Delta x} \right) + u'_k \left( \frac{z - z_k}{\Delta z} \right) \]

where

\[ u'(j, k) = \text{minmod} \left( \alpha \Delta_+, x \bar{u}_{j,k}^n, \frac{1}{2} \Delta_{0,x} \bar{u}_{j,k}^n, \alpha \Delta_-, x \bar{u}_{j,k}^n \right), \]

\[ u'(j, k) = \text{minmod} \left( \alpha \Delta_+, z \bar{u}_{j,k}^n, \frac{1}{2} \Delta_{0,z} \bar{u}_{j,k}^n, \alpha \Delta_-, z \bar{u}_{j,k}^n \right), \]

with \( 1 \leq \alpha < 2 \), and

\[ \text{minmod}(x_1, x_2, \ldots, x_n) = \begin{cases} 
\min x_i, & \text{if } x_i > 0 \ \forall i \\
\max x_i, & \text{if } x_i < 0 \ \forall i \\
0 & \text{otherways}
\end{cases} \]
Semi-discrete Central Schemes – Time Evolution

Solution evolved with SSP RK Schemes (Gottlieb et. al., 2001),

Example: Third-order scheme

\begin{align*}
    u^{(1)} &= u^{(0)} + \Delta t C[u^{(0)}], \\
    u^{(2)} &= u^{(1)} + \frac{\Delta t}{4} (-3 C[u^{(0)}] + C[u^{(1)])}, \\
    u^{n+1} := u^{(n+1)} &= u^{(n)} + \frac{\Delta t}{12} (-C[u^{(0)}] - C[u^{(1)}] + 8 C[u^{(2)}]),
\end{align*}

where

\[
    C[w(t)] = -\frac{H_x^{j+\frac{1}{2},k}(w(t)) - H_x^{j-\frac{1}{2},k}(w(t))}{\Delta x} - \frac{H_z^{j,k+\frac{1}{2}}(w(t)) - H_z^{j,k-\frac{1}{2}}(w(t))}{\Delta z}
\]
Ideal MHD Equations

- conservation of mass:
  \[ \rho_t = -\nabla \cdot (\rho \mathbf{v}), \]

- conservation of momentum:
  \[ (\rho \mathbf{v})_t = -\nabla \cdot [\rho \mathbf{vv}^\top + (p + \frac{1}{2} B^2) \mathbb{I}_{3 \times 3} - \mathbf{BB}^\top], \]

- conservation of energy:
  \[ e_t = -\nabla \cdot \left[ \left( \frac{\gamma}{\gamma - 1} p + \frac{1}{2} \rho \mathbf{v}^2 \right) \mathbf{v} - (\mathbf{v} \times \mathbf{B}) \times \mathbf{B} \right], \]

- transport equation:
  \[ B_t = \nabla \times (\mathbf{v} \times \mathbf{B}) \]
Ideal MHD Equations

- conservation of mass:
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- conservation of momentum:
  \[ (\rho \mathbf{v})_t = -\nabla \cdot [\rho \mathbf{v} \mathbf{v}^\top + (p + \frac{1}{2} B^2) \mathbb{I}_{3 \times 3} - \mathbf{B} \mathbf{B}^\top], \]

- conservation of energy:
  \[ e_t = -\nabla \cdot \left[ \left( \frac{\gamma}{\gamma - 1} p + \frac{1}{2} \rho v^2 \right) \mathbf{v} - (\mathbf{v} \times \mathbf{B}) \times \mathbf{B} \right], \]

- solenoidal constraint:
  \[ \nabla \cdot \frac{\partial \mathbf{B}}{\partial t} = \nabla \cdot [\nabla \times (\mathbf{v} \times \mathbf{B})] \quad \Rightarrow \quad \frac{\partial}{\partial t} (\nabla \cdot \mathbf{B}) = 0 \]
Solenoidal Constraint

- Numerical results indicate central schemes maintain $\nabla \cdot \mathbf{B}$ small
Solenoidal Constraint

- Numerical results indicate central schemes maintain $\nabla \cdot \mathbf{B}$ small
- Also, using the constraint transport approach and notation (Evans and Hawley, 1988), it can be shown that the magnetic field, as evolved by the second order fully-discrete staggered scheme (JT), can be written as

$$B_{\frac{j+1}{2}, \frac{k+1}{2}}^{x,n+1} = \frac{1}{2} (b_{j, \frac{k+1}{2}}^{x,n+1} + b_{j+1, \frac{k+1}{2}}^{x,n+1})$$

with

$$b_{j, \frac{k+1}{2}}^{x,n+1} = \tilde{b}_{j, \frac{k+1}{2}}^{x,n} - \frac{\Delta t}{\Delta z} \left( \Omega_{\frac{j+1}{2}}^{n+\frac{1}{2}} - \Omega_{\frac{j+1}{2}}^{n+\frac{1}{2}} \right)$$

and a similar expression for $B_{\frac{j+1}{2}, \frac{k+1}{2}}^{z,n+1}$.
MHD: Brio-Wu Rotated Shock Tube

- One-dimensional Riemann problem with initial states given by

\[
(\rho, v_x, v_y, v_z, B_x, B_y, B_z, p)^\top = \begin{cases} 
(1, 0, 0, 0, 0.75, 0, 1, 1)^\top & \text{for } x < 0 \\
(0.125, 0, 0, 0, 0.75, 0, -1, 0.1)^\top & \text{for } x > 0
\end{cases}
\]

- Solved over a two dimensional domain with the direction of the flow rotated 45°

- Solution computed up to \( t = 0.2 \), \( x \in [-1, 1] \), with 600 \times 600 grid points, \( \gamma = 2 \).
MHD: Brio-Wu Rotated Shock Tube

Solution at $t = 0.2$

From top to bottom and from left to right: density, transverse velocity, transverse magnetic field, parallel magnetic field, and pressure. The divergence of the reconstructed polynomial $\sim 10^{-13}$. Results computed with Jacobian free formulation of 2nd order JT scheme.
MHD: Shock – Cloud Interaction

- Disruption of a high density cloud by a strong shock
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- Disruption of a high density cloud by a strong shock

- Initial conditions

\[
(\rho, v_x, v_y, v_z, B_x, B_y, B_z, p)^\top = \begin{cases} 
(3.86, 0, 0, 0, -2.18, 2.18, 167.34)^\top & \text{for } x < 0.6 \\
(1, -11.25, 0, 0, 0.564, 0.564, 1)^\top & \text{for } x > 0.6 
\end{cases}
\]

high density cloud – \( \rho = 10, \ p = 1 \) – centered at \( x = 0.8, \ y = 0.5 \), with radius 0.15,
MHD: Shock – Cloud Interaction

- Disruption of a high density cloud by a strong shock

- Initial conditions

$$(\rho, v_x, v_y, v_z, B_x, B_y, B_z, p)^T = \begin{cases} 
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\end{cases}$$

- high density cloud $- \rho = 10, p = 1$ – centered at $x = 0.8, y = 0.5$, with radius 0.15,

- Solved up to $t = 0.06$, $(x, z) \in [0, 1] \times [0, 1]$, with $256 \times 256$ grid points, CFL number 0.5 and $\gamma = 5/3$
Solution of shock-cloud interaction, left: density at $t=0$, center: density at $t=0.06$, right: magnetic field lines at $t=0.06$. Results computed with 3rd order semi-discrete scheme.
Euler Equations of Gas Dynamics

- conservation of mass:
  \[ \rho_t = - \nabla \cdot (\rho \mathbf{v}), \]

- conservation of momentum:
  \[ (\rho \mathbf{v})_t = - \nabla \cdot (\rho \mathbf{v} \mathbf{v}^\top + p \mathbb{I}_{3 \times 3}), \]

- conservation of energy:
  \[ e_t = - \nabla \cdot \left[ \left( \frac{\gamma}{\gamma - 1} p + \frac{1}{2} \rho \mathbf{v}^2 \right) \mathbf{v} \right], \]

- equation of state:
  \[ p = (\gamma - 1) \left[ e - \frac{1}{2} \rho \mathbf{v}^2 \right] \]
Euler Equations: 2d Riemann Problem

Solution of a 2d Riemann problem, left: density at t=0 and initial conditions, center: density at t=0.3 \((S_{21}, S_{32}, S_{34}, S_{41})\), right: pressure at t=0.3. Results computed with 3rd order semi-discrete scheme using 400 × 400 grid cells.